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EACH COMPACT ORIENTABLE SURFACE OF POSITIVE GENUS ADMITS AN EXPANSIVE HOMEOMORPHISM

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It is known that the torus and the orientable surface of genus 2 admit expansive homeomorphisms. In this paper it is shown that all compact orientable surfaces of positive genus admit such homeomorphisms. It remains unknown whether S^2 admits such a map. By taking products expansive homeomorphisms on higher dimensional manifolds are exhibited. Finally dynamical properties of these examples are discussed. Among these are occurrence and nature of periodic points, topological entropy and existence of interesting minimal sets.

A homeomorphism f of a compact metric (d) space X onto itself will be called *expansive with expansive constant* $c > 0$ (or just *expansive*) provided that for each pair of distinct points x, y in X there is an integer n such that $d[f^n(x), f^n(y)] > c$. We denote by M_k the compact orientable surface of genus k. In [8] Reddy exhibited an expansive homeomorphism, f, on the torus, M_1. This mapping is that induced on the torus by the linear mapping of the plane whose matrix is:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

In [6] O'Brien exhibited an expansive homeomorphism, g, on M_2. This map was obtained by lifting f^3 through a branched covering mapping, ϕ, from M_2 onto the torus. If we consider these spaces as spheres with handles imbedded in R^3 then in each horizontal plane ϕ is the mapping which sends z into z^2 (z a complex number). Thus the expansive homeomorphism g is a lift of the expansive homeomorphism on M_1 induced by the matrix.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^3 = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}.$$

The triple iterate of the torus homeomorphism has a fixed point, m, which is not a branch point image. In [6] this point is chosen as a base point. The lift to M_2 can be chosen to leave $n \in \phi^{-1}(m)$ fixed. We will use n as base point for the fundamental group of M_2 in §2.

2. The examples. In this section we prove the existence of expansive homeomorphisms on many compact manifolds. The technique is to exhibit, for any $n > 2$, an expansive homeomorphism of M_2 (which will be an iterate of the above-mentioned map on M_2) which lifts through a covering map to M_k. According to [5, Corollary 3.5] the lift will be expansive. Then since the product of expansive homeomorphisms is expansive, it will follow that any product of orientable surfaces of positive genus admits an expansive homeomorphism.

THEOREM 2.1. *Each compact orientable surface of positive genus admits an expansive homeomorphism.*

Proof. Let f, g and ϕ be the maps mentioned in the preliminaries. Choose generators α, β for $\pi_1(M_1, m)$ where α is covered by the segment from $(0, 0)$ to $(1, 0)$ in the plane and β by the segment from $(0, 0)$ to $(0, 1)$. This represents $\pi_1(M_1, m)$ as $Z \oplus Z$ (where Z denotes the integers) in such a way that $f_*: \pi_1(M_1, m) \to \pi_1(M_1, m)$ is given by the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} .$$

An easy induction shows that, for each positive integer j, f_*^{3j} is given by the matrix

$$\begin{pmatrix} f_{6j-1} & f_{6j} \\ f_{6j} & f_{6j+1} \end{pmatrix} ,$$

where f_i denotes the i^{th} Fibonacci number ($f_1 = 1, f_2 = 1, f_{i+1} = f_i + f_{i-1}$). Now choose generators $\alpha_1, \alpha_2, \beta_1, \beta_2$ for $\pi_1(M_2, n)$ such that $\phi_*(\alpha_i) = \alpha$ and $\phi_*(\beta_i) = \beta$. Define a homomorphism $P: \pi_1(M_2, n) \to Z \oplus Z$ by $P(\gamma) = (a, b)$ where a is the sum of the exponents of α_1 and α_2 and b is the sum of the exponents of β_1 and β_2 in a word representing γ. Because of the defining relation for the group $\pi_1(M_2, n)$, P is independent of the word representing γ. Now if $P(\gamma) = (a, b)$ then $\phi_*(\gamma) = \alpha^a\beta^b$. Also

$$f_*^{3j}(\alpha^a\beta^b) = (\alpha^{f_{6j-1}a+f_{6j}b}\beta^{f_{6j}a+f_{6j+1}b}) .$$

Thus by commutativity ($f^{3j}\phi = \phi g^j$) and the fact that kernel $P =$ kernel ϕ_* it follows that $P(g_*^j(\gamma)) = (f_{6j-1}a + f_{6j}b, f_{6j}a + f_{6j+1}b)$.

Let a surface $M_{k+1} (k \geqq 2)$ be given. Consider the normal subgroup G_k of $\pi_1(M_2, n)$ given by

$$G_k = \{\gamma \in \pi_1(M_2, n) \colon P(\gamma) = (ka, b)\} .$$

The index of G_k is k. Thus $g_*^j(G_k)$ is also a subgroup of index k in $\pi_1(M_2, n)$. We now determine conditions on k such that $g_*^j(G_k) = G_k$. When this happens, g^j lifts to the covering space of M_2 associated with the subgroup G_k. Through consideration of the Euler characteristic, we infer that this space is M_{k+1}.

If $P(\gamma) = (ka, b)$ then $P(g_*^j\gamma) = (f_{6j-1}ka + f_{6j}b, f_{6j}ka + f_{6j+1}b)$. Thus a necessary and sufficient condition for $g_*^j(\gamma)$ to be in G_k is that k divide f_{6j}. Therefore, for existence of a lifting of an expansive homeomorphism on M_2 to the surface of genus $k+1$ it is sufficient that k divide f_{6j} for some j. According to [9], for any k, the Fibonacci sequence mod k is periodic and if j is the period then $f_j \equiv 0 \pmod k$. Thus, by the periodicity, $f_{6j} \equiv 0 \pmod k$. Therefore k does divide some $6j^{\text{th}}$ Fibonacci number. It follows that g^j lifts to a homeomorphism on M_{k+1}. This homeomorphism is expansive by [5, Corollary 3.5].

