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DIMENSION THEORY IN POWER SERIES RINGS

DAVID E. FIELDS

Let V be a valuation ring of finite rank n. If V is discrete, then $V[[X]]$ has dimension $n+1$. If V is not discrete, then the dimension of $V[[X]]$ is at least $n+k+1$, where k is the number of idempotent proper prime ideals of V.

Let R be a commutative ring with identity. If there exists a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of $n+1$ prime ideals of R, where $P_n \subset R$, but no such chain of $n+2$ prime ideals, then we say that R has *dimension* n and we write $\dim R = n$ [3]. In [3] and [4], Seidenberg has investigated the dimension theory of $R[X_1, X_2, \cdots, X_m]$ where R has finite dimension and $X_1, X_2, \cdots, X_m$ are indeterminates over R. We investigate the dimension theory of $V[[X]]$ where V is a valuation ring.

Throughout this paper, R denotes a commutative ring with identity; ω is the set of natural numbers; ω_0 is the set of non-negative integers; and Z is the set of integers. If

$$f(X) = \sum_{i=0}^{\infty} f_i X^i \in R[[X]] ,$$

we denote by A_f the ideal of R generated by the coefficients of $f(X)$: $A_f = \{f_0, f_1, \cdots, f_k, \cdots\}R$. If A is an ideal of R, we let

$$A[[X]] = \{f(X) = \sum_{i=0}^{\infty} f_i X^i \colon f_i \in A \text{ for each } i \in \omega_0\}$$

and we define $A \cdot R[[X]]$ to be the ideal of $R[[X]]$ which is generated by A. Then $A \cdot R[[X]] = \{f(X)\colon A_f \subseteqq B$ for some finitely generated ideal B of R with $B \subseteqq A\}$. It is clear that $A \cdot R[[X]] \subseteqq A[[X]]$; equality holds if and only if each countably generated ideal of R contained in A is contained in a finitely generated ideal of R contained in A. In particular, if V is a valuation ring containing an ideal A which is countably generated but not finitely generated, then $A \cdot V[[X]] \subset A[[X]]$. Finally, we note that if A is an ideal of R, then $R[[X]]/A[[X]] \cong (R/A)[[X]]$; hence $A[[X]]$ is a prime ideal of $R[[X]]$ if and only if A is a prime ideal of R.

2. Discrete valuation rings. Let V be a valuation ring of rank k with associated valuation v and value group G; let $\{0\} = G_0 \subset G_1 \subset \cdots \subset G_k = G$ be the chain of isolated subgroups of G together with G. In [2], Iwasawa proves that for $1 \leqq i \leqq k$,

$G_i/G_{i-1} \cong H_i$ where H_i is a subgroup of the additive group of real numbers, this being an order-preserving isomorphism of groups. If for $1 \leqq i \leqq k$, $H_i \cong Z$, we shall say that V is a *discrete valuation ring* of rank k. This is equivalent to the condition that V contains no idempotent proper prime ideal.

LEMMA 2.1. *Let V be a valuation ring and let P be a proper prime ideal of V. If P is not idempotent, then in $V[[X]]$, $\sqrt{(P \cdot V[[X]])} = P[[X]]$ and $(P[[X]])^2 \subseteqq P \cdot V[[X]]$.*

Proof. Let $\alpha \in P$, $\alpha \notin P^2$. Then

$$(P[[X]])^2 \subseteqq P^2[[X]] \subseteqq (\alpha)V[[X]] \subseteqq P \cdot V[[X]] .$$

Hence $P[[X]] \subseteqq \sqrt{(P \cdot V[[X]])}$ and the reverse containment is clear.

LEMMA 2.2. *Let V be a valuation ring with quotient field K and let P be a proper prime ideal of V. Let*

$$D = V[[X]][K] = (V[[X]])_{V\setminus\{0\}} .$$

Then $D = (V_P[[X]])_{V_P\setminus\{0\}}$.

