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CELL-LIKE MAPPINGS, II

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This paper is an addendum to the previous paper, Cell-like Mappings, I. Therein, the category of cell-like maps between ENR's was established, homotopy-theoretic characterizations of cell-like maps were given, and the image of a cell-like map on an ENR was studied. In the present paper, three related topics are considered: the relationship between (sometimes global) properties of a map and local properties of its mapping cylinder; limits of cell-like maps; and preservation of tameness properties under cell-like maps. Loose descriptions of some of the results follow.

If an onto map between metric spaces has its image locally collared in its mapping cylinder, then the two spaces are stably homeomorphic. If a proper, onto map between ENR's has its mapping cylinder locally k-connected mod its image for all k, then the map is cell-like (hence a proper homotopy equivalence).

The limit of a sequence of cell-like maps between ENR's is cell-like. Likewise, if a proper map between ENR's is "concordantly" approximated by cell-like maps, it is cell-like.

The property of having ULC^1 complements (for compact sets in ENR's) is preserved under monotone maps.

In an appendix, the nonexistence of two types of isolated singularities is proved.

Since this paper is a continuation of [17], none of the definitions from [17] will be restated. All conventions, notation, etc., from [17] carry over. When referencing [17], we will use (I. i. j) to mean "result i.j. of [17]."

As usual, R is the real line, $I = [0, 1]$, R^n is euclidean n-space, B^n is the unit ball in R^n, and $S^n = \partial B^{n+1}$.

1. Locally trivial mapping cylinders. If $f: X \to Y$ is a map, the mapping cylinder Z_f of f is the quotient space of $(X \times I) \cup (Y \times 2)$ obtained by identifying $(x, 1)$ with $(f(x), 2)$ for each $x \in X$. X is identified with the image of $X \times 0$ in Z_f and Y is identified with the image of $Y \times 2$ in Z_f. We have the natural projection $p: Z_f \to Y$ (a cell-like map when f is proper) and the map $q: X \times I \to Z_f$, the quotient map restricted to $X \times I$. (q is cell-like if and only if f is cell-like.)

Local collars. Let Y be a closed subset of the space Z, $y \in Y$. A local collar of Y in Z at y is an embedding $\gamma: V \times [0, 1) \to Z$ such that $\gamma(V \times [0, 1))$ is open in Z and $\gamma(v, 0) = v$ for all $v \in V$, where V

is some neighborhood of y in Y. (See [4].)

We say Y is *locally collared* in Z if it is locally collared in Z at each point of Y.

The following theorem serves to illustrate the power of certain types of hypotheses concerning the way the range of a map is attached to its mapping cylinder.

THEOREM 1.1. *Let X and Y be metric spaces, $f: X \to Y$ an onto map. If Y is locally collared in Z_f (e.g., if Z_f is a topological manifold with boundary $X \cup Y$) then $X \times R \approx Y \times R$.*

Proof. Y is closed in Z_f, so by [4] Y is collared in Z_f. That is, there is an embedding γ of $Y \times [0, 1)$ onto an open subset of Z_f such that $\gamma(y, 0) = y$ for all $y \in Y$. Note that, by the definition of Z_f, X is also collared in Z_f. In fact, if we let $\lambda = q \mid X \times [0, 1)$, then $\lambda(x, 0) = x$ for all $x \in X$ and $\lambda(X \times [0, 1)) = Z_f - Y$.

SUBLEMMA. *For any $0 < \eta < 1$, there is a homeomorphism h: $Z_f \approx Z_f$ such that*

(1) $h \mid (X \cup Y) = $ *identity, and*

(2) $h\gamma(Y \times [0, 1)) \supset Y \cup \lambda(X \times [\eta, 1))$.

