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GROUPS WITH FREE NONABELIAN SUBGROUPS

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Conditions are given which characterize the class of groups with free nonabelian subgroups. It is also shown that a generalization of the conditions characterize the class of nonamenable groups. Neither of these two (possibly identical) classes of groups is the class of models of any finite set of first-order axioms.

THEOREM 1. *A group G has a free nonabelian subgroup if and only if there are two subsets M, N of G and two elements a, b of G such that:*

$$M \cup N = G \tag{1}$$

$$aM \cap bN = \varnothing \tag{2}$$

$$aM \cup bN \subset M \cap N . \tag{3}$$

LEMMA. *Let X be a set of elements of a group G such that if $x \in X$ then $x^{-1} \notin X$ and let Y be the set of x and x^{-1} for $x \in X$. Suppose that to each $y \in Y$ is associated a subset U_y of G such that $y \in U_y$, $1 \notin U_y$, and for all $y, z \in Y$ such that $yz \neq 1$, we have $yU_z \subset U_y$. Then the subgroup generated by X is free.*

Proof of the lemma. By induction on length we show that any reduced word of positive length in elements of X is in the union of the U_y and thus it is not the identity.

Proof of Theorem 1. Assume G has a free subgroup H on two free generators a, b and let R be a set consisting of one element from each right coset of H in G such that $1 \in R$. Let M be the set of all elements wr for $r \in R$ and w a reduced word in H not beginning with a^{-1} and let N be the set of all elements zr for $r \in R$ and z a reduced word in H not beginning with b^{-1}. Then the conditions of the theorem are satisfied.

Conversely, suppose that the conditions of the theorem are satisfied for subsets M, N and elements a, b. First we observe that we may assume

$$a^{-1} \notin M \text{ and } b^{-1} \notin N . \tag{4}$$

For example, if $a^{-1} \in M$, then $b^{-1} \notin N$ by (2) and we replace a with ba and N with N-aM. The resulting subsets $M, N - aM$ and corresponding elements ba, b satisfy (4) and the three conditions of Theorem 1. (The proof, which we omit, is straightforward. To establish condition (3) for the replacement sets we use $baM \subset bN$, which is a consequence of (3) for the given sets M, N.) Next we observe that we may assume

$$(5) \qquad 1 \in M \cap N .$$

For example, if $1 \notin M$ then $1 \in N$ by (1). Also by (1) and (3) either $a \in M$ or $ba \in M$ and by (3) $M \subset a^{-1}M$ and $M \subset a^{-1}b^{-1}M$. If $a \in M$, replace a with a^2 and M with $a^{-1}M$ and if $ba \in M$, replace a with aba and M with $a^{-1}b^{-1}M$. Under these replacements, or similar ones involving N if $1 \notin N$, condition (4) remains valid. The conditions in the lemma are satisfied with $X = \{a, b\}$, $U_a = aM$, $U_b = bN$, $U_{a^{-1}} = a^{-1}(G\text{-}aM)$, and $U_{b^{-1}} = b^{-1}(G\text{-}bN)$. We conclude that G has a free nonabelian subgroup.

A problem posed in 1957 (cf. [1, p. 520]) is to determine whether the class of groups with free nonabelian subgroups is identical with the class of nonamenable groups. Evidence suggests that the two classes coincide (cf. [2, p. 12] and [3, p. 9]). By Theorem 1 the two classes do coincide if and only if the finite sequences mentioned in the following theorem may be limited to two terms.

THEOREM 2. *A group G is not amenable if and only if there is a finite sequence $(a_1, M_1), (a_2, M_2), \cdots, (a_n, M_n)$ of (not necessarily distinct) ordered pairs where $a_i \in G$ and $M_i \subset G$ for $i = 1, \cdots, n$, and such that for every $x \in G$ the number of terms with $x \in M_i$ is strictly greater than the number of terms with $x \in a_iM_i$.*

Proof. This theorem is a consequence of a theorem by Dixmier (cf. [3, p. 4]). The conditions in Dixmier's theorem are equivalent to the same conditions with the functions restricted to characteristic functions.

A class of groups is a *generalized elementary class* in case it is the class of models of a (possibly infinite) set of first-order sentences (cf. [5, p. 92, problem 2]). An easy application of the compactness theorem (cf. 5, p. 70]) shows that any generalized elementary class of groups that contains all finite groups also contains an extension of every residually finite group. Also every free group is residually finite (cf. [4, p. 116]), so every free group has an extension in any generalized elementary class that contains all finite groups. It follows that neither the class of groups with free nonabelian subgroups nor the class of nonamenable groups is the class of models of a finite set of first-order axioms because in each case the complementary class is not a generalized elementary class.

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