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A CONDITIONALLY COMPACT POINT SET WITH NONCOMPACT CLOSURE

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Sometime in 1930, Leo Zippin showed that there exists a complete Moore space that contains a conditionally compact point set whose closure is not compact. It is the object of this paper to show that if the hypothesis of the continuum is true then there exists a separable, complete Moore space which contains such a point set and, furthermore, satisfies R. L. Moore's Axioms 2, 3, 4, 5, and 6. Theorem 1, concerning the existence of certain subsets of the Cartesian plane, is fundamental to the construction of this example and its proof constitutes a major portion of this paper.

A complete Moore space is one satisfying Axioms 0 and 1 of [1]. The terms compact and conditionally compact are as defined in [2] and other definitions and notation are as in [1].

In a Cartesian plane E, let M denote the point set to which the point P belongs if and only if each coordinate of P is a positive integer. Let H denote the collection of point sets to which the point set h belongs if and only if the points of h are the points of an infinite sequence $P_1, P_2, P_3, \cdots$ of points of M such that (a) P_1 is the point (1,1) and (b) for each positive integer n, P_{n+1} is at a distance of 1 from P_n and either above it or to the right of it. For each integer k, let T_k denote a translation in E such that $T_k(0, 0) = (k, -k)$. For each point set h of H and each point P of M, let A_{hP} denote the set to which X belongs if and only if X is a point of $M \cdot T_k(h)$ where k is the integer j such that $T_j(h)$ contains P. For each point set h of H, let G_h denote the collection to which g belongs if and only if for some point P of M, g is A_{hP}.

LEMMA 0. *If h is a point set of H and P and Q are two points of M, then either $A_{hP} = A_{hQ}$ or A_{hP} and A_{hQ} do not intersect.*

LEMMA 1. *Suppose h and h' are two point sets of H such that h' does not contain infinitely many points of any point set of G_h. Then if g and g' are two sets of G_h and G'_h respectively, g does not contain infinitely many points of g'.*

Proof. Suppose P is a point of $g \cdot g'$. There exist integers j and k such that $T_j(P)$ is a point X of h and $T_k(P)$ is a point Y of h'. There exists an integer i such that $T_i(X) = Y$. If P' is any point of $g \cdot g'$, $T_i(T_j(P'))$ is a point of h', thus if g contains infinitely many points of g', then h' contains infinitely many points of the set $T_i(T_j(g))$ of G_h.

Let $h_1(h_2)$ denote the set of all points of M with abscissa (ordinate) 1. Suppose h is a point set of H, $P_1, P_2, P_3, \cdots$ are the points of a point set g of G_h, and P and Q are two points of M not in g. Let R denote the point set to which the point X belongs if and only if X is a point of either (a) the straight line ray with endpoint P_1 and slope 1 which contains no point of $M - P_1$ or (b) the straight line interval from P_i to P_{i+1} for some position integer i. Then P and Q are said to lie on opposite sides of g or to lie on the same side of g according as R does or does not separate P from Q in the plane.

LEMMA 2. *Suppose* (1) *H' is a finite subcollection of H such that* (*a*) *h_1 and h_2 belong to H' and* (*b*) *if h and h' are two point sets of H', no point set of G_h has infinitely many points in common with any one point set of $G_{h'}$,* (2) *P is a point of M, and* (3) *K is an infinite subset of M such that for no point set h of H' does any point set of G_h contain infinitely many points of K. Then there exists an infinite subset K' of K and a point Q of K' such that* (1) *for each point set h of H', each point of K' lies on the same side of A_{hP} as Q,* (2) *for each two point sets h and h' of H', each point of K' lies above and to the right of every point of $(A_{hP}) \cdot (A_{h'P})$, and* (3) *if Q' is a point of K' distinct from Q, there exists a point set h of H containing Q and Q' such that no point of h between Q and Q' lies on the opposite side of $A_{h'P}$ from Q for any point set h' of H'.*

Proof. Let $h_1, h_2, h_3, \cdots, h_n$ denote the point sets of the collection H'. Let K_1 denote some infinite subset of K such that each point of K_1 lies to the right of A_{h_1P}. There exists a sequence $K_1, K_2, K_3, \cdots, K_n$ such that K_1 is an infinite subset of K and for each positive integer i greater than 1 but not greater than n, K_i is an infinite subset of K_{i-1} and each two points of K_i lie on the same side of A_{h_iP}.

