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ON THE IDEAL STRUCTURE OF SOME ALGEBRAS OF ANALYTIC FUNCTIONS

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Using the Beurling-Lax description of invariant subspaces of $H^2(R)$, we describe the ideal structure of two large classes of convolution algebras whose Fourier-Laplace Transforms are entire functions. A closed ideal will be characterized by its cospectrum or by its cospectrum together with a nonnegative number related to the "rate of decrease at infinity"; in the latter case, the closed ideals having the same cospectrum form a totally ordered family $\{I_\xi\}$, $\xi \in [0, \infty)$, with $I_\xi \supsetneqq I_\eta$ whenever $\xi < \eta$. New examples of algebras to which the results apply are given.

The familiar notation for the spaces considered by Schwartz ([9]) is adopted and each space is equipped with its usual topology. Let $\mathscr{K}$ be the subspace of $\mathscr{E}(R)$ of functions ϕ for which

$$||\phi||_k = \sup_{x \in R, p \leqq k} \exp(k|x|)|D^p\phi(x)|$$

is finite for each $k = 0, 1, \cdots$; the topology on $\mathscr{K}$ will be the one induced by the semi-norms $||(\cdot)||_k$, $k = 0, 1, \cdots$. Under this topology $\mathscr{K}$ is a convolution algebra with separately continuous multiplication. A detailed discussion of $\mathscr{K}$ along with associated spaces is given in [4], [12] and [13] (note that Zielézny uses $\mathscr{K}_1$ instead of $\mathscr{K}$). We recall some of the results in the form most convenient for applications here.

Denote by $\mathscr{O}_c'(\mathscr{K})$ the convolution operators on $\mathscr{K}$, i.e., the distributions $S \in \mathscr{D}'(R)$ for which the convolution operator $\phi \rightarrow S * \phi$ is well-defined and continuous from $\mathscr{K}$ into $\mathscr{K}$. $\mathscr{O}_c'(\mathscr{K})$ is given the topology it inherits as a subspace of $\mathscr{L}_b(\mathscr{K}, \mathscr{K})$, the continuous linear mappings from $\mathscr{K}$ into $\mathscr{K}$, when $\mathscr{L}_b(\mathscr{K}, \mathscr{K})$, has the topology of uniform convergence on bounded subsets of $\mathscr{K}$. Alternatively, if $\mathscr{K}'$ is the strong dual of $\mathscr{K}$, $\mathscr{O}_c'(\mathscr{K})$ can be defined as the space $\mathscr{O}_c'(\mathscr{K}', \mathscr{K}')$ of convolution operators on $\mathscr{K}'$ in the sense of Schwartz ([10], exposé 10) and given the topology acquired as a subspace of $\mathscr{L}_b(\mathscr{K}', \mathscr{K}')$. These two definitions of $\mathscr{O}_c'(\mathscr{K})$ are, however, entirely equivalent (cf. [13, Ths. 2(d'), 4]).

THEOREM 1. *The space $\mathscr{O}_c'(\mathscr{K})$ is a convolution algebra for which*

(i) $(S, T) \rightarrow S * T$ *is a separately continuous mapping from $\mathscr{O}_c'(\mathscr{K}) \times \mathscr{O}_c'(\mathscr{K})$ into $\mathscr{O}_c'(\mathscr{K})$,*

(ii) $(S, \phi) \rightarrow S*\phi$ *is a separately continuous mapping from* $\mathscr{O}_c'(\mathscr{K}) \times \mathscr{K}$ *into* $\mathscr{K}$.

Proof. (i) See [12, p. 319] for instance, or, more directly, use the definition of the $\mathscr{L}_b(\mathscr{K}', \mathscr{K}')$ topology.

(ii) The continuity of $\phi \rightarrow S*\phi$ follows immediately from the definition of S while the continuity of $S \rightarrow S*\phi$ follows from the definition of the $\mathscr{L}_b(\mathscr{K}, \mathscr{K})$ topology on $\mathscr{O}_c'(\mathscr{K})$.

