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UNBOUNDED INVERSES OF HYPONORMAL OPERATORS

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It is shown that certain unbounded inverses of hyponormal operators have Cartesian representations in which the real part is absolutely continuous and the imaginary part is bounded. An example is given which shows that in general the imaginary part is not absolutely continuous.

A bounded operator T on a Hilbert space $\mathfrak{H}$ is said to be hyponormal if

$$T^*T - TT^* \geqq 0 . \tag{1.1}$$

For properties of such operators, see Putnam [2]. Such an operator is said to be completely hyponormal if there exists no nontrivial subspace of $\mathfrak{H}$ which reduces T and on which T is normal. Recall that a self-adjoint operator A with the spectral resolution $A = \int \lambda dE_\lambda$ is said to be absolutely continuous if $\|E_\lambda x\|^2$ is an absolutely continuous function of λ for all x in $\mathfrak{H}$. If T is completely hyponormal with the Cartesian representation

$$T = H + iJ , \tag{1.2}$$

then both H and J are absolutely continuous; see [2, p. 42].

In case 0 is not in the spectrum of T then T^{-1} is also hyponormal; Stampfli [7]. Further,

$$\|T^{-1}\| = d^{-1} \text{ and } \|Tx\| \geqq d\|x\|, x \in \mathfrak{H} \text{ and } d = \operatorname{dist}(0, \operatorname{sp}(T)) . \tag{1.3}$$

Suppose however that 0 is in the continuous spectrum of T, so that T^{-1} exists as an unbounded operator, is closed, and $\mathfrak{D}_{T^{-1}} = \mathfrak{R}_T$ is dense in $\mathfrak{H}$; cf. Stone [9, pp. 40, 129]. Then it was shown by Stampfli [8] that

$$\mathfrak{D}_{T^{-1}} \subset \mathfrak{D}_{T^{-1*}} \text{ and } \|T^{-1*}x\| \leqq \|T^{-1}x\| \text{ for } x \in \mathfrak{D}_{T^{-1}} . \tag{1.4}$$

Thus T^{-1} still behaves to a certain extent as does T. The question arises however as to whether T^{-1} admits a Cartesian representation $T^{-1} = K + iL$, where K and L are self-adjoint, and also, if such a representation exists, whether these operators are absolutely continuous when T is completely hyponormal. A partial answer is contained in the theorem below.

It may be noted that if 0 belongs to the spectrum of T and if T is completely hyponormal, then 0 cannot be in the point spectrum of T, so that T^{-1} exists. To see this, note that if $Tx = 0$, $x \neq 0$, then (1.1) implies that $T^*x = 0$, and so the vector x determines a normal reducing subspace of T.

THEOREM. *Let T be a (bounded) completely hyponormal operator and suppose that $0 \in \operatorname{sp}(T)$. In addition, suppose that there exists a number $\alpha > 0$ for which the two open disks $|z \pm i\alpha| < \alpha$ contain no points of $\operatorname{sp}(T)$. Then 0 is in the continuous spectrum of T, and T^{-1} has the representation*

$$(1.5)\qquad T^{-1} = K + iL,\ K \text{ and } L \text{ self-adjoint and } L \text{ bounded}\,.$$

In particular, $T^{-1*} = K - iL$ *and* (cf. (1.4))

$$(1.6)\qquad \|T^{-1*}x\| \leqq \|T^{-1}x\|,\ x \in \mathfrak{D}_{T^{-1}} (= \mathfrak{D}_{T^{-1*}} = \mathfrak{D}_K)\,.$$

Further,

$$(1.7)\qquad K \text{ is absolutely continuous}\,,$$

but

$$(1.8)\qquad L \text{ need not be absolutely continuous}\,.$$

REMARKS. The hypothesis of the theorem assures that 0 is in the spectrum of T but that a small neighborhood of 0 does not contain too much spectrum. In view of the complete hyponormality assumption, however, any neighborhood of 0 necessarily intersects $\operatorname{sp}(T)$ in a set of positive measure; Putnam [5].

2. Proof of (1.5). It was noted above that 0 cannot be in the point spectrum of T. In view of the hypothesis concerning the disks it also follows that 0 is not in the residual spectrum of T. For, otherwise, $\mathfrak{D}_{T^{-1}} = \mathfrak{R}_T$ is not dense and $T^*x = 0$ for some $x \neq 0$. But, if $z \neq 0$ and if z is real and sufficiently small, then $\|(T^* - izI)^{-1}\| = 1/\operatorname{dist}(iz, \operatorname{sp}(T^*)) = |z|^{-1}$, and this implies that $Tx = 0$, so that x determines a normal reducing subspace of T, a contradiction; see Putnam [3], Stampfli [8], Sz.-Nagy and C. Foias [11].

