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CHARACTERIZATION OF THE STEINER POINT

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Let f be a mapping which associates with each compact convex set in E^n a point of E^n. If f is linear (in terms of the vector addition of convex sets), uniformly continuous and commutes with a substantial enough set of congruences of E^n, then $f(K)$ is the Steiner point of K for all compact convex sets K.

Let $\mathscr{K}^n$ denote the collection of compact convex sets in E^n, endowed with the algebraic structure of vector addition [1, p. 29] and the topology of the Hausdorff metric [1, p. 34]. For each $A \in \mathscr{K}^n$ we define its Steiner point, denoted $s(A)$, by

$$s(A) = \frac{1}{\sigma_n} \int_{S^{n-1}} h(A, u)\, u dm$$

where m is Lebesgue measure on S^{n-1}, the unit $n-1$ sphere centered at $\bar{O}$, the origin; σ_n is the volume of the unit n-ball; u is a variable vector ranging over S^{n-1}; and $h(A, u)$ is the support function of A, defined as [1, p. 23]:

$$h(A, u) = \sup_{x \in A} x \cdot u \,.$$

It is apparent that the mapping from $\mathscr{K}^n$ to E^n which associates with each set A the point s(A) is linear, continuous, and commutes with congruence transformations of E^n. In 1963 Grunbaum [3, p. 239] asked whether these properties characterize the Steiner point mapping. Shephard answered this affirmatively for the case $n = 2$ [7]. K. A. Schmitt made an attempt [6] at the general problem but his paper contains an error (in proving the continuity of a certain function-see p. 390) which is apparently serious. Our contribution is the following.

THEOREM. *Let T denote a set of orthogonal transformations of E^n onto itself such that:*

(a) *T is transitive on points of S^{n-1}, that is, if u_1, $u_2 \in S^{n-1}$ then there exists t in T such that $t(u_1) = u_2$.*

(b) *For each u_0 in S^{n-1} there exists a nonempty set*

$$T(u_0),\ T(u_0) \subset T \,,$$

such that each t in $T(u_0)$ fixes u_0 and, on the other hand, if u is fixed by each t in $T(u_0)$ then u must be λu_0 for some scalar λ.

(1) *If* $f\colon \mathscr{K}^n \to E^n$ *is linear, uniformly continuous and satisfies*

$$f(t(K)) = t\, f(K) \tag{*}$$

for each t *in* T *and each* K *in* $\mathscr{K}^n$ *then*

$$f(K) = \lambda s(K)$$

for some fixed λ.

(2) *If* (*) *also holds for each* $K \in \mathscr{K}^n$ *and* t *a reflection in some point* $b \neq \bar{0}$, *then* $\lambda = 1$.

2. In outline the proof goes as follows. Instead of dealing with the space $\mathscr{K}^n$, we imbed $\mathscr{K}^n$ into a normed vector space $\mathscr{H}^n$. We extend f to f^* a linear continuous mapping of $\mathscr{H}^n$ into E^n and then follow this by projection mappings to get f_i^* a set of n continuous linear functionals. We represent each of these as integrals with respect to measures μ_i and then show that the vector valued measure $\mu = (\mu_1, \cdots, \mu_n)$ has commutativity properties analogous to those assumed for f. To apply the commutativity properties we need to consider a derivative, $D\mu$, of μ with respect to Lebesgue measure m and show it has certain commutativity properties. This derivative is a point to point mapping of S^{n-1} to E^n and for this reason we can characterize it from its commutativity properties. As this last fact is the point of our method we do this first and then proceed in the order indicated by the outline.

(1) LEMMA 1. *Let* $f\colon S^{n-1} \to E^n$ *be such that* $f(tu) = tf(u)$ *for each* u *in* S^{n-1} *and each* t *in* T, *where* T *is as described in the theorem. Then there exists* λ *so that* $f(u) = \lambda u$ *for all* u *in* S^{n-1}.