The Fourier-Laplace Transform $\Phi(z)$ of $\phi \in \mathscr{K}$ defined by

$$\Phi(z) = \hat{\phi}(z) = \int_{-\infty}^{\infty} \phi(x) e^{-ixz} dx\ , \qquad z = u + iv\ ,$$

can be extended to $\mathscr{O}_c'(\mathscr{K})$ via the Parseval formula in the usual way since $\mathscr{O}_c'(\mathscr{K}) \subset \mathscr{K}'$. For both $\mathscr{K}$ and $\mathscr{O}_c'(\mathscr{K})$ the corresponding spaces K, $\mathscr{O}_M(K)$ of Fourier-Laplace Transforms $\hat{\phi}$, $\hat{S}$ respectively, are algebras of entire functions under pointwise multiplication; more precisely, if S_α denotes the strip $\{z: |Rl(z)| \leqq \alpha\}$ in the complex plane:

THEOREM 2. *An entire function* Φ

(i) *belongs to* K *if and only if for each positive integer* n

$$\sup_{z \in S_n} (1 + |z|)^n |\Phi(z)| < \infty\ ,$$

(ii) *belongs to* $\mathscr{O}_M(K)$ *if and only if there corresponds to each positive integer* n *an integer* l *for which*

$$\sup_{z \in S_n} (1 + |z|)^{-l} |\Phi(z)| < \infty\ .$$

Proof. See [4], [13].

These spaces K, $\mathscr{O}_M(K)$ are given the topology carried over from $\mathscr{K}$, $\mathscr{O}_c'(\mathscr{K})$ respectively by the Fourier-Laplace Transform. Just as $\mathscr{O}_c'(\mathscr{K})$ is the algebra of convolution operators on $\mathscr{K}$, so $\mathscr{O}_M(K)$ is the algebra of multiplication operators on K. This is in complete analogy with the spaces $\mathscr{O}_c'$, $\mathscr{O}_M$ introduced by Schwartz ($[9]_{\text{II}}$, p. 99) where the space corresponding to $\mathscr{K}$ is then the space $\mathscr{S}$ of indefinitely differentiable functions of rapid decay at infinity (see [12] for elaboration).

Finally, $\mathscr{K}_+$ (respectively $\mathscr{O}_c'(\mathscr{K})_+$) denotes the subspace of functions in $\mathscr{K}$ (respectively distributions in $\mathscr{O}_c'(\mathscr{K})$) with support in $R_+ = [0, \infty)$.

2. Throughout the paper $\mathscr{A}$ will denote a topological convolution subalgebra of $\mathscr{O}_c'(\mathscr{K})$ in which the convolution operation is assumed to be separately continuous. We shall further assume that

$\mathscr{A}$ contains an approximate identity of functions $\{\phi_k\}$ in $\mathscr{K}$ or $\mathscr{K}_+$ in the sense that $S*\phi_k$ converges to S in $\mathscr{A}$ for each $S \in \mathscr{A}$. Now associated with each closed ideal I in $\mathscr{A}$ is the cospectrum cosp (I) of I consisting of the zeros counted according to multiplicity common to the Fourier-Laplace Transform of elements in I. If, in addition, $\mathscr{A} \subset \mathscr{O}_c'(\mathscr{K})_+$ so that $S \in \mathscr{A}$ has support in $[0, \infty)$, a_S will denote the largest nonnegative number such that S has support in $[a_S, \infty)$, i.e., the convex support of S lies in $[a_S, \infty)$ but not in $[c, \infty)$ for any $c > a_S$. It is known that a_S can be characterized as the largest number for which

$$|\exp(a_S z)\hat{S}(z)| = O(1 + |z|^n)\,, \qquad Rl(z) > u_0\,, \tag{1}$$

for some integer n and every $u_0 > 0$ (cf. [2, p. 52]). Thus a_S is a measure of the rapidity of decay of $\hat{S}$ at infinity. This definition makes equally good sense for any $S \in \mathscr{S}'(R)$ with support in $[0, \infty)$.