Thus, 0 is in the continuous spectrum of T, T^{-1} is closed, and $\mathfrak{D}_{T^{-1}} \subset \mathfrak{D}_{T^{-1*}}$ (cf. (1.4)). Clearly,

$$(2.1)\qquad \begin{aligned} &T^{-1} = K + iL_1,\ \text{where } K = \tfrac{1}{2}(T^{-1} + T^{-1*}) \text{ and} \\ &L_1 = (1/2i)(T^{-1} - T^{-1*})\,. \end{aligned}$$

Note that

(2.2) $$\mathfrak{D}_K = \mathfrak{D}_{L_1} = \mathfrak{D}_{T^{-1}} \cap \mathfrak{D}_{T^{-1*}} = \mathfrak{D}_{T^{-1}}\,.$$

Moreover, $K^* = \frac{1}{2}(T^{-1} + T^{-1*})^* \supset \frac{1}{2}(T^{-1*} + T^{-1**}) = \frac{1}{2}(T^{-1*} + T^{-1}) = K$, and, similarly, $L_1^* \supset L_1$ (see, e,g., Sz.-Nagy [10, p. 29], so that both K and L_1 are symmetric.

Suppose that $z \notin \mathrm{sp}(T)$ (and hence $z \neq 0$). Then $z(zI - T)^{-1}T = (I - z^{-1}T)^{-1}T = [T^{-1}(I - z^{-1}T)]^{-1} = [T^{-1} - z^{-1}I]^{-1}$ and hence $z^{-1} \notin \mathrm{sp}(T^{-1})$. (Concerning the spectrum of an unbounded operator, see Stone [9, p. 129]; Taylor [12, pp. 199–200]. Further, if $z \neq 0$ and $z^{-1} \notin \mathrm{sp}(T^{-1})$, then $(zI - T)^{-1} = z^{-1}I - z^{-2}(z^{-1}I - T^{-1})^{-1}$, so that for $z \neq 0$, $z \in \mathrm{sp}(T)$ if and only if $z^{-1} \in \mathrm{sp}(T^{-1})$. (Cf. Taylor [12], *loc. cit.*) Since the mapping $w = 1/z$ sends the circles $|z + i\alpha| < \alpha$ and $|z - i\alpha| < \alpha$ $(\alpha > 0)$ respectively onto half-planes $\mathrm{Im}(w) > \beta$ and $\mathrm{Im}(w) < -\beta$ for some $\beta = \mathrm{const.} > 0$, it follows that $\mathrm{sp}(T^{-1})$ lies between the two lines $\mathrm{Im}(w) = \pm\beta$.

Next, it will be shown that if $z \notin \mathrm{sp}(T^{-1})$ (hence $z^{-1} \notin \mathrm{sp}(T)$), then

(2.3) $$\|(T^{-1} - zI)x\| \geqq \mathrm{dist}(z, \mathrm{sp}(T^{-1}))\|x\| \text{ for } x \in \mathfrak{D}_{T^{-1}}\,.$$

To see this, note that $(T^{-1} - zI)^{-1} = [T^{-1}(I - zT)]^{-1} = (I - zT)^{-1}T = z^{-1}[(I - zT)^{-1} - I]$, so that $(T^{-1} - zI)^{-1}$ is hyponormal. Also, by (1.3), $\|(T^{-1} - zI)^{-1}\| = 1/\mathrm{dist}(z, \mathrm{sp}(T^{-1}))$, and relation (2.3) then follows. See also Clancey [1].

Let $W_{T^{-1}}$ denote the closure of the set $\{(T^{-1}x, x); x \in D_{T^{-1}} \text{ and } \|x\| = 1\}$. It will be shown that $W_{T^{-1}}$ is contained in the least closed convex set containing $\mathrm{sp}(T^{-1})$. It is sufficient to show that if $\mathrm{Re}(\mathrm{sp}(T^{-1})) \leqq 0$ and if $a + ib \in W_{T^{-1}}$ then $a \leqq 0$. (Note that for this argument one can replace T^{-1} by $rT^{-1} + sI$, where r and s are constants.) But if this is not the case then there exists some $x \in \mathfrak{D}_{T^{-1}}$, $\|x\| = 1$, such that $T^{-1}x = (a + ib)x + y$ with $(x, y) = 0$ and $a > 0$. Then for $c > 0$, it follows from (2.3) that $c^2 \leqq \|(T^{-1} - cI)x\|^2 = (a - c)^2 + b^2 + \|y\|^2$ and hence $2ac \leqq a^2 + b^2 + \|y\|^2$, which is impossible for large c. (This argument was used by Stampfli [7] for bounded hyponormal operators.)