Proof of sublemma. Since X is paracompact, we can find open covers $\{U_\alpha\}$ and $\{V_\alpha\}$ of X such that $\bar{V}_\alpha \subset U_\alpha$, $\{U_\alpha\}$ being locally finite, and such that there exists a number $0 < \phi_\alpha < 1$ with

$$\lambda(x, t) \in \gamma(Y \times [0, 1))$$

for all $x \in V_\alpha$ and $\phi_\alpha \leqq t < 1$. We extend ϕ_α to a continuous function $\bar{\phi}_\alpha: X \to [0, \phi_\alpha]$ which is zero outside U_α. Now let

$$\phi(x) = \max \bar{\phi}_\alpha(x) .$$

Clearly ϕ is continuous and maps X into $[0, 1)$. Moreover,

$$\lambda(x, t) \in \gamma(Y \times [0, 1))$$

for all $x \in X$ and $\phi(x) \leqq t < 1$.

We can easily find a homeomorphism g of $X \times [0, 1]$ onto itself such that

$$g \mid X \times 0 = \text{identity},$$
$$g \mid x \times [(\phi(x) + 1)/2, 1] = \text{identity, and}$$
$$g(x \times [\phi(x), 1]) = x \times [\eta, 1]$$

hold for all $x \in X$. (g is defined linearly on each interval $x \times [0, \eta]$, $x \times [\eta, \phi(x)]$, $x \times [\phi(x), (\phi(x)+1)/2]$, and $x \times [(\phi(x)+1)/2, 1]$. Continuity of ϕ implies g is a homeomorphism.) Define h by

$$h = \begin{cases} \lambda g \lambda^{-1} \text{ on } Z_f - Y \\ \text{identity on } Y \end{cases} .$$

This completes the proof of the sublemma.

The completion of the proof of (1.1) uses a typical "monotone union" argument to show that $(Z_f - X, Y) \approx (Y \times (0, 1], Y \times 1)$ and hence that $X \times (0, 1) \approx Z_f - X - Y \approx Y \times (0, 1)$. (See [3] or [15].)

Locally shrinkable maps. Let $f: X \to Y$ be an onto map. We say f is locally shrinkable (by pseudo-isotopies) if and only if, for each $y \in Y$, there exist a neighborhood V of y in Y and a proper homotopy $h: f^{-1}(V) \times I \to f^{-1}(V)$ such that h_t is an onto homeomorphism for $0 \leqq t < 1$ and $\{h_1^{-1}(x) \mid x \in f^{-1}(V)\} = \{f^{-1}(y) \mid y \in V\}$.

COROLLARY 1.2. *Let X and Y be locally compact metric spaces, $f: X \to Y$ a proper, onto map. If f is locally shrinkable, then $X \times R \approx Y \times R$.*

Proof. For $y \in Y$, let V and h be given as in the definition of locally shrinkable, $U = f^{-1}(V)$. The conditions imply that $qH^{-1}: U \times I \to Z_f$ is an embedding, where $H(x, t) = h(x, t) \times t$. (See [10].) Also, the image of $U \times I$ under qH^{-1} is $p^{-1}(V)$, a neighborhood of y in Z_f. Finally, note that $h_1 f^{-1}: V \to U$ is a homeomorphism of V onto U, so $\gamma = qH^{-1}(h_1 f^{-1} \times T): V \times I \to Z_f$ is a local collar at y, where $T(t) = 1 - t$. Hence, Y is locally collared in Z_f.

2. Local connectivity for maps.

UV-properties. Let A be a subset of the space X, and let $\mathscr{F}$ be a homotopy invariant covariant functor from a category containing inclusion maps of open sets (respectively based open sets) of X to the category of based sets. We say that $A \subset X$ has property $UV(\mathscr{F})$ if and only if, for any neighborhood U of A in X, there exists a neighborhood V of A in U such that the inclusion-induced map $\mathscr{F}(V) \to \mathscr{F}(U)$ (respectively $\mathscr{F}(V, v) \to \mathscr{F}(U, v)$) is zero (for all $v \in V$). Property $UV(\mathscr{F})$ is a topological property of A in the following sense:

THEOREM 2.1. *Let A be a compact set in the* ANR *X. If there* *exists an embedding* $f: A \to Y$, *where* Y *is an* ANR *and* $f(A) \subset Y$ *has property* $UV(\mathscr{F})$, *then* $A \subset X$ *has property* $UV(\mathscr{F})$. (See the argument that (a) $\Rightarrow$ (d) in [16].)