From the Beurling-Lax theorem describing the invariant subspaces of $H^2(R)$ (see [6, p. 165]; [5, p. 107]), we shall deduce the following results ($\subset$ will always imply continuous embedding):

THEOREM A. *Let $\mathscr{A}$ be a topological convolution subalgebra of $\mathscr{O}_c'(\mathscr{K})$ with*

$$\mathscr{K} \subset \mathscr{A} \subset \mathscr{O}_c'(\mathscr{K})\,. \tag{2}$$

Then each closed ideal in $\mathscr{A}$ is characterized by its cospectrum.

THEOREM B. *Let $\mathscr{A}$ be a topological convolution subalgebra of $\mathscr{O}_c'(\mathscr{K})_+$ with*

$$\mathscr{K}_+ \subset \mathscr{A} \subset \mathscr{O}_c'(\mathscr{K})_+\,.$$

Then each closed ideal I in $\mathscr{A}$ is characterized by its cospectrum together with the number

$$a_I = \inf\{a_S \colon S \in I\}\,. \tag{3}$$

For each $\alpha \in R$ denote by $L^p_\alpha(R)$, $1 \leqq p < \infty$, the usual (equivalence classes of) functions for which

$$\|f\|_{p,\alpha} = \left\{\int_R (|f(x)|\exp(\alpha|x|))^p dx\right\}^{1/p}$$

is finite and by L^p_ω the intersection $\bigcap_{\alpha \geqq 0} L^p_\alpha(R)$ provided with the topology defined by $\|(\cdot)\|_{p,\alpha}$, $\alpha \in R_+$. Then $L^p_\omega(R)$ is a convolution subalgebra of $\mathscr{O}_c'(\mathscr{K})$ satisfying (2) with an approximate identity from $\mathscr{K}$, even from $\mathscr{D}$ (use Theorem 2, for instance). Thus Theorem A applies. Further examples can be obtained by this construction by imposing smoothness conditions, say differentiability or suitable Lipschitz conditions, on the functions. In the opposite direction, denote by $W_{\alpha}^{rp}(R)$ the (Sobolev type) space of functions f in $L_{\alpha}^{p}(R)$ with generalized derivatives $D^{j}f$ in $L_{\alpha}^{p}(R)$, $j = 1, \cdots, r$, and $W_{\omega}^{rp}(R)$ the intersection $\bigcap_{\alpha \geqq 0} W_{\alpha}^{rp}(R)$, both spaces being given the usual topology. Theorem A applies here also to $W_{\omega}^{rp}(R)$, $r = 1, 2, \cdots$, $1 \leqq p < \infty$. Theorem B applies, for instance, to analogously defined algebras with R replaced by R_{+}, extending any function or distribution defined on R_{+} to all of R by zero.

3. This section contains preliminary results the first of which reduces the proof of Theorems A, B to the special case when $\mathscr{A} = L_{\omega}^{2}(R)$, $L_{\omega}^{2}(R_{+})$ respectively.

THEOREM 3. *Let $\mathscr{A}$ be a convolution algebra with an approximate identity $\{\phi_k\}$ from $\mathscr{K}_{+}$ and satisfying*

$$(4) \qquad \mathscr{K}_{+} \subset \mathscr{A} \subset \mathscr{O}_c'(\mathscr{K})_{+} .$$

Then there is a one-to-one correspondence between the closed ideals of $\mathscr{A}$ and the closed ideals of $\mathscr{O}_c'(\mathscr{K})_{+}$. More precisely, every closed ideal $I \subset \mathscr{A}$ is the intersection with $\mathscr{A}$ of a unique closed ideal J in $\mathscr{O}_c'(\mathscr{K})_{+}$ such that

$$(5) \qquad I = J \cap \mathscr{A}, \quad \operatorname{cosp}(I) = \operatorname{cosp}(J), \quad a_I = a_J ;$$

conversely, every such intersection $J \cap \mathscr{A}$ is a closed ideal in $\mathscr{A}$ satisfying (5).