It was noted earlier that $\mathrm{sp}(T^{-1})$ lies in some strip $-\beta \leqq \mathrm{Im}(w) \leqq \beta$, $\beta = \mathrm{const.} > 0$, and it now follows that the set $W_{T^{-1}}$ does also. It follows from (2.1) that $-\beta \leqq (L_1x, x) \leqq \beta$ for $\|x\| = 1$, $x \in \mathfrak{D}_{L_1} (= \mathfrak{D}_{T^{-1}} = \mathfrak{R}_T)$, so that L_1 is bounded on its (dense) domain. Consequently,

(2.4) L_1 has a unique bounded self-adjoint extension L.

Next, it will be shown that K of (2.1) is self-adjoint. To this end, it is sufficient to show that

(2.5) $\mathfrak{R}_{K+ikI} = \mathfrak{H}$ holds for $k = \pm j$ and some $j > 0$.

(See, e.g., Sz.-Nagy [10, pp. 37–38].) Since $\mathfrak{D}_{T^{-1}} = \mathfrak{D}_K = \mathfrak{D}_{L_1}$, then

$K + ikI = T^{-1} + i(kI - L_1) = T^{-1} + i(kI - L)$ and so

$$K + ikI = [I + i(kI - L)T]T^{-1}\,. \tag{2.6}$$

(Note that all equations are interpreted in the strict operator sense.) Since $\mathfrak{R}_{T^{-1}} = \mathfrak{D}_T = \mathfrak{H}$, then, in order to prove (2.5), it is sufficient to show that the (bounded) operator $I + i(kI - L)T$ is nonsingular for $|k|$ sufficiently large.

Choose k so that

$$|k| > ||L||\,, \tag{2.7}$$

and hence $(kI - L)$ is nonsingular. It is then sufficient to show that $Q = T - i(kI - L)^{-1}$ is nonsingular, that is,

$$||Qx|| \geqq c||x||,\ ||Q^*x|| \geqq c||x|| \text{ for some } c = \text{const.} > 0\,, \tag{2.8}$$

whenever k is sufficiently large.

Next, choose m real and satisfying $0 < |m| < \alpha/2$, where α is defined in the statement of the theorem. Then $||(T - imI)x|| \geqq |m|$ for $||x|| = 1$, in view of the present hypothesis and (1.3). Since $Q = (T - imI) - i[(kI - L)^{-1} - mI]$, it follows that, for $||x|| = 1$, $||Qx|| \geqq |m| - ||(kI - L)^{-1} - mI||$. Since $(kI - L)^{-1}$ is definite, it is clear that by choosing m to have the same sign as k, one has $||(kI - L)^{-1} - mI|| < |m|$, provided k is sufficiently large, and hence the first relation of (2.8) is satisfied. Since $Q^* = (T^* + imI) + i[(kI - L)^{-1} - mI]$, the second relation is clear from a similar argument if one notes that $||(T - imI)^{-1}|| = ||(T^* + imI)^{-1}||$ and hence $||(T^* + imI)x|| \geqq |m|$ for $||x|| = 1$, $0 < |m| < \alpha/2$. This completes the proof of (1.5).

3. Proof of (1.7). Let K have the spectral resolution

$$K = \int \lambda dG_\lambda\,. \tag{3.1}$$

For any finite interval Δ and for any operator A (possibly unbounded), let $A_\Delta = G(\Delta)AG(\Delta)$, as an operator on $G(\Delta)\mathfrak{H}$. In order to prove that K is absolutely continuous (on $\mathfrak{H}$) it is clearly sufficient to show that the (bounded) operator K_Δ is absolutely continuous on $G(\Delta)\mathfrak{H}$ for every finite interval Δ.