Let W denote the point set to which the point w belongs if and only if for some two point sets h and h' of H', w is a point of $(A_{hP}) \cdot (A_{h'P})$. Since no point set of G_h has infinitely many points in common with any one point set of $G_{h'}$, it follows from Lemma 1 that W is a finite subset of M. There exist points P_1 and P_2 of W such that no point of W is above $A_{h_2P_2}$ and no point of W is to the right of $A_{h_1P_1}$. Let K_n' denote the set of all points of K_n which lie above $A_{h_2P_2}$ and to the right of $A_{h_1P_1}$. Since no point set of $G_{h_1} + G_{h_2}$ contains infinitely many points of K_n, K_n' is an infinite point set.

Let Q denote some point of K'_n and let K' denote the set whose points are Q and those points of K'_n which lie both to the right of A_{h_1Q} and above A_{h_2Q}.

Suppose Q' is a point of K' distinct from Q. Q' lies above and to the right of Q. Let Q'' denote the intersection of A_{h_1Q} with $A_{h_2Q'}$. If no point of M between Q and Q'' belongs to $A_{h'P}$ for any point set h' of H', let Z denote the subset of M whose points are Q'', the points of M, if any, between Q and Q'', and the points of M, if any, between Q'' and Q'. If, however, for some point set h' of H', $A_{h'P}$ contains a point between Q and Q'', let P' denote the lowest such point. Since Q is above and to the right of every point of W, h' is the only point set h of H' such that A_{hP} contains P'. Since Q' lies on the same side of $A_{h'P}$ as Q, some point of $A_{h'P}$ belongs to $A_{h_2Q'}$ and all such points lie to the left of Q'. Let P'' denote the right-most point of $(A_{h'P})\cdot(A_{h_2Q'})$. In this case, let Z denote the subset of M whose points are P'; the points of M, if any, between Q and P'; the points of $A_{h'P}$, if any, between P' and P''; P''; and the points of M, if any, between P'' and Q'.

There exists point sets h_q and $h_{q'}$ of H containing Q and Q' respectively. There exist positive integers i and j such that Q and Q' are respectively the i^{th} point of h_q and the j^{th} point of $h_{q'}$. Let Z' denote the point set whose points are the first i points of h_q. Let Z'' denote the point set whose points are those points of $h_{q'}$ other than the first $j-1$ points. The point set $Z + Z' + Z''$ is a point set h of H such that no point of h between Q and Q' lies on the opposite side of $A_{h'P}$ from Q for any point set h' of H'.

LEMMA 3. *Suppose* (1) H' *is a countable subcollection of* H *such that* (a) h_1 *and* h_2 *belong to* H' *and* (b) *if* h *and* h' *are two point sets of* H', *no point set of* G_h *has infinitely many points in common with any one point set of* $G_{h'}$ *and* (2) K *is an infinite subset of* M *such that for no point set* h *of* H' *does any point set of* G_h *contain infinitely many points of* K. *Then there exists a point set* h *of the collection* H *such that* (1) h *contains an infinite subset of* K *and* (2) h *does not contain infinitely many points of any point set of* $G_{h'}$ *for any point set* h' *of the collection* H'.

Proof. If H' is infinite, let $h_1, h_2, h_3, \cdots$ denote the point sets of H'. If H' is finite, let $h_1, h_2, h_3, \cdots, h_n$ denote the point sets of H' and for each positive integer i greater than n, let h_i denote the point set h_n. For each positive integer i, let H_i denote the collection whose members are the first $i+1$ point sets of the sequence $h_1, h_2, h_3, \cdots$. Let P_0 denote the point $(1, 1)$ and let K_0 denote the point set K.

It follows from Lemma 2 that there exist sequences $K_1, K_2, K_3, \cdots$ and $P_1, P_2, P_3, \cdots$ such that for each positive integer i, K_i is an infinite subset of K_{i-1} and P_i is a point of K_i below and to the left of each point of $K_i - P_i$ such that (1) for each point set h of H_i, K_i lies on the same side of $A_{hP_{i-1}}$ as P_i, (2) for each two point sets h and h' of H_i, each point of K_i lies above and to the right of every point of $(A_{hP_{i-1}})\cdot(A_{h'P_{i-1}})$, and (3) if Q_i is a point of K_i distinct from P_i, there exists a point set $h_i(Q_i)$ of the collection H containing P_i and Q_i such that no point of $h_i(Q_i)$ between P_i and Q_i lies on the opposite side of $A_{hP_{i-1}}$ from P_i for any point set h of H_i.