Proof. We first show that $V_P[[X]] \subseteqq D$. Let

$$f(X) = \sum_{i=0}^{\infty} f_i X^i \in V_P[[X]] .$$

For each $i \in \omega_0$, there exists $r_i \in V\setminus P$ such that $r_i f_i \in V$. Let $a \in P\setminus\{0\}$; then for each $i \in \omega_0$, $a/r_i \in PV_P = P \subseteqq V$, implying that $af_i = (a/r_i)(r_i f_i) \in V$; that is, $af(X) \in V[[X]]$. This implies that $f(X) \in (V[[X]])_{V\setminus\{0\}} = D$, showing that $V_P[[X]] \subseteqq D$.

Since $D \supseteqq K$, each nonzero element of V_P is a unit in D. Thus $D \supseteqq (V_P[[X]])_{V_P\setminus\{0\}}$ and the reverse containment is obvious.

COROLLARY 2.3. *Let V be a valuation ring and let P be a proper prime ideal of V. There is a one-to-one correspondence between prime ideals of $V[[X]]$ which contract to* (0) *in V and prime ideals of $V_P[[X]]$ which contract to* (0) *in V_P; this correspondence preserves containment.*

Proof. Lemma 2.2 assures that there is a one-to-one, containment preserving correspondence between each of these classes of prime ideals and the class of prime ideals of D.

LEMMA 2.4. *Let R be a quasi-local ring having maximal ideal*

M. Let $f(X) \in R[[X]]$, $f(X) \notin M[[X]]$ — say $f_k \in R \backslash M$, k minimal. There exists $g(X)$, a unit of $R[[X]]$, such that $f(X)g(X)$ has exactly one unit coefficient, namely $(fg)_k$.

Proof. For $u(X) \in R[[X]]$, denote by $\bar{u}(X)$ the canonical image of $u(X)$ in $(R/M)[[X]]$. By choice of k,

$$\bar{f}(X) = \bar{f}_k X^k + \bar{f}_{k+1} X^{k+1} + \cdots = X^k(\bar{f}_k + \bar{f}_{k+1} X + \cdots),$$

where $\bar{f}_k \neq 0$. Then $\bar{f}_k + \bar{f}_{k+1}X + \cdots$ is a unit of $(R/M)[[X]]$, and we can choose $g(X) \in R[[X]]$ such that $\bar{g}(X) \cdot (\bar{f}_k + \bar{f}_{k+1}X + \cdots) = 1$. Thus $\bar{f}(X) \cdot \bar{g}(X) = X^k$, and $f(X)g(X) - X^k \in M[[X]]$. This implies that only the coefficient of X^k in $f(X)g(X)$ is not in M.

COROLLARY 2.5. *Let V be a valuation ring and let P be a proper prime ideal of V. If Q is an ideal of $V_P[[X]]$ and if $Q \nsubseteq (PV_P)[[X]]$, then $Q \cap V[[X]] \nsubseteq P[[X]]$.*

Proof. Lemma 2.4 assures that there is a power series $g(X)$ in Q with $g(X)$ having exactly one unit coefficient, g_k. Since g_k is a unit of V_P, there in no loss of generality in assuming that, in fact, $g_k = 1$. Then for $i \neq k$, $g_i \in PV_P = P \subsetneqq V$, implying that $g(X) \in Q \cap V[[X]]$ while $g(X) \notin P[[X]]$.

LEMMA 2.6.[1] *Let R be a Noetherian ring having dimension n. Then $R[[X_1, X_2, \cdots, X_m]]$ is Noetherian and has dimension $n + m$.*

Proof. It is well known that if R is Noetherian, then $R[[X_1, X_2, \cdots, X_m]]$ is Noetherian. We shall show that the dimension of $R[[X]]$ is $n + 1$; the lemma follows immediately by induction on m.

Let M be a maximal ideal of $R[[X]]$. Then $M = M_1 + (X)$ for some maximal ideal M_1 of R. Since $\dim R = n$, the height of M_1 is k where $k \leqq n$. There exists an ideal $A = (a_1, a_2, \cdots, a_k)$ of R which admits M_1 as an isolated prime ideal [5; 242]. It is straightforward to verify that $M = M_1 + (X)$ is an isolated prime ideal of $A + (X) = (a_1, a_2, \cdots, a_k, X)R[[X]]$. This implies that the height of M is at most $k + 1$ [5; 240]; since $k \leqq n$, the height of M is at most $n + 1$. Since M was an arbitrary maximal ideal of $R[[X]]$, we conclude that $\dim R[[X]] \leqq n + 1$; the reverse inequality is clear.

