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ON COMMUTATIVE ENDOMORPHISM RINGS

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This note deals with a finitely generated faithful module E over a commutative semi-prime noetherian ring R, with commutative endomorphism ring $\mathrm{Hom}_R(E, E) = \Omega(E)$. It is shown that E is identifiable to an ideal of R whenever $\Omega(E)$ lacks nilpotent elements; a class of examples with $\Omega(E)$ commutative but not semi-prime is discussed.

1. Main result. Throughout R will denote a commutative noetherian ring and modules will be finitely generated. In order to use the full measure of the ring, we shall consider mostly faithful modules. As for notation, unadorned $\otimes$ and Hom are taken over the base ring.

In case R is semi-prime (meaning here: no nilpotent elements distinct from 0) we recall that its total ring of quotients K is semi-simple, and thus a direct sum of fields $K = \oplus\sum K_i$, $1 \leqq i \leqq n$. Any ideal I of R has the property that $\mathrm{Hom}(I, I)$ is commutative and semi-prime: for if S denotes the set of regular elements of R,

$$\mathrm{Hom}(I, I) \subseteqq \mathrm{Hom}(I, I)_S = \mathrm{Hom}_{R_S}(I_S, I_S) .$$

But this last is a subring of K. The content of the next theorem is precisely a converse to this observation.

THEOREM 1.1. *Let E be a finitely generated faithful module over the semi-prime ring R. Then, if* $\mathrm{Hom}(E, E)$ *is commutative and semi-prime, E is isomorphic to an ideal of R.*

Proof. Denote by T the torsion submodule of E, i.e., let T be the set of elements of E annihilated by a regular element of R. If $T = 0$, then $\mathrm{Hom}(E, E) \subseteqq \mathrm{Hom}(E, E)_S = \mathrm{Hom}_{R_S}(E_S, E_S)$; using the decomposition of $R_S = K$ as a direct sum of fields,

$$\mathrm{Hom}_K(E \otimes K, E \otimes K) = \oplus\sum \mathrm{Hom}_{K_i}(E \otimes K_i, E \otimes K_i) .$$

Since $\mathrm{Hom}_K(E \otimes K, E \otimes K)$ is commutative, we must have, for each i, $E \otimes K_i = 0$ or isomorphic to K_i. This allows identification of E_S to a submodule of K and consequently of E to an ideal of R, since E is finitely generated.

Assume then, by way of contradiction, $T \neq 0$ and consider the exact sequence

$$0 \longrightarrow T \longrightarrow E \xrightarrow{\pi} F \longrightarrow 0 .$$

It yields

$$(1)\qquad 0 \longrightarrow \operatorname{Hom}(E, T) \longrightarrow \operatorname{Hom}(E, E) \xrightarrow{\pi_*} \operatorname{Hom}(F, F)$$

as T is a characteristic submodule of E; observe also that π_* is an R-algebra homomorphism. Let P be a prime ideal of R minimal over the annihilator J of T. Then $T_P \neq 0$ and can be viewed as a R_P/J_P-module; by the choice of P this last ring is artinian [2; Chap. IV, p. 147] and T_P has finite length as an R_P-module. On the other hand, localization at P does not introduce nilpotent elements in either R_P or $\Omega = \operatorname{Hom}_{R_P}(E_P, E_P)$ $(= \operatorname{Hom}(E, E)_P)$. Let I denote $\operatorname{Hom}(E, T)_P$; since T_P has finite length, I also has finite length and the sequence

$$I \supseteqq I^2 \supseteqq \cdots \supseteqq I^n \supseteqq \cdots$$

must eventually become stationary. Say $I^n = I^{2n}$ for some n; by [2; Chap. I, p. 83] I^n is generated by an idempotent e of Ω. Actually, $I = \Omega e$, for Ω lacks nilpotent elements and $(I(1-e))^n = 0$. The idempotent e induces the direct sum decomposition $M = eM \oplus (1-e)M$, with $M = E_P$. Thus

$$\Omega = \begin{bmatrix} \operatorname{Hom}_{R_p}(eM, eM) & \operatorname{Hom}_{R_p}(eM, (1-e)M) \\ \operatorname{Hom}_{R_p}((1-e)M, eM) & \operatorname{Hom}_{R_p}(1-e)m, (1-e)M) \end{bmatrix}.$$

Since is semi-prime, $\operatorname{Hom}_{R_p}((1-e)M, eM) = 0$. Observe that $eM \subseteqq T_p$ and thus $(1-e)M$ is a faithful R_p-module. To conclude we need the

LEMMA 1.2. *If A is a finitely generated faithful module over the commutative ring R, then every simple R-module is a homomorphic image of A.*

Proof. Just note that for each maximal ideal P, $PA \neq A$ [2; Chap. I, p. 83 again].