There exists a sequence $Z_1, Z_2, Z_3, \cdots$ such that (1) for each positive integer i, Z_i is a point set whose points are P_{i-1}, P_i, and those points of $h_{i-1}(P_i)$, where h_0 is some point set of H containing P_0 and P_1, between P_{i-1} and P_i and (2) no point of Z_{i+1} between P_i and P_{i+1} lies on the opposite side of $A_{hP_{i-1}}$ from P_i for any point set h of H_i.

$Z_1 + Z_2 + Z_3 + \cdots$ is a point set h of the collection H which contains the infinite subset $P_1 + P_2 + P_3 + \cdots$ of K.

Suppose h contains infinitely many points of some point set g of $G_{h'}$ for some point set h' of H'. For some positive integer j, h' is the term h_j of the sequence $h_1, h_2, h_3, \cdots$. $A_{h_jP_j}, A_{h_jP_{j+1}}, A_{h_jP_{j+2}}, \cdots$ is an infinite sequence such that for each positive integer i, $A_{h_jP_{j+i}}$ and $A_{h_jP_{j+i+1}}$ lie on the same side of $A_{h_jP_j}$ and Z_{j+i+1} lies on the same side of $A_{h_jP_{j+i-1}}$ as P_{j+i}. Thus for each positive integer i, only finitely many points of h are on the same side of $A_{h_jP_{j+i}}$ as P_{j-1}. Therefore, contrary to supposition, no point set of G_{h_j} contains infinitely many points of h.

LEMMA 4. *If H' is a countable subcollection of H, then there exists an infinite subset K of M such that for each point set h of H', no point set of G_h contains infinitely many points of K.*

Proof. Let G denote the collection to which g belongs if and only if for some point set h of H', g is a point set of G_h. Let $g_1, g_2, g_3, \cdots$ denote the point sets of the collection G. There exists a sequence $P_1, P_2, P_3, \cdots$ such that for each positive integer i, P_i is a point of M which does not belong to any point set of the sequence $g_1, g_2 + P_1, g_3 + P_2, \cdots, g_i + P_{i-1}$. $P_1 + P_2 + P_3 \cdots$ is an infinite subset K of M such that no point set of G contains infinitely many points of K.

THEOREM 1. *If the hypothesis of the continuum is true, then there exists an uncountable subcollection of H' of the collection H*

such that (1) *if h and h' are two point sets of H', no point set of G_h contains infinitely many points of any one point set of $G_{h'}$ and* (2) *if K is an infinite subset of M, there exists a point set h of H' such that some point set of the collection G_h contains infinitely many points of K.*

Proof. Let W denote the collection of all infinite subsets of M. W is equally numerous with the number interval [0, 1]. Thus since no uncountable subset of [0, 1] is less numerous than [0, 1], there exists a meaning P_W of the word precedes with respect to which W is well ordered such that h_1 is the first point set of W, h_2 is the second point set of W, and no point set of W is preceded by uncountably many point sets of W. There exists a meaning P_H of the word preceded with respect to which the collection H is well ordered such that h_1 is the first point set of H and h_2 is the second point set of H.

It follows from Lemmas 1 and 3 that there exists a transformation T of W into a subcollection H' of H such that (1) $T(w_1) = h_1$ and $T(w_2) = h_2$, (2) if w is a point set of W distinct from w_1 and w_2 such that for some point set w' of W preceding w, some point set of the collection $G_{T(w')}$ contains infinitely many points of w, then for the first such point set w'' of W in the P_W sense, $T(w) = T(w'')$ and (3) if w is a point set of W distinct from w_1 and w_2 such that for each point set w' of W preceding w, no point set of the collection $G_{T(w')}$ contains infinitely many points of w, $T(w)$ is the first point set h of the collection H in the P_H sense such that h contains infinitely many points of w and for each point set w'' of W that precedes w in the P_W sense, h does not contain infinitely many points of any point set of the collection $G_{T(w'')}$.

It follows from Lemma 4 that H' is an uncountable subcollection of H. The collection H' fulfills the requirements of Theorem 1.