THEOREM 2.7. *Let V be a discrete valuation ring of rank n and let $(0) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ be the nonunit prime ideals of*

[1] The proof of Lemma 2.6 was pointed out to the author by William Heinzer.

V. *Then* $\dim V[[X]] = n + 1$.

Proof. We use induction on n, the case $n = 1$ following from Lemma 2.6 since a rank one discrete valuation ring is Noetherian.

Assuming the result for discrete valuation rings of rank less than n, let V be a discrete valuation ring of rank n and let $(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t$ be a chain of prime ideals of $V[[X]]$. We consider two cases.

Case 1. $Q_1 \cap V \neq (0)$. Here $Q_1 \cap V \supseteqq P_1$, so that $Q_1 \supseteqq P_1 \cdot V[[X]]$, implying that $Q_1 \supseteqq \sqrt{(P_1 \cdot V[[X]])} = P_1[[X]]$, the latter equality being a consequence of Lemma 2.1. But the depth of $P_1[[X]]$ cannot exceed $\dim (V/P_1)[[X]] = n$; we conclude that $t \leqq n + 1$.

Case 2. $Q_1 \cap V = (0)$. Corollary 2.3 asserts that $Q_1 = Q^* \cap V[[X]]$, where Q^* is a prime ideal of $V_{P_1}[[X]]$ and $Q^* \cap V_{P_1} = (0)$. Since $\dim V_{P_1}[[X]] = 2$, $Q^* \nsubseteq (P_1 V_{P_1})[[X]]$. By Corollary 2.5, $Q_1 \nsubseteq P_1[[X]]$. Since $V_{P_1}[[X]]$ is two-dimensional and local, each proper prime ideal of $V_{P_1}[[X]]$ which contracts to (0) in V_{P_1} is a minimal prime ideal of $V_{P_1}[[X]]$. Corollary 2.3 now assures that each proper prime ideal of $V[[X]]$ which contracts to (0) in V is a minimal prime ideal of $V[[X]]$. It follows that $Q_2 \cap V \neq (0)$, implying that $Q_2 \supseteqq P_1[[X]]$. Since also $Q_2 \supseteqq Q_1$ and $Q_1 \nsubseteq P_1[[X]]$, we conclude that $Q_2 \supset P_1[[X]]$. Thus we have a chain $(0) \subset P_1[[X]] \subset Q_2 \subset Q_3 \subset \cdots \subset Q_t$. It follows, as in Case 1, that $t \leqq n + 1$.

Thus $\dim V[[X]] \leqq n + 1$ and the reverse inequality is clear.

3. Rank one nondiscrete valuation rings.

We note that if V is a rank one valuation ring, then the value group of v is Archimedian.

LEMMA 3.1. *Let V be a valuation ring and let B be an ideal of V. If B is not finitely generated, then the following conditions are equivalent*:

(a) $f(X) \in B \cdot V[[X]]$.

(b) $A_f \subseteqq (b)$ *for some* $b \in B$.

(c) $f(X) = bg(X)$ *for some* $b \in B$, $g(X) \in V[[X]]$.

(d) $A_f \subset B$.

Proof. We establish that (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (a) and that (b) $\leftrightarrow$ (d).

(a) $\rightarrow$ (b): Let $f(X) \in B \cdot V[[X]]$; then we can write

$$f(X) = b_1[g^{(1)}(X)] + b_2[g^{(2)}(X)] + \cdots + b_t[g^{(t)}(X)]$$

where for $1 \leqq i \leqq t$, $b_i \in B$ and $g^{(i)}(X) = \sum_{j=0}^{\infty} g_{ij} X^j \in V[[X]]$. Thus $f(X) = \sum_{i=0}^{\infty} f_i X^i$ where $f_i = \sum_{k=1}^{t} b_k g_{ki}$. In V, $(b_1, b_2, \cdots, b_t) = (b_s)$ for some s, $1 \leqq s \leqq t$. Now for $i \in \omega_0$, $f_i = \sum_{k=1}^{t} b_k g_{ki} \in (b_s)$, implying that $A_f \subseteqq (b_s)$ where $b_s \in B$.