REMARK. An entirely analogous result holds when $\mathscr{A}$ contains an approximate identity from $\mathscr{K}$ and satisfies (2).

Proof of Theorem 3. The final assertion is almost obvious in view of (4). On the other hand, if I is a closed ideal in $\mathscr{A}$, certainly there exists at least one closed ideal J in $\mathscr{O}_c'(\mathscr{K})_{+}$ satisfying (5); for let J be the closure of I in $\mathscr{O}_c'(\mathscr{K})_{+}$. Then, clearly, $I \subset J \cap \mathscr{A}$, $\operatorname{cosp}(I) = \operatorname{cosp}(J)$ and $a_I = a_J$. Now, when $\{f_n\}$ is a net in I converging in $\mathscr{O}_c'(\mathscr{K})_{+}$ to $g \in J \cap \mathscr{A}$, by Theorem 1(ii) the net $\{f_n * \phi_k\}$ converges for each k to $g * \phi_k$ in $\mathscr{K}_{+}$ and hence in $\mathscr{A}$. But then $g * \phi_k \in I$ and so g itself belongs to I, i.e., $I \supset J \cap \mathscr{A}$.

To check the uniqueness, suppose J_1, J_2 are closed ideals in $\mathscr{O}_c'(\mathscr{K})_{+}$ for which $J_1 \cap \mathscr{A} = I = J_2 \cap \mathscr{A}$. Now I contains $g * \mathscr{K}_{+}$ for each $g \in J_1, J_2$ so I contains dense subsets of both J_1 and J_2 since $\mathscr{O}_c'(\mathscr{K})_{+}$ has an approximate identity from $\mathscr{K}_{+}$. Hence, with the notation of the previous paragraph, $J_1 = J = J_2$.

Assuming Theorem B we obtain very easily the characterization mentioned in the introduction of the closed ideals in $\mathscr{A}$ having the same cospectrum.

COROLLARY. *Under the hypotheses of Theorem* 3 *the closed ideals in* $\mathscr{A}$ *having the same cospectrum form a totally ordered family* $\{I_\xi\}$, $\xi \in [0, \infty)$, *with* $I_\xi \supsetneqq I_\eta$ *whenever* $\xi < \eta$.

Proof. It is enough to prove the result for $\mathscr{A} = \mathscr{O}_c'(\mathscr{K})_+$ (cf. (5)). Let I be any closed ideal in $\mathscr{O}_c'(\mathscr{K})_+$. If $a_I \neq 0$, say $a_I = \lambda$, the set I_0 of λ-left translates

$$I_0 = \{S_{-\lambda}: S \in I, S_{-\lambda}(x) = S(x + \lambda)\}$$

(obvious modifications if S is not a function) is a closed ideal in $\mathscr{O}_c'(\mathscr{K})_+$ with $\operatorname{cosp}(I_0) = \operatorname{cosp}(I)$ and $a_{I_0} = 0$. When $a_I = 0$ merely set $I_0 = I$. Now define I_ξ, $\xi \in [0, \infty)$ by

$$I_\xi = \{S_\xi: S \in I_0, S_\xi(x) = S(x - \xi)\} ,$$

the ξ-right translates of elements in I_0. This family $\{I_\xi\}$, $\xi \in [0, \infty)$, of closed ideals in $\mathscr{O}_c'(\mathscr{K})_+$ certainly satisfies $\operatorname{cosp}(I_\xi) = I$, $a_{I_\xi} = \xi$ as is easy to see; hence it is totally ordered by reverse inclusion. Of course, the original ideal I is I_λ in the family. By Theorem B any closed ideal having the same cospectrum as I belongs to $\{I_\xi\}$.