Proof. Suppose $f(u_0) \neq \bar{0}$ for some u_0 in S^{n-1}. For each $t \in T(u_0)$,

$$tf(u_0) = f(tu_0) = f(u_0)$$

whence $f(u_0) = \lambda(u_0)u_0$ where $\lambda(u_0)$ is some scalar possibly depending on u_0. Now let u_1 be any other member of S^{n-1}. Then there exists t in T so that $tu_0 = u_1$. Then

$$\begin{aligned}\lambda(u_1)u_1 = f(u_1) = f(tu_0) &= tf(u_0)\\ &= t\,[\lambda(u_0)u_0] = \lambda(u_0)tu_0 = \lambda(u_0)u_1\end{aligned}$$

whence $\lambda(u_1) = \lambda(u_0)$.

(2) The mapping $A \to h(A, u)$ is an isomorphism of $\mathscr{K}^n$ onto

the space of support functions of compact convex sets in E^n [1, p. 26, 29.]. Because support functions are positively homogeneous we can consider instead their restrictions to S^{n-1} and define an isomorphism of $\mathscr{K}^n$ onto this class in the obvious way. Let $\mathscr{H}^n$ denote the set of all differences of these restricted support functions. Because the support functions are closed under addition and scalar multiplication, $\mathscr{H}^n$ is a vector space in which $\mathscr{K}^n$ is isomorphically imbedded. If we provide $\mathscr{H}^n$ with the supremum norm, this isomorphism is an isometry [1, p. 35]. (A slightly different description of this can be found in [4].) Now we can extend f to f^*, a linear function on $\mathscr{H}^n$ by defining

$$f^*(g) = f(h_1) - f(h_2)$$

whenever $g = h_1 - h_2$ where h_1, h_2 are restricted support functions. The uniform continuity of f yields continuity for f^*. Composing this with the projection mapping onto the i^{th} coordinate of E^n we have f_i^*, a real continuous liner functional on the space $\mathscr{H}^n$. Now $\mathscr{H}^n$ is dense in the space of real continuous functions on S^{n-1} [2, p. 10]. Let f_i^* be extended to be defined on this larger space so that we may apply the Riesz Representation Theorem [5, p. 131] to get, for $i = 1, 2, \cdots, n$, real regular Borel measures μ_i on S^{n-1} with the property that:

$$f_i^*(g) = \int_{S^{n-1}} g(u) d\mu_i(u) .$$

To compactify our notation we can write:

$$f^*(g) = \int_{S^{n-1}} g(u) d\mu(u)$$

where μ denotes the vector valued measure defined by

$$\mu(E) = (\mu_1(E), \cdots, \mu_n(E)) .$$

(3) To investigate the effect of the hypothesis requiring commutativity with certain orthogonal transformations, note that if t is orthogonal,

$$\begin{aligned} h(tA, u) &= h(tA, t(t^{-1} u)) \\ &= \sup_{x \in tA} [x \cdot t(t^{-1} u)] \\ &= \sup_{y \in A} [ty \cdot t(t^{-1} u)] \\ &= \sup_{y \in A} [y \cdot t^{-1} u] \\ &= h(A, t^{-1} u) . \end{aligned}$$

Consequently for each $g \in \mathscr{H}^n$

$$f^*[g_{t'}] = t\, f^*[g]$$

where $g_{t'}(u) = g(t^{-1} u)$. In terms of our integral representations,

$$t\int_{S^{n-1}} g(u)d\mu(u) = \int_{S^{n-1}} g(t^{-1} u)\, d\mu(u)$$

$$= \int_{S^{n-1}} g(u)d\mu(tu) .$$

But since t is linear and continuous, for any function g,

$$t\int_{S^{n-1}} g(u)d\mu(u) = \int_{S^{n-1}} g(u)d(t\mu)(u) .$$

Therefore, for each $g \in \mathscr{H}^n$:

$$\int_{S^{n-1}} g(u)d(t\mu)(u) = \int_{S^{n-1}} g(u)d\mu(tu) .$$

Since $\mathscr{H}^n$ is dense in the space of continuous functions on S^{n-1}, the last equation shows that for each Borel set E,