Relative local properties. Let Z be a space, Y a closed subset of Z, and let $\mathscr{F}$ be a homotopy invariant covariant functor from a category containing inclusion maps of open subsets of Z to the category of based sets. For $y \in Y$, we say Z is $LC(\mathscr{F})$ mod Y at y if and only if, for any neighborhood U of y in Z, there is a neighborhood V of y in U such that the inclusion-induced map $\mathscr{F}(V - Y) \to \mathscr{F}(U - Y)$ is zero. If Z is $LC(\mathscr{F})$ mod Y for all $y \in Y$, we say that Z is $LC(\mathscr{F})$ mod Y.

Local properties of maps. Let $f: X \to Y$ be a map. We say that f is $LC(\mathscr{F})$ if and only if Z_f is $LC(\mathscr{F})$ mod Y.

Standard examples. There are special notations for $UV(\mathscr{F})$ and $LC(\mathscr{F})$ for certain $\mathscr{F}$. When $\mathscr{F} = \tilde{H}_k(-; G)$ (reduced singular homology with coefficients in the group G) we get $k - uv(G)$ and $k - lc(G)$, respectively. When $\mathscr{F} = \tilde{H}_0(-; G) \oplus \cdots \oplus \tilde{H}_k(-; G)$, we get $uv^k(G)$ and $lc^k(G)$, respectively. When $\mathscr{F} = [S^k, -]$ (based homotopy classes of maps $S^k \to -$) we get $k - UV$ and $k - LC$, respectively. And when $\mathscr{F} = [S^0, -] \oplus \cdots \oplus [S^k, -]$, we get UV^k and LC^k, respectively.

For an appropriate space K, $\mathscr{F} = [K, -]$ also yields UV^∞ and LC^∞, respectively. When we are dealing with ANR's, $K =$ disjoint union of all homotopy types of ANR's, would do. When dealing with separable metric spaces, $K =$ disjoint union of all separable metpic spaces works. And other examples can be worked up.

THEOREM 2.2. *Let* X *and* Y *be locally compact metric spaces, let* $f: X \to Y$ *be a proper, onto map, and let* $\mathscr{F}$ *be a homotopy invariant covariant functor from a category containing inclusion maps of open sets of* X *and* $X \times I$ *to the category of based sets. Then the following are equivalent*:

(a) *Each inclusion* $f^{-1}(y) \subset X$ *has property* $UV(\mathscr{F})$, $y \in Y$.

(b) f *is* $LC(\mathscr{F})$.

REMARKS. Theorem 2.2 relates the (recently defined) UV-properties to the (classical) local connectivity properties. For results on the latter, see [11], [22] and [26], among others. The results from [22] can be translated, using (2.2) above, into an "Hurewicz" theorem: Let $f: X \to Y$ be a proper, onto map between locally compact ANR's; if each $f^{-1}(y)$ has property UV^k then each has $uv^k(G)$ for all G; and, if each $f^{-1}(y)$ has UV^1

and $uv^k(Z)$, where $Z =$ integers, then each has UV^k. (Compare with Theorems 4.1 and 4.2 of [18].) Theorems 2 and 3 of [11] can be translated, again using (2.2), to yield (I.3.1). (See the remark on page 617.)

Proof. First assume that each inclusion $f^{-1}(y) \subset X$ has $UV(\mathscr{F})$. Let $y \in Y$, and suppose that U is a given neighborhood of y in Z_f. Find a neighborhood U_1 of y in Y and a number t, $0 < t < 1$, such that $p^{-1}(U_1) \cap q(X \times (t, 1])$ is contained in U. Then, we can find a neighborhood V_1 of y in U_1 such that the inclusion-induced map

$$\mathscr{F}(f^{-1}(V_1)) \to \mathscr{F}(f^{-1}(U_1))$$

is zero. Define $V = p^{-1}(V_1) \cap q(X \times (t, 1])$. Since $\mathscr{F}$ is homotopy invariant, the inclusion-induced

$$\text{map}\ \mathscr{F}(f^{-1}(V_1) \times (t, 1)) \to \mathscr{F}(f^{-1}(U_1) \times (t, 1))$$

is zero. But the restriction of q to $f^{-1}(U_1) \times (t, 1)$ takes the pair

$$(f^{-1}(U_1) \times (t, 1), f^{-1}(V_1) \times (t, 1))$$

homeomorphically into $(U - Y, V - Y)$, with $f^{-1}(V_1) \times (t, 1)$ mapping onto $V - Y$. Thus it is clear that $\mathscr{F}(V - Y) \to \mathscr{F}(U - Y)$ is zero, so Z_f is $LC(\mathscr{F})$ mod Y at y.