COROLLARY 2.2. *Let M be a topological product whose factors are compact orientable surfaces of positive genus. Then M admits an expansive homeomorphism.*

3. Properties. In this section we prove several propositions concerning dynamical properties of the examples just constructed.

PROPOSITION 3.1. *In all of the examples the set of periodic points is dense.*

Proof. For the torus case this is well known. See [7, p. 758]. Any lift to M_n through a pseudo-covering map must have dense periodic points since the fibre over a periodic point will consist of periodic points. For the higher dimensional manifolds the set of periodic points is just the product of the periodic sets in the factors and is therefore dense.

DEFINITION 3.2. A fixed point, x, of an expansive homeomorphism, ϕ, is called a saddle point if there exist $p \neq x$ and $q \neq x$ such that p is positively assymptotic to x and q is negatively assymptotic to x. If x is a periodic point with period m and x is a saddle point of f^m we will say that x is of saddle type.

PROPOSITION 3.3. *For the examples in §2, all periodic points are of saddle type.*

Proof. According to Theorem 9 in [3] all periodic points will be of saddle type if the homeomorphism preserves a continuous Borel probability measure which is positive on open sets. Since an automorphism of an abelian group space preserves Haar measure we have our result in the toral case. We can use the pseudo-covering mappings to lift the measure on the torus to the higher genus spaces so that the lifted expansive homeomorphism preserves the measure. Thus all periodic points on the M_k are of saddle type. Clearly when we take products all periodic points will be of saddle type.

Next we show that all of our examples have nonzero topological entropy. See [1] for definitions and results about topological entropy.

According to K. R. Berg [2] the entropy of our toral maps is not 0. We wish to show that all of our examples have non-zero topological entropy. The following is a special case of Theorem 5 in [1].

THEOREM 3.5. *Suppose* $f: M \rightarrow M$ *and* $g: N \rightarrow N$ *are continuous,* M *and* N *are compact and* $\phi: M \rightarrow N$ *is open and onto and that* $g\phi = \phi f$. *Then* $h(g) \leqq h(f)$.

PROPOSITION 3.6. *The examples of* §2 *all have nonzero topological entropy.*

Proof. It follows from the construction in Theorem 2.1., Theorem 3.5. and Berg's result that this conclusion holds for the homeomorphisms constructed on each of the surfaces M_k. Since entropy satisfies the relation $h(f \times g) = h(f) + h(g)$ ([1]), the proposition is valid.

Finally we consider minimal sets. All of our examples have nonperiodic minimal sets. For each space M we exhibit an expansive homeomorphism (an iterate of one of those given § 2) such that some subspace restriction is a Sturmiam minimal set [4; pp. 111-113].

PROPOSITION 3.7. *Each space considered in* §2 *admits an expansive homeomorphism* f *with nonperiodic minimal sets.*

Proof. Each of our examples f on M_k projects through a pseudocovering mapping, ϕ, onto a torus automorphism, g. According to [7, Th. 5.5] there is a Cantor set $\varLambda \subset M_1$ and an integer m such that $g^m(\varLambda) = \varLambda$ and g^m restricted to $\varLambda$ is topologically a shift automorphism. There is a subset L of $\varLambda$ which is totally minimal with respect to g^m [4, 12.63]. L is compact and contains no fixed points. In particular $\phi(B_\phi) \cap L$ is empty. Thus L is contained in a simply connected subset U of $M_1 - \phi(B_\phi)$. Each arc component of $\phi^{-1}(U)$ in $M_k - B_\phi$ maps homeomorphically onto U. There arc $2k - 2$ such arc components. Thus there are $2k - 2$ copies, L_i, of L in M_k. The set $\bigcup_{i=1}^{2k-2} L_i$ is invariant under f. We can cover L by a finite collection of disjoint elementary neighborhoods U_j. These lift to open sets

V_{ij} such that $L_i \subset \bigcup_j V_{ij}$. The effect of f on the collection $\{V_{ij}\}$ is, essentially, to permute these sets. Thus for some iterate f^n of f there is an invariant copy of L and f^n is topologically an iterate of the shift automorphism. Since the Sturmian minimal sets are totally minimal we have exhibited nonperiodic minimal sets for each M_k. For a product $M \times N$ we can choose a subset $L \times \{x_0\}$ where L is a minimal orbit closure of M and x_0 is a fixed point. Thus all of our examples contain nonperiodic minimal sets.

REFERENCES

1. R. L. Adler, A. G. Konheim, M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309-319.
2. K. R. Berg, *Entropy of torus automorphisms*, Topological Dynamics, An International Symposium, Ed. J. Auslander and W. Gottschalk, W. A. Benjamin (1968).
3. B. F. Bryant, P. Walters, *Asymptotic properties of expansive homeomorphisms*, Math. Systems Theory **3** (1969), 60-66.
4. W. H. Gottschalk and G. A. Hedlund, *Topological Dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R. I., 1955.
5. E. Hemmingsen and W. L. Reddy, *Lifting and projecting expansive homeomorphisms*, Math. Systems Theory **2** (1968), 7-15.
6. T. O'Brien, *Expansive homeomorphisms on compact manifolds*. Proc. Amer. Math. Soc. **24** (1970), 767-771.
7. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.
8. W. Reddy, *The existence of expansive homeomorphisms on manifolds*, Duke Math. J. **32** (1965), 627-632.
9. D. D. Wall, *Fibonacci series modulo m*, Amer. Math. Monthly **67** (1960), 525-532.

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