$$t\mu(E) = \mu(tE) .$$

(4) The next step is to consider a Lebesgue decomposition of μ with respect to m, and then to show that the derivative of the components of μ with respect to m and a suitable Vitali covering have the appropriate commutativity property. For each of μ_1, $\mu_2, \cdots, \mu_n$ write its Lebesgue decomposition:

$$\mu_i(E) = u_i^*(E) + \mu_i^{**}(E)$$

[8, p. 187] where μ_i^* is absolutely continuous with respect to m and μ_i^{**} is singular with respect to m. We can then write

$$\mu(E) = \mu^*(E) + \mu^{**}(E) \tag{**}$$

where μ^* and μ^{**} are vector measures whose respective components are the μ_i^* and μ_i^{**}.

We will now consider derivatives of μ^* and μ^{**} with respect to Lebesgue measure m and the Vitali covering of S^{n-1} consisting of spherical caps of the form $\{x \in S^{n-1} \mid x \cdot a \geqq r\}$, $o < r < 1$, $a \in S^{n-1}$ using the derivation process described below [8, p. 221].

DEFINITION. (1) If $E_1, E_2, \cdots$ is a sequence of Borel sets, they are said to converge regularly to x provided there exists a sequence of Vitali sets $A_1, A_2, \cdots$ such that: $x \in A_i$ and $E_i \subset A_i$ for each i,

$m(A_i) \to 0$, and there exists a fixed constant $c > 0$ such that $m(E_i) \geqq cm(A_i)$.

(2) Let ν be any countably additive set function (real or vector valued). Then

$$D\nu(x) = \lim_{i\to\infty} \frac{\nu(E_i)}{m(E_i)}$$

where E_i is any sequence reqularly converging to x, provided the limit exists and is independent of the choice of the E_i.

LEMMA 2. *Let ν be any vector valued measure on S^{n-1} where $\nu(tE) = t\nu(E)$ holds for a set of orthogonal transformations which is transitive on S^{n-1}. Then $D\nu(x)$ exists for all x and $D\nu(tx) = tD\nu(x)$.*

Proof. First observe that our Vitali system is closed under orthogonal transformations. Moreover, so is the family $\mathscr{B}$ of Borel sets. For this reason, and because $m(tE) = m(E)$ for any orthogonal t, if E_i converge regularly to x, tE_i converge regularly to tx. Furthermore, any sequence converging regularly to tx arises this way.

Now suppose $D\nu(x)$ exists and y is any other point of S^{n-1}. For some t we have $tx = y$ and:

$$\begin{aligned} tD\nu(x) &= t \lim_{i\to\infty} \frac{\nu(E_i)}{m(E_i)} \\ &= \lim_{i\to\infty} \frac{t\nu(E_i)}{m(E_i)} \\ &= \lim_{i\to\infty} \frac{\nu(tE_i)}{m(tE_i)} \\ &= D\nu(tx) . \end{aligned}$$

However $D\nu(x)$ exists except for a set of Lebesgue measure 0 [8, p. 222]. This proves the lemma.

Now, to make the connection between this derivative and (**), we have [8, p. 222]:

$$\mu(E) = \mu^{**}(E) + \int_E D\mu(u)dm .$$

$D\mu(u)$ satisfies the hypotheses of Lemma 1 so this becomes:

$$\mu(E) = \mu^{**}(E) + \int_E \lambda u \, dm .$$

Comparing this with (*) we see that the first assertion of the Theorem will follow if u^{**} is identically $\bar{0}$.