(b) $\rightarrow$ (c): We assume that $A_f \subseteqq (b)$; then for $i \in \omega_0$, $f_i = bg_i$ where $g_i \in V$. Let $g(X) = \sum_{i=0}^{\infty} g_i X^i$; it then is clear that $f(X) = bg(X)$.

(c) $\rightarrow$ (a): This is obvious.

(b) $\rightarrow$ (d): This is immediate from the assumption that B is not finitely generated.

(d) $\rightarrow$ (b): Assuming that $A_f \subset B$, let $b \in B$, $b \notin A_f$. Then $(b) \not\subseteq A_f$ so $A_f \subseteqq (b)$ since V is a valuation ring.

THEOREM 3.2. *Let V be a rank one nondiscrete valuation ring having maximal ideal M. Then $M \cdot V[[X]] = \sqrt{(M \cdot V[[X]])}$.*

Proof. Let $f(X) \in \sqrt{(M \cdot V[[X]])}$ – say $[f(X)]^k \in M \cdot V[[X]]$; we then can write $[f(X)]^k = rg(X)$ where $r \in M$ and $g(X) \in V[[X]]$. There exists an element s of M with $0 < v(s) \leqq v(r)/k$; then $r = s^k t$ where $t \in V$, implying that $[f(X)]^k = rg(X) = s^k tg(X)$, so that

$$[f(X)]^k / s^k = [f(X)/s]^k = tg(X) \in V[[X]] .$$

Therefore $f(X)/s$ is a root of $Z^k - tg(X) \in V[[X]][Z]$, whereby $f(X)/s$ is integral over $V[[X]]$. Also $f(X)/s$ clearly is in the quotient field of $V[[X]]$. But V is completely integrally closed, implying that $V[[X]]$ is completely integrally closed, hence is integrally closed [1; 150]. Thus $f(X)/s = h(X) \in V[[X]]$ and $f(X) = sh(X) \in M \cdot V[[X]]$ since $s \in M$. Hence $\sqrt{(M \cdot V[[X]])} \subseteqq M \cdot V[[X]]$, so that equality holds.

THEOREM 3.3. *Let R be a quasi-local ring having maximal ideal M and let Q be a prime ideal of $R[[X]]$. If $Q \supseteqq M \cdot R[[X]]$, then either $Q \supseteqq M[[X]]$ or $Q \subseteqq M[[X]]$.*

Proof. We assume that $Q \not\subseteq M[[X]]$ and show that $Q \supseteqq M[[X]]$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \notin M[[X]]$. Let t be the smallest integer k for which f_k is a unit of R. Let $g(X) = \sum_{i=0}^{t-1} f_i X^i$ if $t > 0$; let $g(X) = 0$ if $t = 0$. Then $g(X) \in M \cdot R[[X]] \subseteqq Q$, implying that $f(X) - g(X) \in Q$. If $f(X) - g(X)$ has order zero, then $g(X) = 0$, so that f_0 is a unit of R, implying that $f(X)$ is a unit of $R[[X]]$, whence $Q = R[[X]] \supseteqq M[[X]]$. If $f(X) - g(X)$ has positive order n, then $[f(X) - g(X)]_n$ is a unit of R and $f(X) - g(X) = X^n h(X)$ where $h_0 = [f(X) - g(X)]_n$ is a unit of R, implying that $h(X)$ is a unit of $R[[X]]$.

Since $f(X) - g(X) = X^n h(X) \in Q$ and Q is a prime ideal of $R[[X]]$, either $X^n \in Q$ or $h(X) \in Q$. If $X^n \in Q$, then $X \in Q$, implying that $Q \supseteqq M \cdot R[[X]] + (X) \supseteqq M[[X]]$. If $h(X) \in Q$, then $Q = R[[X]] \supseteqq M[[X]]$. Hence if $Q \not\subseteqq M[[X]]$, then $Q \supseteqq M[[X]]$.