In order to show this, it is sufficient to show that the bounded operator $S(\Delta) \equiv G(\Delta)T^{-1}G(\Delta) = K_\Delta + iL_\Delta$ is completely hyponormal (cf. §1 above). If this is not the case however, then there exists a subspace $\mathfrak{H}_1 \subset G(\Delta)\mathfrak{H}$, $\mathfrak{H}_1 \neq 0$, with the property that $\mathfrak{H}_1$ reduces $S(\Delta)$ and $S(\Delta)/\mathfrak{H}_1$ is normal. It will be shown that this implies that

$$\mathfrak{H}_1 \text{ reduces } T \text{ and } T/\mathfrak{H}_1 \text{ is normal}\,, \tag{3.2}$$

thus contradicting the hypothesis that T is completely hyponormal.

To this end, note that in view of (1.5) and (1.6),

$$\|T^{-1}x\|^2 - \|T^{-1*}x\|^2 = 2i[(Lx, Kx) - (Kx, Lx)] \geqq 0,\ x \in \mathfrak{D}_K\ . \tag{3.3}$$

Since, for every finite interval Δ, $G(\Delta)x \in \mathfrak{D}_K$ $(x \in \mathfrak{H})$, then (3.3) implies that

$$i(K_\Delta L_\Delta - L_\Delta K_\Delta) \equiv M^{(\Delta)} \geqq 0\ , \tag{3.4}$$

where now all operators of (3.4) are bounded. It is clear that $M^{(\Delta)} = G(\Delta)M^{(\Delta)}G(\Delta)$ and that if η is any interval containing Δ then $G(\Delta)M^{(\eta)}G(\Delta) = M^{(\Delta)} \geqq 0$. Since $M^{(\Delta)}x = 0$ for $x \in \mathfrak{H}_1$, then $M^{(\eta)}G(\Delta)\mathfrak{H}_1 = 0$, that is, $M^{(\eta)}\mathfrak{H}_1 = 0$. Since $\mathfrak{H}_1$ reduces $S(\Delta)$, and hence also K_Δ, then $\mathfrak{H}_1$ is invariant under K_η (note that $K_\eta\mathfrak{H}_1 = K_\Delta\mathfrak{H}_1 \subset \mathfrak{H}_1$), thus $M^{(\eta)}K_\eta\mathfrak{H}_1 = 0$. But (3.4) (with Δ replaced by η) implies that $i(K_\eta^2 L_\eta - L_\eta K_\eta^2) = (K_\eta M^{(\eta)} + M^{(\eta)}K_\eta)$, so that $K_\eta^2 L_\eta x = L_\eta K_\eta^2 x$ for x in $\mathfrak{H}_1$. In the same way, one obtains $K_\eta^n L_\eta x = L_\eta K_\eta^n x$ $(n = 0, 1, 2, \cdots)$ for x in $\mathfrak{H}_1$. Since $G(\Delta)$ is the strong limit of polynomials in K_η (note that η contains Δ), this implies that $G(\Delta)Lx = G(\eta)Lx$ for x in $\mathfrak{H}_1$. Since η is any interval containing Δ, it follows that $G(\Delta)L\mathfrak{H}_1 = L\mathfrak{H}_1$. But $\mathfrak{H}_1$ reduces $S(\Delta)$, hence also $G(\Delta)LG(\Delta)$, so that $G(\Delta)L\mathfrak{H}_1 = G(\Delta)LG(\Delta)\mathfrak{H}_1 \subset \mathfrak{H}_1$. Thus $\mathfrak{H}_1$ reduces L (as well as K, since $K\mathfrak{H}_1 = K_\Delta\mathfrak{H}_1 \subset \mathfrak{H}_1$). Thus $\mathfrak{H}_1$ reduces T^{-1}. Also, since $M^{(\Delta)}\mathfrak{H}_1 = 0$, it follows from (3.3) and (3.4) that $\|T^{-1}x\| = \|T^{-1*}x\|$ for x in $\mathfrak{H}_1$ $(\subset G(\Delta)\mathfrak{H})$. Thus T^{-1} is normal on $\mathfrak{H}_1$ (cf. Sz.-Nagy [10, p. 33]) and hence (3.2) follows. As noted earlier, this yields a contradiction, and, as also noted before, relation (1.7) follows.