For the strip S_α, $H^2(S_\alpha)$ denotes the space of functions analytic in the interior of S_α for which

$$||F|| = \sup_{|u|<\alpha} \left\{\int_R |F(u + iv)|^2 dv\right\}^{1/2}$$

is finite, $\tilde{H}^2(S_\alpha)$ then denotes the space

$$\tilde{H}^2(S_\alpha) = \left\{G: G = \left(\cos \frac{\pi z}{4\alpha}\right)F, F \in H^2(S_\alpha)\right\} .$$

It is well known that $L^2_\alpha(R)$ is isomorphic to $H^2(S_\alpha)$ under the Fourier-Laplace Transform (cf. [11, p. 130]). On the other hand, $\tilde{H}^2(S_\alpha)$ consists of those functions L^2-integrable on the boundary ∂S_α of S_α with respect to the measure $(\cosh(\pi v/2\alpha))^{-1}dv$ whose Poisson integrals are analytic in the interior of S_α. This can be checked by considering for instance the mapping $\zeta \rightarrow z = (4\alpha/\pi)\tan^{-1} i\zeta$ of the closed unit disc onto S_α. When $\tilde{H}^2(S_\alpha)$ is given the norm

$$||G|| = \left\{\int_{\partial S_\alpha} |G(\pm\alpha + iv)|^2 \left(\cosh \frac{\pi v}{2\alpha}\right)^{-1} dv\right\}^{1/2} ,$$

it is easy to see the mapping $z \rightarrow w = \exp(i\pi z/2\alpha)$ of S_α onto the right hand half-plane $Rl(w) \geqq 0$ induces an isomorphism between $\tilde{H}^2(R)$ (cf. [5, p. 107])[1] and $\tilde{H}^2(S_\alpha)$. Since $\tilde{H}^2(R)$ is isomorphic with the usual H^2 space for the unit disc ([5, p. 105]) the significance of $\tilde{H}^2(S_\alpha)$ is not surprising.

The spaces $H^\infty(S_\alpha)$, $H^\infty(R)$ of functions bounded and analytic in the strip S_α and the right half-plane respectively are isometrically isomorphic under the mapping $z \to \exp(i\pi z/2\alpha)$. Thus, each $F \in H^\infty(S_\alpha)$ admits a factorization in the form

$$F(z) = \lambda \exp(-\rho_- e^{i\pi z/2\alpha} - \rho_+ e^{-i\pi z/2\alpha}) F_I(z) F_0(z) \tag{6}$$

with $|\lambda| = 1$, ρ_- and ρ_+ in R_+, F_I an "inner" function and F_0 an "outer" function by transferring the usual factorization for $H^\infty(R)$ to $H^\infty(S_\alpha)$ (cf. [5, p. 133]). Each "inner" function can be further decomposed again by transferring the analogous decomposition for the half-plane case; at the risk of confusion the same terminology is used as in the half-plane case—Blaschke product, $\cdots$.

We shall denote by $H^2_+(S_\alpha)$ the closed subspace of $H^2(S_\alpha)$ corresponding under the Fourier-Laplace Transform to the closed subspace $L^2_\alpha(R_+)$ of $L^2_\alpha(R)$. A doubly-invariant subspace I of $H^2(S_\alpha)$ will mean one invariant under multiplication by e^{az}, $a \in R$, a simply invariant subspace of $H^2_+(S_\alpha)$ one invariant under multiplication by e^{-az}, $a \in R_+$.

THEOREM 4. (a) *Each closed doubly-invariant subspace I of $H^2(S_\alpha)$ is of the form $I = FH^2(S_\alpha)$ for some inner function $F \in H^\infty(S_\alpha)$.*

(b) *If I is a closed simply-invariant subspace of $H^2_+(S_\alpha)$ then*

$$I = e^{-\rho z} G H^2_+(S_\alpha) \tag{7}$$

for some $\rho \in R_+$ and G a function bounded and analytic in $Rl(z) > -\alpha$ having measurable boundary values of modules 1 a.e. on $Rl(z) = -\alpha$.

A simple lemma is needed in the proof of Theorem 4.