Now suppose Z_f is $LC(\mathscr{F})$ mod Y. Let $y \in Y$, and let U_1 be a neighborhood of y in Y, $U = p^{-1}(U_1)$. By assumption, there is a neighborhood V_1 of y in Y such that $\mathscr{F}(V - Y) \to \mathscr{F}(U - Y)$ is zero, where $V = p^{-1}(V_1) \cap q(X \times (t, 1])$ for some $t < 1$. Let $U_0 = f^{-1}(U_1)$ and $V_0 = f^{-1}(V_1)$. Then q maps $(U_0 \times [0, 1), V_0 \times (t, 1))$ homeomorphically onto $(U - Y, V - Y)$, so it is clear that $\mathscr{F}(V_0) \to \mathscr{F}(U_0)$ is zero. Since $f^{-1}(y)$ has arbitrarily small neighborhood pairs of the form (U_0, V_0), we see that $f^{-1}(y) \subset X$ has property $UV(\mathscr{F})$.

REMARK. If one were to desire a definition of "locally homotopically collared" pertaining to $Y \subset Z$, "LC^∞ mod Y" might be a good choice, at least when Y is locally contractible. This interpretation, combined with known facts, yields some interesting analogues (as well as conjectures) about cell-like maps and locally shrinkable maps.

COROLLARY 2.3. *Let X and Y be locally compact metric spaces, $f: X \to Y$ a proper, onto map. If f is LC^k then*

$$f_*: \pi_q(X, x) \to \pi_q(Y, f(x))$$

is an isomorphism for $0 \leqq q \leqq k$ (and an epimorphism for $q = k + 1$, provided Y is an ANR).

Proof. Apply (I.3.1). If Y is an ANR, use the technique in the proof of (I.2.4) to see that f_* is epic when $q = k + 1$.

COROLLARY 2.4. *Let X and Y be ENR's, $f: X \to Y$ a proper, onto map. f is cell-like if and only if it is LC^∞.*

3. Limits of cell-like maps.

The following was obtained independently by R. Finney. (See [12].)

THEOREM 3.1. *Let X and Y be ENR's and let $f: X \to Y$ be a proper, onto map. If there exists a sequence of proper, cell-like maps of X onto Y which converges to f (in the compact-open topology) then f is cell-like.*

Before proving (3.1), we need the following lemma on monotone maps. (A map $f: X \to Y$ is *monotone* if $f^{-1}(y)$ is compact and connected for each $y \in Y$.)

LEMMA 3.2. *Let X and Y be locally compact metric spaces, f a proper map of X onto Y. Let $\{f_n\}$ be a sequence of proper monotone maps of X onto Y which converges to f. Suppose that $y \in V \subset \bar{V} \subset U$, where V and U are open sets in Y, V is connected, and U has compact closure. Then there exists an integer m such that*

$$f^{-1}(y) \subset f_n^{-1}(V) \subset f^{-1}(U)$$

for $n \geqq m$.

Proof. Suppose that $f^{-1}(y) \not\subset f_n^{-1}(V)$ for infinitely many n. Then, there is an infinite sequence $\{x_i\}$, with $x_i \in f^{-1}(y) - f_{n_i}^{-1}(V)$. Since $f^{-1}(y)$ is compact, we may assume that $\{x_i\}$ converges to some $x \in f^{-1}(y)$. Thus $f_{n_i}(x_i)$ converges to $f(x) = y$, so almost all of $f_{n_i}(x_i)$ must lie in V, contrary to the choice of $x_i \notin f_{n_i}^{-1}(V)$. Hence there is an integer k such that $f^{-1}(y) \subset f_n^{-1}(V)$ for $n \geqq k$.