THEOREM 3.4. *Let V be a rank one nondiscrete valuation ring having maximal ideal M.*

(a) *There is a prime ideal P of $V[[X]]$ satisfying $M \cdot V[[X]] \subseteqq P \subset M[[X]]$.*

(*b*) $\dim V[[X]] \geqq 3$.

Proof. Theorem 3.2 asserts that

$$\sqrt{(M \cdot V[[X]])} = M \cdot V[[X]] \subset M[[X]] .$$

Hence there is a prime ideal P of $V[[X]]$ satisfying $P \supseteqq M \cdot V[[X]]$, $P \not\supseteqq M[[X]]$. Theorem 3.3 then asserts that $P \subset M[[X]]$; hence (a) holds.

We now have a chain $(0) \subset P \subset M[[X]] \subset M \cdot V[[X]] + (X)$ of prime ideals of $V[[X]]$, implying (*b*).

4. Valuation rings of finite rank.

LEMMA 4.1. *Let V be a valuation ring and let P be a proper prime ideal of V. Then $PV_P = P$; hence P is idempotent if and only if PV_P is idempotent.*

The proof of Lemma 4.1 is straightforward and will therefore be omitted.

LEMMA 4.2. *Let V be a valution ring and let P be an idempotent proper prime ideal of V. Then $P \cdot V[[X]] = (PV_P) \cdot V_P[[X]]$.*

Proof. Let $f(X) \in (PV_P) \cdot V_P[[X]]$ — say $f(X) = rh(X)$ where $r \in PV_P$ and $h(X) \in V_P[[X]]$. Since $P = PV_P$ is idempotent, we can write $r = st$ where $s, t \in P = PV_P$; then for $i \in \omega_0$, there exists $a_i \in V \backslash P$ such that $a_i h_i \in V$. Since $a_i \in V \backslash P$ and $t \in P$, we have that $(t) \subseteqq (a_i)$ so that $t/a_i \in V$ for each $i \in \omega_0$, implying that $th_i = (t/a_i)(a_i h_i) \in V$ for each $i \in \omega_0$ — that is, $th(X) \in V[[X]]$. Since $s \in P$, we conclude that $f(X) = rh(X) = s(th(X)) \in P \cdot V[[X]]$, establishing that

$$(PV_P) \cdot V_P[[X]] \subseteqq P \cdot V[[X]] .$$

The reverse containment is obvious.

THEOREM 4.3. *Let V be a valuation ring and let P be a proper prime ideal of V. If Q is a prime ideal of $V[[X]]$ and if $Q \supseteq P \cdot V[[X]]$, then either $Q \supseteq P[[X]]$ or $Q \subseteq P[[X]]$.*

Proof. Assuming that $Q \not\subseteq P[[X]]$, we first establish that either $X \in Q$ or Q contains $h(X)$, where $h(X) \in V[[X]]$ and $h_0 \notin P$. Let $f(X) = \sum_{i=0}^{\infty} f_i X^i \in Q$, $f(X) \notin P[[X]]$. Let t be the smallest integer k for which $f_k \notin P$. If $t = 0$, then we let $h(X) = f(X)$. If $t > 0$, then we let $g(X) = \sum_{i=0}^{t-1} f_i X^i$. Then $g(X) \in P \cdot V[[X]] \subseteq Q$, implying that $f(X) - g(X) \in Q$. Further, $f(X) - g(X) = X^t h(X)$ where $h_0 = f_t \notin P$. Since Q is prime, either $X \in Q$ or $h(X) \in Q$. Hence if $Q \not\subseteq P[[X]]$, then either $X \in Q$ or Q contains $h(X)$ where $h(X) \in V[[X]]$ and $h_0 \notin P$.