LEMMA 1. *A closed doubly-invariant subspace I of $H^2(S_\alpha)$ is invariant under multiplication by every $\Psi \in H^\infty(S_\alpha)$.*

Proof. The subspace J of $L^2_\alpha(R)$ corresponding to I is invariant under translation both to the left and to the right. Now, by Plancherel's theorem, the mapping $F \to \Psi F$ for $F \in H^2(S_\alpha)$ gives rise to a mapping $f \to f_\psi$ of $L^2_\alpha(R)$ commuting with translation. To prove the lemma therefore, it is enough to show that whenever $\phi \in L^2_{-\alpha}(R)$ and $\phi * f^* = 0$ for all $f \in J$, then $\phi * (f_\psi)^* = 0$ the convolution $\phi * g^*$ be-

[1] $\tilde{H}^2(R) = \{(1 + w)f : f \in H^2(R),\ H^2(R)$ the Hardy space for the right half-plane$\}$.

ing defined by

$$\phi * g^*(x) = \int_R \phi(x + y)g(y)dy .$$

But, if $h \in L^1_\alpha(R) \cap L^2_\alpha(R)$,

$$(\phi * f^*_\psi) * h^* = \phi * (f_\psi * h)^* = (\phi * f^*) * h^*_\psi = 0$$

as an easy calculation shows. Such functions h are dense in $L^2_\alpha(R)$ so $\phi * f^*_\psi = 0$.

Proof of Theorem 4. (a) Since $|\cos(\pi z/4\alpha)|^2 = \frac{1}{2}\cosh(\pi v/2\alpha)$ on ∂S_α the set $\tilde{I} = (\cos(\pi z/4\alpha))I$ is a closed subspace of $\tilde{H}^2(S_\alpha)$ invariant under multiplication by every $\Psi \in H^\infty(S_\alpha)$. Thus the subspace of $\tilde{H}^2(R)$ corresponding to $\tilde{I}$ under the isomorphism of $\tilde{H}^2(S_\alpha)$ and $\tilde{H}^2(R)$ is of the form $F_1\tilde{H}^2(R)$ for some inner function $F_1 \in H^\infty(R)$ applying the Beurling-Lax result (cf. [5, p. 107]). Consequently, for some inner function $F \in H^\infty(S_\alpha)$,

$$\left(\cos\frac{\pi z}{4\alpha}\right)I = F\left(\cos\frac{\pi z}{4\alpha}\right)H^2(S_\alpha) .$$

Since $\cos(\pi z/4\alpha)$ is zero-free throughout S_α the result follows.

(b) Under the mapping $F \to F_\alpha$, $F_\alpha(z) = F(z - \alpha)$, $Rl(z) \geqq 0$, $H^2_+(S_\alpha)$ is isomorphic with $H^2(R)$. In addition, the image of any closed simply invariant subspace I of $H^2_+(S_\alpha)$ is an invariant subspace of $H^2(R)$ in the terminology of Hoffman ([5, p. 106]). The expression (7) now follows from the result of Lax ([6]; [5, p. 107]).

As mentioned earlier, if F is the Fourier-Laplace Transform of a distribution in $\mathscr{S}'(R)$ with support in $[0, \infty)$, the mapping $F \to a_F$ with a_F the largest number for which (1) holds, is well-defined. This applies in particular to functions in $H^2(R)$ or $H^\infty(R)$.

THEOREM 5. *If* $F = \lambda e^{-\rho z}F_I F_0$ *is the usual factorization of a function* $F \in H^2(R)$ *or* $H^\infty(R)$, *then* $\rho = a_F$.

THEOREM 6. *When* $F \in H^\infty(S_\alpha)$ *is factorized in the form* (6) *the numbers* ρ_+, ρ_- *satisfy*

$$\lim_{v\to-\infty} \frac{\log|F(u + iv)|}{\exp\left(-\frac{\pi v}{2\alpha}\right)} = -\rho_- \cos\frac{\pi u}{2\alpha}$$

(8)

$$\lim_{v\to\infty} \frac{\log|F(u + iv)|}{\exp\left(\frac{\pi v}{2\alpha}\right)} = -\rho_+ \cos\frac{\pi u}{2\alpha}$$

for almost all u, $|u| < \alpha$. *In particular, if* F *belongs also to* $H^\infty(S_\beta)$ *for some* $\beta > \alpha$, *then* $\rho_+ = \rho_- = 0$.