THEOREM 2. *If the hypothesis of the continuum is true, there exists a separable space satisfying Axions* 0, 1, 2, 3, 4, 5, *and* 6 *of* [1] *and containing a conditionally compact point set whose closure is not compact.*

Proof. Let M, G_h, h_1, and h_2 be as previously defined. Let H' denote some collection of point sets containing h_1 and h_2 and satisfying conditions (1) and (2) of Theorem 1. Let G denote the collection to which g belongs if and only if g is a point set of G_h for some point set h of the collection H'.

Suppose P is a point (x, y) of M. Let A_P and B_P denote the endpoints of an interval I such that (1) A_P is above B_P, (2) P is the midpoint of I, (3) the length of I is $(x+y)^{-1}$, and (4) I has slope -1. For each number k between 0 and 1 let A_{Pk} denote the point X of the interval PA_P such that the length of the interval XP is the product of k and the length of the interval PA_P. Let B_{Pk} denote point X of the interval PB_P such that P is the midpoint of the sub-interval XA_{Pk} of the interval A_PB_P.

Suppose g is a point set of G and k is a number between 0 and 1. Let L_{gk} denote the point set to which the point w belongs if and only if for some positive integer i, w is either a point of the interval $A_{P_ik}A_{P_{i+1}k}$ or a point of the interval $B_{P_ik}B_{P_{i+1}k}$, where P_i is the i^{th} point of g. Let L_{g0} denote the point set to which the point w belongs if and only if for some positive integer i, w is a point of the interval P_iP_{i+1}, where P_i is the i^{th} point of g.

Suppose g is a point set of G, k is a number between 0 and 1, and n is a positive integer. Let R_{gkn} denote the set to which w belongs if and only if either (1) w is L_{g0}, (2) for some number c between 0 and k, w is L_{gc}, or (3) w is a point of E which is separated from $(0, 0)$ by the point set consisting of L_{gk} and the interval A_PB_P for the n^{th} point P of g.

Suppose g is a point set of G, k_1 and k_2 are numbers between 0 and 1, and n is a positive integer. Let $R_{gk_1k_2n}$ denote the set to which w belongs if and only if either (1) for some number k between k_1 and k_2, w is L_{gk} or (2) w is a point of E which is separated from $(0, 0)$ by the sum of L_{gk_1}, L_{gk2}, and the sub-intervals $A_{Pk_1}A_{Pk_2}$ and $B_{Pk1}B_{Pk_2}$ of the interval A_PB_P for the n^{th} point P of g.

Let Σ denote a space such that (1) P is a point of Σ if and only if either (a) P is a point of E or (b) for some point set g of G and some number k between 0 and 1, P is either L_{gk} or L_{g0} and (2) R is a region in Σ if and only if either (a) for some point P in E and some positive integer n, R is the interior of a circle with center P and radius $1/n$, (b) R is R_{gkn} for some g, k, and n, or (c) R is $R_{gk_1k_2n}$ for some g, k_1, k_2, and n.

The set S of all points of Σ is the sum of two mutually exclusive point sets E and F. If P is a point of F, then P is L_{gk} for some g and k (including $k=0$); indeed, P is a limit point of the infinite subset $A_{P_1k}+A_{P_2k}+A_{P_3k}+\cdots$ of E (in case $k=0$, $A_{P_jk}=P_j$) where $P_1+P_2+P_3\cdots=g$. Since each point of F is a limit point of E, S is separable. For each point set g of G, let R_g denote the ray in Σ whose points are the points L_{gk} for each nonnegative number k less than 1. If P is a point of R_g for some point set g of G and g' is a point set of G distinct from g, no region containing P contains a point of $R_{g'}$.

It follows from Theorem 1 that if K is an infinite subset of M, the set of all points P of E such that each coordinate of P is a positive integer, then some point set g of the collection G contains infinitely many points of K. Thus in Σ, L_{g0} is a limit point of K and M is conditionally compact. $\bar{M} - M$ is the set of all points L_{g0} for all point sets g of G. If P is a point of $\bar{M} - M$ and R is a region containing P, R does not contain any point of R_g for any point set g of G distinct from the one that converges to P. No point of E is a limit point of F and $\bar{M} - M$ is a subset of F, thus $\bar{M} - M$ has no limit point. Therefore M is a conditionally compact point set whose closure is not compact.

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