Now suppose that $f_n^{-1}(\bar{V}) \not\subset f^{-1}(U)$ for infinitely many n. Since $f_n^{-1}(\bar{V}) \cap \overline{f^{-1}(U)} \neq \emptyset$ for all $n \geqq k$, and each $f_n^{-1}(\bar{V})$ is compact and connected, we can find an infinite sequence $\{z_i\}$, with $z_i \in f_{n_i}^{-1}(\bar{V}) \cap B$, where $B = \overline{f^{-1}(U)} \cap (X - f^{-1}(U))$. B is compact, so we may assume that $\{z_i\}$ coverges to some $z \in B$. Since $\{f_{n_i}(z_i)\}$ converges to $f(z)$, and $z_i \in f_{n_i}^{-1}(\bar{V})$, we see that $f(z) \in \bar{V} \subset U$, contradicting the fact that $z \notin f^{-1}(U)$. We conclude that there is an integer $m \geqq k$ such that

$$f_n^{-1}(V) \subset f^{-1}(U)$$

for $n \geqq m$.

Proof of (3.1). We want to show that each inclusion $f^{-1}(y) \subset X$ has property UV^∞. The result will then follow from (I.1.1). Let U, V, and W be connected neighborhoods of $y \in Y$, chosen so that $\bar{W} \subset V \subset \bar{V} \subset U$ and $\bar{U}$ is compact. Let $\{f_n\}$ be a sequence of proper, cell-like maps which converges to f. Choose W so that W is contractible in V. By (3.2), we can find an integer m such that

$$f^{-1}(y) \subset f_m^{-1}(W) \subset f_m^{-1}(V) \subset f^{-1}(U) .$$

By (I.1.2), $f_m^{-1}(W)$ is contractible in $f_m^{-1}(V)$. Since $f^{-1}(y)$ has arbitrarily small neighborhoods of the form $f^{-1}(U)$, $f^{-1}(y)$ has property UV^∞.

THEOREM 3.3. *Let $f: X \to Y$ be an onto map between* ENR's. *Suppose $f \times 0$ extends to a proper map f': $X \times [0, 1) \to Y \times [0, 1)$ such that $f' | X \times (0, 1)$ is a cell-like map of $X \times (0, 1)$ onto $Y \times (0, 1)$. Then f is cell-like.*

Proof. When $\alpha: X \to (0, 1)$ is continuous, let ϕ_α be the map

$$\phi_\alpha(x, t) = q\left(x, 1 - \frac{t}{\alpha(x)}\right), x \in X, 0 \leqq t \leqq \alpha(x) .$$

Let $g_\alpha = f'\phi_\alpha^{-1}$. Then g_α is a cell-like map of Z_f onto a subset of $Y \times [0, 1)$. (g_α is a homeomorphism on Y and is f' "twisted" on $x \times [0, 1)$.) Moreover, $g_\alpha(y) = y \times 0$, and $g_\alpha(Z_f - Y) \subset Y \times (0, 1)$.

Let α be chosen so that

$$Y \times [0, \tfrac{1}{2}] \subset \bigcup_{x \in X} f'(x \times [0, \alpha(x)]) = g_\alpha(Z_f) .$$

Then Y is collared in $g_\alpha(Z_f)$. Also, g_α is cell-like, $g_\alpha^{-1}(g_\alpha(Z_f) - Y) = Z_f - Y$, and $g_\alpha | Y$ is a homeomorphism, so it follows as in the proof of (4.2) below that Z_f is $LC^\infty \bmod Y$. By (2.4), f is cell-like.

4. Preservation of tameness properties. The tameness properties referred to involve the LC^k properties. We think of the compact set A in the ANR X as being "locally k-tame" in X whenever X is $LC^k \bmod A$; clearly, this is a topological property of the pair (X, A).

It should be noted that, when A is closed in the *compact* ANR X, X is $LC^k \bmod A$ if and only if $X - A$ is ULC^k (uniformly locally k-connected).