A proof of Theorem 5 appears, for instance, in [8, Lemma 4]. Actually, the Ahlfors-Heins theorem [1, Th. A] gives an even stronger result since

$$\lim_{r\to\infty} \frac{\log |F(re^{i\theta})|}{r} = -\rho\cos\theta \tag{9}$$

for almost all θ, $-\pi/2 < \theta < \pi/2$.[2] To prove Theorem 6 it is enough to establish the first of the limits since the second follows after a transformation $z \to \bar{z}$. But, when S_α is mapped onto $Rl(w) \geqq 0$ via the mapping $z \to w = \exp(i\pi z/2\alpha)$, the limit (8) is precisely the analogue for the strip S_α of (9). Finally, when ρ_-, ρ'_- are corresponding numbers in the factorization of F as a function in $H^\infty(S_\alpha)$, $H^\infty(S_\beta)$ respectively, we deduce

$$\lim_{v\to-\infty} \frac{\log |F(u+iv)|}{\exp\left(-\dfrac{\pi v}{2\beta}\right)} = -\rho'_- \cos\frac{\pi u}{2\beta}, \tag{10}$$

for almost all u, $|u| < \beta$, in addition to (8). Choosing any u, $|u| < \alpha$, on which (8) and (10) hold simultaneously we can soon check that ρ_- must be zero if $\beta > \alpha$. Similarly $\rho_+ = 0$.

4. The proofs of Theorems A and B can now be given.

Proof of A. In view of the remark following Theorem 3, Theorem A need be proved only in the case $\mathscr{A} = L^2_\omega(R)$.

Let I be a closed ideal in $L^2_\omega(R)$, I_α the closure of I in $L^2_\alpha(R)$. Then $I = \bigcap_{\alpha>0} I_\alpha$. For certainly $I \subset \bigcap_{\alpha\geqq 0} I_\alpha$; on the other hand, the topology on $L^2_\omega(R)$ being the topology defined by the semi-norms $||(\cdot)||_\alpha$, i.e., the projective limit topology, each $f \in \bigcap_{\alpha\geqq 0} I_\alpha$ is a limit point of I in $L^2_\omega(R)$ hence $\bigcap_{\alpha\geqq 0} I_\alpha = I$. The set J_α of Fourier Laplace Transforms of functions in I_α is a closed doubly-invariant subspace of $H^2(S_\alpha)$. Thus $J_\alpha = FH^2(S_\alpha)$ where F is an inner function in $H^\infty(S_\alpha)$ depending on α of course. In the factorization of F

$$F = \exp(-\rho_- e^{i\pi z/2\alpha} - \rho_+ e^{-i\pi z/2\alpha})BS, \tag{11}$$

with B a Blaschke product, S a singular function, the Blaschke product is formed with the elements of $\operatorname{cosp}(I)$ lying in $S_\alpha \backslash \partial S_\alpha$. On the

[2] In the application of (9) we have in mind the singular function in F is identically 1. A proof of (9) in this case avoiding the Ahlfors-Heins theorem is given in [7] (for the upper half-plane) on page 243.

other hand, if α is chosen so that ∂S_α does not intersect cosp (I), the singular function in (11) is identically 1; for if $z_0 \in \partial S_\alpha$, there exists $f \in I$ with $\hat{f}$ continuous on ∂S_α and nonzero at z_0 in which case z_0 does not belong to the support of the singular measure defining S (cf. [5, p. 70]). Furthermore, as each $\hat{f}$, $f \in I$, belongs to $H^\infty(S_\beta)$ for every $\beta > \alpha$, the constants ρ_+, ρ_- in the factorization of $\hat{f}$, and hence in (11), are both zero. Thus, with this choice of α, the inner function reduces to the Blaschke product formed by the elements of cosp (I) in S_α.