4. Proof of (1.8). Let Q denote the Hilbert transform on $\mathfrak{H} = L^2(-\infty, \infty)$ defined by

$$(Qx)(t) = (i\pi)^{-1}\int_{-\infty}^{\infty}(s - t)^{-1}x(s)ds\ , \tag{4.1}$$

the integral being a Cauchy principal value. It is well-known that Q is both unitary and self-adjoint and that its spectrum consists of ± 1, each of infinite multiplicity. Define the self-adjoint operators K and L on $L^2(-\infty, \infty)$ by

$$K = t,\ L = -2I - Q\ . \tag{4.2}$$

The spectrum of the multiplication operator K is $(-\infty, \infty)$, while that of the (bounded) operator L consists of the two numbers -1 and -3, each of infinite multiplicity. Further, K is absolutely continuous, but L is not. Next, define S by

$$S = K + iL\ . \tag{4.3}$$

Since L is bounded, $S^* = K - iL$ and $\mathfrak{D}_S = \mathfrak{D}_{S^*} = \mathfrak{D}_K$. It will be shown that $T = S^{-1}$ exists, is bounded and is hyponormal.

First, note that 0 is not in the point spectrum of either S or S^*. For, if $x \in \mathfrak{D}_S$ and if $Sx = 0$, then $0 = (Sx, x) = (Kx, x) + i(Lx, x)$ and hence $(Lx, x) = 0$. But this is impossible since $-3I \leqq L \leqq -I$. Similarly, 0 is not in the point spectrum of S^*.

It follows that $T = S^{-1}$ exists and that $\mathfrak{D}_T = \mathfrak{D}_{S^{-1}} = \mathfrak{R}_S$ is dense. Further, if $x \in \mathfrak{D}_S$, then $(Sx, x) = (Kx, x) + i(Lx, x)$ and so $\|x\|^2 \leqq |(Lx, x)| \leqq |(Sx, x)| \leqq \|Sx\|\,\|x\|$, that is, $\|Sx\| \geqq \|x\|$. If $y = Sx$, this implies that $\|Ty\| \leqq \|y\|$ for $y \in \mathfrak{D}_T$, so that T is bounded on its domain. Since K is closed and L is bounded, then S is closed and hence (cf. Stone [9, p. 40]), $S^{-1} = T$ is closed. It follows that T must be bounded (with $\mathfrak{D}_T = \mathfrak{H} = L^2(-\infty, \infty)$).

Next, it will be shown that

$$\|Sx\| \geqq \|S^*x\| \text{ for } x \in \mathfrak{D}_S . \tag{4.4}$$

To see this, note that (cf. (3.3))

$$\|Sx\|^2 - \|S^*x\|^2 = 2i[(Lx, Kx) - (Kx, Lx)],\ x \in \mathfrak{D}_S . \tag{4.5}$$

For any finite interval Δ, let $x = G(\Delta)x$, where $K = t = \int \lambda dG_\lambda$. Then one obtains $\|Sx\|^2 - \|S^*x\|^2 = 2(M^{(\Delta)}x, x)$, where (cf. (3.4)) $M^{(\Delta)} = i(K_\Delta L_\Delta - K_\Delta L_\Delta) = i(Q_\Delta K_\Delta - K_\Delta Q_\Delta)$ (cf. (4.2)) and

$$(M^{(\Delta)}x, x) = \pi^{-1}\left|\int_\Delta x(s)ds\right|^2 \geqq 0 .$$

Thus,

$$\|SG(\Delta)x\|^2 - \|S^*G(\Delta)x\|^2 \geqq 0 \text{ for } x \in \mathfrak{H} , \tag{4.6}$$

where Δ is any finite interval. If $x \in \mathfrak{D}_S(=\mathfrak{D}_{S^*})$ and $\Delta_n = (-n, n)$, $n = 1, 2, \cdots$, then $G(\Delta_n)x \to x$ as $n \to \infty$. Moreover, since L is bounded, it is clear from (4.3) that $SG(\Delta_n)x \to Sx$ and $S^*G(\Delta_n)x \to S^*x$. Relation (4.4) now follows from (4.6).

By an argument similar to that used by Stampfli [8] (cf. (1.4) above), it follows that $T = S^{-1}$ is hyponormal; see Clancey [1, p. 33]. Since $-3I \leqq L \leqq -I$, it follows that $\operatorname{sp}(S)$ lies between the lines $\operatorname{Im}(w) = -3$ and $\operatorname{Im}(w) = -1$ of the w-plane (cf. Taylor [12, p. 199]; Clancey [1, p. 34]) and hence $\operatorname{sp}(T)$ lies between the circles in the z-plane which are images of these lines under the mapping $z = 1/w$. (These circles are centered on the positive imaginary axis and have the real axis as a common tangent at $z = 0$.) In case T is completely hyponormal, it can clearly be identified with the operator occurring in the statement of the Theorem.