THEOREM 4.1. *Let X and Y be locally compact* ANR's, *and let A and B be compact sets in X and Y, respectively. Suppose that $f: X \to Y$ is a proper, onto UV^k-map such that $f^{-1}(B) = A$ and $f | A$ is one-to-one. If X is $LC^{k+1} \bmod A$ then Y is $LC^{k+1} \bmod B$.*

THEOREM 4.2. *Let X, Y, A, B, and f be as in the hypothesis of Theorem* 4.1. *If Y is $LC^k \bmod B$ then X is $LC^k \bmod A$.*

Proof of (4.1.). We will show that Y is $LC^{k+1} \bmod B$ at each point of B. Let U be a neighborhood of $y \in B$ in Y, and let V be a neighborhood of y in U such that any map $S^q \to f^{-1}(V - B)$ extends to a map $B^{q+1} \to f^{-1}(U - B)$, $0 \leqq q \leqq k + 1$, using the fact X is $LC^{k+1} \bmod A$ at $f^{-1}(x)$.

Since f induces epimorphisms on π_q when restricted to any inverse open set, it follows that any map $S^q \to (V - B)$ extends to a map $B^{q+1} \to (U - B)$, $0 \leqq q \leqq k + 1$, and Y is $LC^{k+1} \bmod B$ at y.

Proof of (4.2.). We want to show that X is $LC^k \bmod A$ at each point of A. Let U be a neighborhood of $x \in A$ in X. Since $f^{-1}f(x) = x$, there is a neighborhood W of $f(x)$ in Y such that $f^{-1}(W) \subset U$. Now, Y is $LC^k \bmod B$ at $f(x)$, so there is a neighborhood W' of $f(x)$ in Y such that any map $S^q \to (W' - B)$ extends to a map $B^{q+1} \to (W - B)$, $0 \leqq q \leqq k$. Let $V = f^{-1}(W')$. Again, since f induces isomorphisms on π_q when restricted to any inverse open set in $X - A$, we see that any map $S^q \to (V - A)$ extends to a map $B^{q+1} \to (U - A)$, $0 \leqq q \leqq k$, and X is $LC^k \bmod A$ at x.

COROLLARY 4.3. *Let M^m and N^n be unbounded PL manifolds with $m \geqq 5$ and $n \geqq 5$. Further, let P^p and Q^q be compact polyhedra topologically embedded in M and N, respectively, $m - p \geqq 3$, $n - q \geqq 3$. Finally, let $f\colon M \to N$ be a proper, onto monotone map such that $f^{-1}(Q) = P$ and $f \mid P$ is one-to-one.*

(1) *If P is tame in M then Q is tame in N.*

(2) *If each $f^{-1}(y)$ has property $1 - UV$ and Q is tame in N then P is tame in M.*

Proof. When $n - p \geqq 3$, M is $LC^1 \bmod P$ if and only if P is tame in M. The difficult part of this statement is due to Bryant and Seebeck [5].

5. Maps on Euclidean space. In [8], Cohen pointed out that, if $f\colon R^3 \to R^3$ is the example described by Bing in [2] of a proper, monotone noncell-like map, there is no proper, onto map $f'\colon R^4 \to R^4$ whose nondegenerate point-inverses are the same as those of $f \times 0$. A higher-dimensional analogue of this result can be obtained.

THEOREM 5.1. *Let $f\colon R^n \to R^n$ be a proper, onto UV^1-map. If there exists a proper, onto map $f'\colon R^{n+1} \to R^{n+1}$ whose nondegenerate point-inverses are the same as those of $f \times 0$, then f is cell-like* (*and hence cellular if $n \neq 4$*).

Thus, for example, f could not be any of the generalizations of Bing's example described in § 6 of [18], with $k \geqq 2$ and $l \geqq 2$.

Proof. Note that the condition implies that Z_f embeds in R^{n+1} with the crinkled end being $f'(R^n \times 0)$. Thus Z_f is $lc^k \operatorname{rel} f(R^n)$ for all k, by Theorem II. 5. 35 [26], and hence f is UV^1-trivial and uv^k-trivial for all k by Theorem 2.2. Using the "Hurewicz" theorem mentioned in the remark following (2.2), we see that f is UV^k-trivial for all k, hence cell-like.

For $n \leqq 3$, it is not necessary to assume that f is UV^1-trivial in the above proposition. (See [23] and Theorem 2 of [19].)