If $X \in Q$, then $Q \supseteq P[[X]]$; hence we consider the case where $h(X) \in Q$ with $h_0 \notin P$. Observe now that $h(X) \in V_P[[X]]$ and that h_0 is a unit of V_P, implying that $h(X)$ is a unit of $V_P[[X]]$ — that is $1/h(X) \in V_P[[X]]$. Now let $r(X) \in P[[X]]$; then

$$r(X)[1/h(X)] \in P[[X]] \cdot V_P[[X]] \subseteq P[[X]]$$

— in particular, $r(X)[1/h(X)] \in V[[X]]$. Since $h(X) \in Q$, we see that $r(X) = h(X)[r(X)/h(X)] \in Q$. Hence $Q \supseteq P[[X]]$.

LEMMA 4.4. *Let V be a valuation ring having a minimal prime ideal P. If P is idempotent, then $P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}$.*

Proof. Let $f(X) \in \sqrt{(P \cdot V[[X]])}$. Then in

$$V_P[[X]],\ f(X) \in \sqrt{((PV_P) \cdot V_P[[X]])}$$

by Lemma 4.2. Since V_P is a rank one nondiscrete valuation ring, Theorem 3.2 asserts that $\sqrt{((PV_P) \cdot V_P[[X]])} = (PV_P) \cdot V_P[[X]]$. Hence $f(X) \in (PV_P) \cdot V_P[[X]] = P \cdot V[[X]]$, the latter equality following from Lemma 4.2.

THEOREM 4.5. *Let V be a valuation ring and let P be a proper prime ideal of V. If P is idempotent, then*

$$P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}\ .$$

Proof. We shall say that P is *branched* provided there exists a *P-primary* ideal distinct from P [1; 173]. We consider two cases.

Case 1. P is branched. Then there is a prime ideal Q of V with $Q \subset P$ and such that there are no prime ideals of V properly between Q and P [1; 173]. Then P/Q is a minimal prime ideal of V/Q and P/Q is idempotent. Lemma 4.4 assures that

$$(P/Q) \cdot (V/Q)[[X]] = \sqrt{((P/Q) \cdot (V/Q)[[X]])} .$$

By considering the natural homomorphism from $V[[X]]$ to $(V/Q)[[X]]$, we conclude that $P \cdot V[[X]] = \sqrt{(P \cdot V[[X]])}$.

Case 2. P is not branched. Then $P = \bigcup_\lambda M_\lambda$ where $\{M_\lambda\}_{\lambda \in \Lambda}$ is the collection of prime ideals of V properly contained in P [1; 173]. Let $f(X) \in \sqrt{(P \cdot V[[X]])}$ – say $f(X)^k \in P \cdot V[[X]]$. Then $f(X)^k = rg(X)$ where $g(X) \in V[[X]]$ and $r \in P$, implying that $r \in M_{\lambda_1}$ for some $\lambda_1 \in \Lambda$. Thus $f(X)^k = rg(X) \in M_{\lambda_1}[[X]]$, implying that $f(X) \in M_{\lambda_1}[[X]]$. There exists $\lambda_2 \in \Lambda$ such that $M_{\lambda_1} \subset M_{\lambda_2}$. Let $s \in M_{\lambda_2}$, $s \notin M_{\lambda_1}$; then $(s) \supseteqq M_{\lambda_1} \supseteqq A_f$, so that $f(X) = sh(X)$ where $h(X) \in V[[X]]$. Since $s \in M_{\lambda_2}$, $s \in P$; hence $f(X) = sh(X) \in P \cdot V[[X]]$.

COROLLARY 4.6. *Let V be a valuation ring having a proper prime ideal P. If P is idempotent, then there is a prime ideal Q of $V[[X]]$ satisfying $P \cdot V[[X]] \subseteqq Q \subset P[[X]]$.*

Proof. Theorem 4.5 assures that

$$\sqrt{(P \cdot V[[X]])} = P \cdot V[[X]] \subset P[[X]] .$$

Hence there is a prime ideal Q of $V[[X]]$ satisfying $Q \supseteqq P \cdot V[[X]]$, $Q \not\supseteqq P[[X]]$. Theorem 4.3 then asserts that $Q \subset P[[X]]$.

THEOREM 4.7. *Let V be a valuation ring of rank n having k distinct idempotent proper prime ideals. Then* $\dim V[[X]] \geqq n + k + 1$.