Suppose then that T is not completely hyponormal. Then T

cannot be normal. For, otherwise (cf. (3.3)), $(Lx, Kx) = (Kx, Lx)$ for all $x \in L^2(-\infty, \infty)$, where L, K are defined by (4.2). Thus, $(Qx, Kx) = (Kx, Qx)$ for all $x \in L^2(-\infty, \infty)$ and, in particular, if Δ is any finite interval and if x is replaced by $G(\Delta)x$, it follows that $K_\Delta Q_\Delta - Q_\Delta K_\Delta = 0$, where $A_\Delta = G(\Delta)AG(\Delta)$. Thus, for any $x \in L^2(-\infty, \infty)$, $\int_\Delta x(t)dt = 0$, Δ arbitrary, which is clearly impossible.

Since T is not normal, its normal part (if it exists, i.e., if T is not completely hyponormal) can be split off, so that T can be represented as

$$T = T_1 \oplus T_2\,, \tag{4.6}$$

where T_1 is normal and T_2 is completely hyponormal. Further, it is clear that $T^{-1} = T_1^{-1} \oplus T_2^{-1}$ and that T_2 has the properties of T in the statement of the theorem. If $T_2^{-1} = K_2 + iL_2$ as in the theorem, then, since $T^{-1} = K + iL = T_1^{-1} \oplus (K_2 + iL_2)$, it follows that $L = (1/2i)(T_1^{-1} - T_1^{-1*}) \oplus L_2$. In particular, $\operatorname{sp}(L_2)$ is a subset of $\operatorname{sp}(L)$, so that $\operatorname{sp}(L_2)$ contains at most the two numbers -1 and -3, and hence L_2 is not absolutely continuous.

Thus, an operator T satisfying the conditions of the theorem has been constructed for which L of (1.5) is not absolutely continuous. This completes the proof of the theorem.

5. Remarks. In case T is a (bounded) hyponormal operator and if the spectrum of its imaginary part has measure zero then T must be normal; see [2, p. 43]. That the corresponding assertion can be false if T is unbounded, even if T admits the representation (1.5) (with T^{-1} replaced there by T), is clear from the example constructed above. Also, if T is (bounded and) hyponormal, the spectra of its real and imaginary parts are precisely the projections onto the coordinate axes of the spectrum of T; [2, p. 46]. This is not in general true in the unbounded case. (Concerning the connection between the spectrum of T and that of its real or imaginary part when T is unbounded and is, in some sense, "hyponormal," see Clancey [1].) Roughly speaking, if T has a representation $T = K + iL$, where K and L are self-adjoint, then, even if L is bounded, its spectrum is largely unpredictable unless $\operatorname{sp}(K) \neq (-\infty, \infty)$. (In this connection, see Clancey [1, Th. 4.2.6]; Putnam [2, p. 39], [4]; Rosenblum [6].) Thus, it is not completely accidental that the operator S of §4 above has a real part with spectrum equal to the entire real axis.

References

1. K. F. Clancey, *Spectral properties of semi-normal operators*, Thesis, Purdue University, January, 1969.

2. C. R. Putnam, *Commutation properties of Hilbert space operators and related topics*, Ergebnisse der Math. **36** (1967), Springer.
3. ———, *Eigenvalues and boundary spectra*, Illinois J. Math. **12** (1968), 278-282.
4. ———, *Absolute continuity of singular integral operators*, Amer. J. Math. **91** (1969), 453-462.
5. ———, *An inequality for the area of hyponormal spectra*, Math. Zeits. **116** (1970), 323-330.
6. M. Rosenblum, *A spectral theory for self-adjoint singular integral operators*, Amer. J. Math. **86** (1966), 314-328.
7. J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc. **117** (1965), 469-476.
8. ———, *Analytic extensions and spectral localization*, J. Math. Mech. **16** (1966), 287-296.
9. M. H. Stone, *Linear transformations in Hilbert space and their application to analysis*, Amer. Math. Soc., New York, 1932.
10. B. Sz.-Nagy, *Spectraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergebnisse der Math. **39** (1967), Springer.
11. B. Sz.-Nagy and C. Foias, *Une relation parmi les vecteurs propres d'un operateur de l'espace de Hilbert et de l'operateur adjoint*, Acta Szeged **20** (1959), 91-96.
12. A. E. Taylor, *Spectral theory of closed distributive operators*, Acta Math. **84** (1951), 189-224.

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