THEOREM 5.2. (*Lacher and Wright.*) *Let $f: R^n \to R^n$ be a proper, onto map, $n \leqq 3$. If there exists a proper, onto map $f': R^{n+1} \to R^{n+1}$ whose nondegenerate point-inverses are the same as those of $f \times 0$, then f is shrinkable by a pseudo-isotopy.*

THEOREM 5.3. *Let $f: R^{n+1} \to R^{n+1}$ be a proper, onto map,*

$$S_f = \{x \in R^{n+1} \mid f^{-1}f(x) \neq x\} .$$

If $S_f \subset R^n \times 0, f(R^n \times 0) \approx R^n$, and $\overline{f(S_f)}$ is a polyhedron of dimension $\leqq n - 3$, tame in both $f(R^n \times 0)$ and R^{n+1}, then f is cell-like.

Proof. By Theorem 6.1 of [7], $f(R^n)$ is locally flat in R^{n+1}. Assuming $n \geqq 3$, and applying [6], we obtain a homeomorphism h of R^{n+1} onto itself such that $hf(R^n) = R^n$. By Theorem 3.3, $hf|R^n$ is cell-like, hence $f|R^n$ is cell-like. Since $S_f \subset R^n$, f is cell-like.

REMARKS. 1. D. R. McMillan has recently proved the following: If $f: R^{n+1} \to R^{n+1}$ is a proper, onto map, and $S(f) \subset R^n \times 0$, then each $f^{-1}(y)$ has property $uv^\infty(Z)$.

2. The results of this section should be compared with [28].

Appendix: The nonexistence of two types of isolated singularities. Let M be a manifold (without boundary), Y a Hausdorff space, and $f: M \to Y$ a proper, onto map. Define two "singular sets" as follows:

$$C_f = \{y \in Y \mid f^{-1}(y) \text{ is not cellular in } M\} ,$$

and

$$E_f = \{y \in Y \mid Y \text{ is not locally euclidean at } y\} .$$

We show below that $C_f - E_f$ has no isolated points (assuming dim $M \geqq 5$, f is strongly acyclic, and M is simply connected), and that $E_f - \bar{C}_f$ has no isolated points (assuming $\dim M \neq 4,5$). (*Note*: E_f is a closed subset of Y. $\bar{C}_f$ denotes the closure of C_f in Y.)

A1. *Noncellular points of a strongly acyclic map.* A map $f: X \to Y$ is *strongly acyclic* (cf. [20]) if and only if each inclusion $f^{-1}(y) \subset X$, $y \in Y$, has property $UV(\mathscr{F})$, where $\mathscr{F} = \tilde{H}_*(-; Z)$. (I.e., $\mathscr{F}$ is reduced singular homology with integral coefficients.) According to Corollary 3.3 of [18], if X is an ENR then a proper map $f: X \to Y$ is strongly acyclic if and only if $\check{\tilde{H}}^*(f^{-1}(y)) = 0$ for all $y \in Y$. ($\check{\tilde{H}}^*$ is reduced Čech cohomology.) McMillan [20] and [21] and Wright [27] have studied strongly acyclic maps on 3-manifolds. In particular, Wright has shown that if $f: M^3 \to N^3$ is a proper, onto, strongly acyclic map between (open or closed) 3-manifolds, then C_f is a closed, locally finite set in N, and, hence, if $M = S^3$ or R^3, $C_f = \varnothing$. Theorem 1 below could be considered a weak analogue of this last statement.

We recall two methods of producing strongly acyclic maps.

EXAMPLE 1. Let H be a topological k-manifold which is a homology k-sphere, and let A be the closure of the complement of a locally flat k-cell in H. By taking the quotient map cross the identity map, we obtain a strongly acyclic map $H \times S^{n-k} \to S^k \times S^{n-k}$ between n-manifolds with $\dim C_f = n - k$, $k \geqq 3$. In particular, one can have C_f a finite set when the domain is not simply connected.

EXAMPLE 2. Let $f: R^n \to R^n$ be a proper, onto map, and let $S_f = \{x \in R^n \mid f^{-1}f(x) \neq x\}$. If $f(S_f)$ is compact and 0-dimensional, then f is strongly acyclic. (See [25].)