The measure $\int_E \lambda u\, dm(u)$ is regular since m is regular, and also inherits the commutativity properties:

$$\int_{tE} \lambda u\, dm(u) = \int_E \lambda(tu)\, dm(tu)$$
$$= \int_E t(\lambda u)\, dm(u)$$
$$= t\int_E \lambda u\, dm(u) .$$

Consequently μ^{**} is regular and commutes with all t in T. Being a singular measure, $D\mu^{**}(u) = \bar{0}$ except for a set of Lebesgue measure 0. But by Lemma 2, this means that $D\mu^{**}(u) = \bar{0}$ everywhere. Now the following lemma completes the proof of the first assertion of the theorem.

LEMMA 3. *Let μ be vector valued measure on S^{n-1} such that:*
(1) *$D\mu(x) = \bar{0}$ for all x in S^{n-1}.*
(2) *μ is regular.*
Then μ is identically $\bar{0}$.

Proof. The idea of this long-winded proof is rather simple. Because of the regularity, it is enough to consider sets of positive Lebesgue measure. If E is a set of positive Lebesgue measure for which $|\mu(E)| = \alpha m(E) > 0$ then we use a Heine-Borel type argument (partitioning sets into subsets of equal Lebesgue measure) to find a nested sequence of subsets $E \supset E_1 \supset E_2 \cdots$ where

$$|\mu(E_i)| > \alpha m(E_i) > 0 ,$$

and where the E_i converge regularly to a point a. Then

$$|D\mu(a)| \geqq \alpha .$$

Before proceeding we need some preliminaries. Let E^{n-1} denote a subspace of E^n. If $x \in S^{n-1}$ we give it a coordinate representation $x = (x_1, \cdots, x_n)$ where $x_1, \cdots, x_{n-1}$ are coordinates with respect to a fixed orthogonal basis of E^{n-1} and x_n is measured along the orthogonal complement of E^{n-1}. Denote by N_ε the set $\{x \in S^{n-1} \mid x_n \geqq \varepsilon\}$. Henceforth let ε be fixed and suppose $0 < \varepsilon < 1$. Let p denote the projection map onto E^{n-1} in the direction of the axis of the n^{th} coordinate. We will show that subsets of N_ε have μ measure $\bar{0}$. The result then follows by covering S^{n-1} by finitely many caps of this type.

We now define a net on N_ε, that is, a countable collection of partitions, $\Omega_1, \Omega_2, \cdots$, of N_ε into countably many Borel sets such that: Ω_{i+1} is a refinement of Ω_i and where, if $B_i \in \Omega_i$ and $B_1 \supset B_2 \supset \cdots$ we have $\bigcap_1^\infty \bar{B}_i$ is a single point. Rudin [5, p. 49, 50] describes a net $\{\Omega'_i\}$ for E^{n-1} which has the further property that each set of Ω'_i is an $n-1$ cube of side 2^{-i} and $n-1$ dimensional Lebesgue measure $2^{-i(n-1)}$ and is composed of 2^{n-1} sets from Ω'_{i+1}. By intersecting all these cubes with $p(N_\varepsilon)$ we get a net for $p(N_\varepsilon)$. Those cubes contained entirely in $p(N_\varepsilon)$ (and are not truncated by intersection with $p(N_\varepsilon)$) are called proper. We can "lift" this net to a net for N_ε by the inverse mapping p^{-1}. Call these partitions $\Omega_1, \Omega_2, \cdots$, and distinguish by the term proper those cubes which arise from proper cubes in $p(N_\varepsilon)$.

Now define a new measure $\bar{m}$ on S^{n-1} by

$$\bar{m}(E) = m'(p(E))$$

where m' is Lebesgue measure in E^{n-1}. Since $m(E) = \int_{p(E)} \frac{1}{x_n} dm'$ and $1 \leqq \frac{1}{x_n} \leqq \frac{1}{\varepsilon}$ we have:

$$\bar{m}(E) = m'(p(E)) \leqq m(E) \leqq \frac{1}{\varepsilon} m'(p(E)) = \frac{1}{\varepsilon} \bar{m}(E) .$$

Consequently m and $\bar{m}$ are absolutely continuous with respect to one another and our earlier defined Vitali system for m is also one for $\bar{m}$.