Proof. We use induction on n, the case $n = 1$ following from Theorem 2.7 and Theorem 3.4.

Assuming the result for valuation rings of rank t, let V be a valuation ring of rank $t + 1$ having k distinct idempotent proper prime ideals and let $(0) \subset P_1 \subset P_2 \subset \cdots \subset P_{t+1}$ be the chain of nonunit prime ideals of V. We consider two cases.

Case 1. P_1 is not idempotent. Here V/P_1 is a valuation ring of rank t which has k distinct idempotent proper prime ideals. By the induction hypothesis, $\dim (V/P_1)[[X]] \geqq t + k + 1$. Since $(V/P_1)[[X]] \cong V[[X]]/P_1[[X]]$, this implies that the depth of $P_1[[X]]$ is at least $t + k + 1$. Since $P_1[[X]] \neq (0)$, $\dim V[[X]] \geqq t + k + 2$.

Case 2. P_1 is idempotent. Here V/P_1 is a valuation ring of rank

t which has $k-1$ distinct idempotent proper prime ideals. By the induction hypothesis. $\dim (V/P_1)[[X]] \geqq t+(k-1)+1 = t+k$; hence the depth of $P_1[[X]]$ is at least $t+k$. Since P_1 is idempotent, Corollary 4.6 asserts that there is a prime ideal Q of $V[[X]]$ satisfying $P_1 \cdot V[[X]] \subseteqq Q \subset P_1[[X]]$ — in particular, $(0) \subset Q \subset P_1[[X]]$. Since the depth of $P_1[[X]]$ is at least $t+k$, we see that $\dim V[[X]] \geqq t+k+2$.

LEMMA 4.8. *Let V be valuation ring and let P be a proper prime ideal of V.*

(*a*) *If Q' is a prime ideal of $V_P[[X]]$ which satisfies $(PV_P) \cdot$ as $V_P[[X]] \subseteqq Q' \subset (PV_P)[[X]]$, then Q' is a prime ideal of $V[[X]]$ which satisfies $P \cdot V[[X]] \subseteqq Q' \subset P[[X]]$.*

(*b*) *Conversely, if Q is a prime ideal of $V[[X]]$ which satisfies $P \cdot V[[X]] \subseteqq Q \subset P[[X]]$, then Q is a prime ideal of $V_P[[X]]$ which satisfies $(PV_P) \cdot V_P[[X]] \subseteqq Q \subset (PV_P)[[X]]$.*

Proof. To establish (a), we observe that $Q' \subseteqq (PV_P)[[X]] = P[[X]] \subseteqq V[[X]]$, whereby $Q' \cap V[[X]] = Q'$.

We now establish (*b*); we begin by proving that Q is an ideal of $V_P[[X]]$. Let $f(X) \in Q$and $g(X) \in V_P[[X]]$; we show that $f(X) \cdot$ as $g(X) \in Q$. Choose $h(X) \in P[[X]]$, $h(X) \notin Q$. For each $i, j \in \omega_0$, $g_i \in V_P$ and $h_j \in P$, implying that $g_i h_j \in PV_P = P$. Hence $g(X)h(X) \in P[[X]] \subseteqq V[[X]]$, implying that $f(X)[g(X)h(X)] \in Q$. Since $f(X) \in Q \subseteqq P[[X]]$, each $f_i \in P$; hence $f(X)g(X) \in P[[X]] \subseteqq V[[X]]$. Since $[f(X)g(X)] \cdot h(X) \in Q$ where $f(X)g(X) \in V[[X]]$, $h(X) \in V[[X]]$, and $h(X) \notin Q$, we conclude that $f(X)g(X) \in Q$. Hence Q is an ideal of $V_P[[X]]$.