Finally, we should point out that there is a strongly acyclic map of S^n onto itself, $n \geqq 4$, with $\dim \bar{C}_f = 1$. (See [21].)

THEOREM 1. *Let M and N be (open or closed) topological n-manifolds, $n \geqq 5$, and let $f: M \to N$ be a proper, onto, strongly acyclic map. If M is simply connected then C_f has no isolated points.*

Proof. Suppose that y_0 is a point of N which has a neighborhood V such that $f^{-1}(y)$ is cellular in M for all $y \neq y_0$ in V. Since

$$\check{H}^4 f^{-1}(y_0) = 0,$$

$f^{-1}(y_0)$ has a PL neighborhood, according to Theorem 2 of [13]. We assume, therefore, that V is an open n-cell and that $f^{-1}(V) = U$ is a PL manifold.

Let W be a closed neighborhood of $f^{-1}(y_0)$ in U, chosen to be a compact PL manifold. Let $W_0 = W - f^{-1}(y_0)$, $U_0 = U - f^{-1}(y_0)$, and $V_0 = V - \{y_0\}$. Since $f \mid U_0$: $U_0 \to V_0$ is cellular, we see that W_0 is 1-connected at infinity. Applying Theorem 3.10 of [24], we can replace

W by another compact manifold W' whose boundary is connected and simply connected. Applying Van-Kampen's Theorem, we see that $0 = \pi_1(\overline{M - W'}) * \pi_1(W')$, so W' is simply connected. Therefore $f^{-1}(y_0)$ has property UV^1.

It follows from Theorem 4.2 of [18] that $f^{-1}(y_0)$ has property UV^∞ and hence is cellular in M.

A2. *Noneuclidean points of a cellular map.* Let $f: S^n \to Y$ be an onto map whose only nondegenerate point-inverse is A. Then A is cellular if and only if Y is a manifold. I.e., $C_f = \emptyset$ if and only if $E_f = \emptyset$. Such a statement is not true about maps in general, but Theorem 2 below is a partial converse to Theorem 1 in this sense.

THEOREM 2. *Let M be a topological n-manifold* (*open or closed*), $n \neq 4,5$, *and let Y be a Hausdorff space. If $f: M \to Y$ is a proper, cellular map, then E_f has no isolated points.*

Proof. Let $y_0 \in Y$, and suppose that y_0 has a neighborhood V such that $V - \{y_0\}$ is an open topological manifold. Then $\dim(V - \{y_0\}) = n$, so V is an ENR by Corollary I.3.3.

Let $W = f^{-1}(V - \{y_0\})$. Since $f|W$ is a proper homotopy equivalence when restricted to any inverse open set (see Theorem I. 1.2). and since $f^{-1}(y_0)$ is cellular in M, it is clear that y_0 has a neighborhood $V' \subset V$ such that the inclusion-induced map $H^4(V - \{y_0\}; Z_2) \to H^4(V' - \{y_0\}; Z_2)$ is zero. It follows from the Kirby-Siebenmann triangulation theorem [14] that y_0 has a neighborhood $V'' \subset V'$ such that $V'' - \{y_0\}$ has a PL structure. (See the remark following Theorem 2 of [13]).

We may as well assume, then, that $V - \{y_0\}$ has a PL structure. Let ε be the end of V determined by y_0. Then $\{f^{-1}(U) \mid U \in \varepsilon\}$ determines an end $f^{-1}(\varepsilon)$ of W, the same one determined by $f^{-1}(y_0)$. Moreover, $f^{-1}(\varepsilon)$ is $(n-2)$-connected, so ε is $(n-2)$-connected. It follows immediately from [24] that, in case $n \geqq 6$, ε has a collar neighborhood (which must be $S^{n-1} \times [0, \infty)$ by [9]). Thus V is a topological (in fact, PL) manifold, and $y_0 \notin E_f$.

If $n \leqq 3$, $f|W$ is properly homotopic to a homeomorphism: $W \approx V - \{y_0\}$, so the result is easy in that case. (See [1].)

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