Now choose a monotonic unbounded sequence of α's for which cosp $(I) \cap \partial S_\alpha$ is empty. Such a choice is always possible since any such sequence is enough to describe $L^2_\omega(R)$ both algebraically and topologically. If f is any function in $L^2_\omega(R)$ for which $\hat{f}(z) = 0$ whenever $z \in$ cosp (I) (with appropriate multiplicities), it is clear that $\hat{f}$ belongs to every J_α because the corresponding inner function (11), merely a Blaschke product, divides $\hat{f}$. Consequently, $f \in \bigcap_{\alpha \geqq 0} I_\alpha = I$ showing that I is determined by cosp (I).

Proof of B. In this case it is enough to consider $L^2_\omega(R_+)$. For a closed ideal I in $L^2_\omega(R_+)$, let I_α be its closure in $L^2_\alpha(R_+)$. By the same argument as in the proof of A we have $I = \bigcap_{\alpha \geqq 0} I_\alpha$. The corresponding set J_α of Fourier-Laplace Transforms is a simply invariant subspace of $H^2_+(S_\alpha)$ so is given by

$$J_\alpha = e^{-\rho z} G H^2_+(S_\alpha) \tag{12}$$

for some $\rho \in R_+$ and "inner" function G. By much the same argument as in the proof of Theorem A, if α belongs to a suitably chosen sequence, G consists only of the Blaschke product for a half-plane formed with the elements of cosp (I) in the half-plane $Rl(z) > -\alpha$. Also, by Theorem 5, the number ρ in (12) is given by

$$\rho = \inf \{a_F \colon F \in J_\alpha\}$$

since $e^{-\rho z} G$ is the greatest common divisor of the inner functions in the factorization of elements in J_α. But then, with the notation of (3), $\rho = a_I$. For certainly $\rho \leqq a_I$ since $I_\alpha \supset I$; on the other hand, the limit in $L^2_\alpha(R^+)$ of any sequence with convex support in $[a_I, \infty)$ again has convex support in $[a_I, \infty)$—hence $\rho = a_I$. Thus any $f \in L^2_\omega(R_+)$ which is zero a.e. outside $[a_I, \infty)$ and whose Fourier-Laplace Transform $\hat{f}$ is zero on cosp (I) (with appropriate multiplicities), belongs to each I_α, hence to $I = \bigcap_{\alpha \geqq 0} I_\alpha$. Thus I is determined by cosp (I) together with the number a_I.

REFERENCES

1. L. Ahlfors and M. Heins, *Questions of regularity connected with the Phragmen-Lindelöf principle*, Ann. of Math. (2) **50** (1949), 341–346.
2. E. Beltrami and M. Wohlers, *Distributions and the Boundary Values of Analytic Functions*, Academic Press, New York, 1966.
3. R. P. Boas, *Entire Functions*, Academic Press, New York, 1954.
4. M. Hasumi, *Note on the n-dimensional tempered ultra-distributions*, Tôhoku Math. J. **13** (1961), 94–104.
5. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
6. P. Lax, *Translation invariant subspaces*, Acta Math. **101** (1959), 163–178.
7. B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc. Trans. of Math. Monographs, vol. 5, Providence, R. I., 1964.
8. B. Nyman, *On the one-dimensional translation group and semi-group in certain function spaces*, Thesis, Uppsala, 1950.
9. L. Schwartz, *Théorie des Distributions*, vols. I, II, Hermann, Paris, 1957, 1959.
10. ———, *Séminaire Schwartz*, Paris, 1953–1954.
11. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford Univ. Press, Oxford, 1948.
12. Z. Zielézny, *Hypoelliptic and entire elliptic convolution equations in subspaces of the space of distributions* (I), Studia Math. **28** (1967), 317–332.
13. ———, *On the space of convolution operators in* $\mathscr{K}_1'$, Studia Math. **31** (1968), 111–124.

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