We now prove that Q is a prime ideal of $V_P[[X]]$. Let $S = V[[X]]\backslash Q$; then S is a multiplicative system in $V[[X]]$, hence also in $V_P[[X]]$, and S clearly does not meet the ideal Q of $V_P[[X]]$. Hence there is a prime ideal Q^* of $V_P[[X]]$ which satisfies $Q \subseteqq Q^*$, $Q^* \cap S = \varnothing$. Since $Q \subseteqq Q^*$, $Q \subseteqq Q^* \cap V[[X]]$; since $Q^* \cap S = \varnothing$, $Q^* \cap V[[X]] \subseteqq Q$. Thus $Q^* \cap V[[X]] = Q$. Observe now that $Q^* \supseteqq Q \supseteqq P \cdot V[[X]] = (PV_P) \cdot V_P[[X]]$. By Theorem 4.3, Q^* compares with $(PV_P)[[X]] = P[[X]]$. Since Q^* lies over Q we must have that $Q^* \subset P[[X]] \subseteqq V[[X]]$, implying that $Q^* = Q$. Hence Q is a prime ideal of $V_P[[X]]$.

That $(PV_P) \cdot V_P[[X]] \subseteqq Q \subset (PV_P)[[X]]$ is clear.

THEOREM 4.9. *The following conditions are equivalent*:

(a) *If V is a rank one nondiscrete valuation ring, then $V[[X]]$ has finite dimension.*

(b) *If V is a valuation ring having finite rank n, then $V[[X]]$ has finite dimension.*

Proof. It is clear that (b) $\rightarrow$ (a). We prove that (a) $\rightarrow$ (b) using induction on n, the case $n = 1$ being a consequence of (a) and Theorem 2.7.

We now assume that if W is a valuation ring of rank k, then $W[[X]]$ has finite dimension. Let V be a valuation ring of rank $k + 1$ which has minimal prime P_1. Let $(0) \subset Q_1 \subset Q_2 \subset \cdots \subset Q_t$ be a chain of prime ideals of $V[[X]]$. Let $d = \dim V_{P_1}[[X]]$. Corollary 2.3 assures that there are at most d proper prime ideals in this chain which contract to (0) in V. Choose m so that $Q_m \cap V = (0)$ and $Q_{m+1} \cap V \neq (0)$; then $m \leqq d$. For $r \geqq m + 1$, $Q_r \cap V \supseteqq P_1$; Theorem 4.3 assures that for $r \geqq m + 1$, Q_r compares with $P_1[[X]]$. Lemma 4.8 assures that at most d of the ideals $Q_{m+1}, Q_{m+2}, \cdots, Q_t$ are contained in $P_1[[X]]$, whereby $Q_{m+d+1} \supset P_1[[X]]$. Since $m \leqq d$, we have that $Q_{2d+1} \supseteqq Q_{m+d+1} \supset P_1[[X]]$.

By the induction hypothesis, $(V/P_1)[[X]]$ has finite dimension. The depth of $P_1[[X]]$ is at most $(\dim (V/P_1)[[X]] - 1)$. It follows that the depth of Q_{2d+1} is at most $(\dim (V/P_1)[[X]] - 1)$, whereby

$$t \leqq (2d + 1) + (\dim (V/P_1)[[X]] - 1) = 2d + \dim (V/P_1)[[X]] .$$

We conclude that $\dim V[[X]] \leqq 2d + \dim (V/P_1)[[X]]$, whereby $V[[X]]$ has finite dimension.

THEOREM 4.10. *The following conditions are equivalent*:

(a) *If V is a rank one nondiscrete valuation ring, then the ascending chain condition for prime ideals holds in $V[[X]]$.*

(b) *If V is a valuation ring having finite rank n, then the ascending chain condition for prime ideals holds in $V[[X]]$.*

The proof of Theorem 4.10 is analogous to the proof of Theorem 4.9 and will therefore be omitted.

Added in proof. Jimmy T. Arnold has recently conveyed to me a paper of his, *On Krull Dimension in Power Series Rings*, in which he has established the following result.

Let R be a commutative ring with identity. If there exists a prime ideal P of R such that $\sqrt{(P \cdot R[[X]])} \neq P[[X]]$, then $R[[X]]$ has infinite dimension.

It follows immediately that if V is a valuation ring which is not discrete, then $V[[X]]$ has infinite dimension.

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FLORIDA STATE UNIVERSITY