Now if we calculate a derivative $\bar{D}\mu$ of μ with respect to $\bar{m}$ in the usual way, with respect to the same Vitali system, namely,

$$\bar{D}\mu(x) = \lim_{i\to\infty} \frac{\mu(E_i)}{\bar{m}(E_i)}$$

we have:

$$|\bar{D}\mu(x)| \leqq \frac{1}{\varepsilon} |D\mu(x)| .$$

Hypothesis 1 then gives $\bar{D}\mu(x) = \bar{0}$.

The machinery for the proof has now been set up. By regularity, it is enough to show μ is $\bar{0}$ on open sets. But open sets are disjoint unions of countably many proper cubes B_i from $\Omega_1 \cup \Omega_2 \cup \cdots$ where $\bar{m}(B_i) = 2^{-j(n-1)}$ whenever $B_i \in \Omega_j$ (this follows easily from the corresponding result for the net $\{\Omega'_i\}$ for E^{n-1} [5, p. 49, 50]). Therefore we need only show that μ is $\bar{0}$ on proper cubes. Suppose the contrary, that is, there exists a proper cube $B_0 \in \Omega_k$ with $|\mu(B_0)| > \alpha \bar{m}(B_0)$

for some positive α. Since B_0 is the union of 2^{n-1} proper cubes from Ω_{k+1}, each with the same $\bar{m}$ measure, we can find $B_1 \in \Omega_{k+1}$ where

$$| \mu(B_1) | \geqq \frac{1}{2^{n-1}} | \mu(B_0) | > \alpha \frac{\bar{m}(B_0)}{2^{n-1}} = \alpha \bar{m}(B_1) .$$

Proceeding in this way we find $B_0 \supset B_1 \supset B_2 \cdots$ with $\dfrac{| \mu(B_i) |}{\bar{m}(B_i)} \geqq \alpha > 0$. Now $\bigcap_1^\infty \bar{B}_i$ is a single point, say a. If we can show that the B_i converge regularly to a, then $| \bar{D}\mu(a) | \geqq \alpha > 0$, a contradiction.

To establish the regular convergence, we examine the numbers $f(i) = \bar{m}(\bar{S}(a, r_i))/\bar{m}(B_i)$ where $\bar{S}(a, r_i)$ is the smallest closed cap centered at a containing B_i. It is sufficient to show that $\limsup_{i\to\infty} f(i) < \infty$. The proof of this, although straight-forward, is tedious and we omit the details. This concludes the proof of Lemma 3 and the first part of the Theorem.

Concerning the second assertion of the theorem, suppose t is a reflection through the point b where $b \neq \bar{0}$ and suppose that f commutes also with t. By part 1, f is of the form λs so $\lambda s(tb) = t[\lambda s(b)]$. But t has b as its one and only fixed point so this gives $\lambda s(b) = t[\lambda s(b)]$ whence $\lambda s(b) = b$. But $s(b) = b \neq 0$ so $\lambda = 1$.

REMARKS. For $n \leqq 2$ the set T of the theorem must include indirect (orientation reversing) congruences if it is to satisfy the hypotheses on T. For $n = 1$ the only orthogonal transformation is reflection in $\bar{0}$. If we drop the commutativity with this reflection and assume instead commutativity with all translations we do not get the result of the theorem. The mapping $[a, b] \rightarrow b$ is a counter-example. For $n = 2$, although our method of proof does not yield the representation $f(K) = \lambda s(K)$ if we only require commutativity with rotations, we have been able to deduce the same representation by connecting the problem with the uniqueness of Haar measures on S^1.

Concerning the continuity requirements, Shephard [7] has shown that continuity (as opposed to uniform continuity) suffices when $n = 2$. If $n = 1$ even continuity is unnecessary. For we can represent a closed interval as a pair of real variables representing the endpoints and then f becomes a linear function of these variables and we can write $f(x, y) = \alpha x + \beta y$. We can easily show $\alpha = \beta$ by using commutativity with reflections, and this gives the result.

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