Returning to the proof of the theorem, observe that eM must contain a simple submodule, unless $e = 0$. Then $I = 0$ and again by the lemma, $T_P = 0$.

2. Examples. In order to construct examples of faithful modules E with commutative $\Omega(E)$ but not isomorphic to ideals, by the preceding it will be necessary to waive the requirement that $\Omega(E)$ be semi-prime.

We shall need a special case of the following result, which has various amusing consequences. Let R, as before, be a commutative noetherian ring and E a finitely generated R-module. Assume that E is faithful; then R can be viewed as a subring of the center C of

$\mathrm{Hom}(E, E)$. E is said to be *balanced* if $R = C$. A mild homological hypothesis will imply that torsion-less modules (i.e., submodules of direct products of R) are, very often, balanced.

To state this condition we recall the notion of grade of an ideal I: it is the smallest integer n such that I contains no R-sequence of length $n + 1$ [3].

PROPOSITION 2.1. *Let E be a finitely generated, torsion-less, faithful R-module. Then if E_P is R_P-free for each prime ideal P with grade $PR_P \leqq 1$ (as R_P ideal), then E is balanced.*

Proof. Consider the exact sequence

$$0 \longrightarrow R \longrightarrow C \longrightarrow L \longrightarrow 0 \tag{2}$$

induced by the inclusion of R into C. With the present finiteness conditions, "C localizes", i.e., for each prime ideal P, C_P is the center of $\mathrm{Hom}(E, E)_P = \mathrm{Hom}_{R_p}(E_P, E_P)$. Thus for each prime ideal P, with grade $PR_P \leqq 1$, $L_P = 0$ as E_P is then R_P-free. Let J be the annihilator of L. The preceding says that J has grade $\geqq 2$. Applying $\mathrm{Hom}(R/J, -)$ to the sequence (2) we get

$$0 \longrightarrow \mathrm{Hom}(R/J, R) \longrightarrow \mathrm{Hom}(R/J, C) \longrightarrow \mathrm{Hom}(R/J, L) \longrightarrow \mathrm{Ext}(R/J, R)\ .$$

Since C is torsion-free, $\mathrm{Hom}(R/J, C) = 0$, while by [3]

$$\mathrm{Ext}(R/J, R) = 0\ .$$

Thus $\mathrm{Hom}(R/J, L) = 0$, which evidently leads to $L = 0$.

The following are cases where the proposition applies:

(i) I ideal of R of grade 2; then $\mathrm{Hom}(I, I) = R$.

(ii) Serre's normality criterion [4; III–13].

(iii) E is a finitely generated, torsion-less, faithful R-module of finite projective dimension; then E is balanced.

(iv) Commutative noetherian rings of finite global dimension are integrally closed.

EXAMPLE 2.3. Let P be a maximal ideal of a commutative domain R, such that grade $P \geqq 2$. Then $\mathrm{Ext}(P, R/P) \neq 0$, as otherwise R_P would be a discrete valuation ring, which is not the case [1]. Let E be a nontrivial extension of P by R/P, that is, consider a non-splitting sequence

$$0 \longrightarrow R/P \longrightarrow E \xrightarrow{\pi} P \longrightarrow 0\ . \tag{3}$$

The exact sequence corresponding to (1) is

$$0 \longrightarrow \operatorname{Hom}(E, R/P) \longrightarrow \operatorname{Hom}(E, E) \xrightarrow{\pi_*} \operatorname{Hom}(P, P) .$$

By (i) above, $\operatorname{Hom}(P, P) = R$ and π_* is actually a surjection with the endomorphisms of E induced by multiplication by elements of R mapping injectively onto $\operatorname{Hom}(P, P)$. Thus

$$\operatorname{Hom}(E, E) = R + I$$

with $I = \operatorname{Hom}(E, R/P)$. By Lemma 1.2 we know that $I \neq 0$. Hom (E, E) will be commutative if $I^2 = 0$. If $I^2 \neq 0$, there would be $f, g \in I$, with $f \circ g \neq 0$. This however says that $f: E \longrightarrow R/P$ is non-trivial on R/P. We could then modify f by multiplication by an element in $R - P$, and thus accomplish a splitting of (3), against the assumption.

In the example above the projective dimension of E is at least 2; it would be interesting to find an example with similar properties but lower projective dimension $(=1)$.

If R is no longer noetherian, then Theorem 1.1 looks still plausible if E is assumed of finite presentation.

As a final remark, in a ligther vein, it should be of interest to determine all commutative rings R in which endomorphism rings of ideals are always commutative. In the noetherian case, we conjecture that the total ring of quotients of R is quasi-Frobenius.

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