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THE DERIVED SET OF THE SPECTRUM OF A DISTRIBUTION FUNCTION

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Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real sequences with $b_n > 0$, $b_n \to 0\,(n \to \infty)$. Let $\{P_n(x)\}_{n=0}^{\infty}$ be the sequence of orthonormal polynomials satisfying the recurrence

$$xP_n(x) = b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x)\,, \quad (n \geqq 0)\,,$$
$$P_{-1}(x) = 0\,, \quad P_0(x) = 1\,.$$

Then there is a substantially unique distribution function ψ with respect to which the $P_n(x)$ are orthogonal. This paper verifies a conjecture of D. P. Maki that the set of all limit points of the sequence $\{a_n\}$ is the derived set of the spectrum of ψ.

Let $\{P_n(x)\}$ be a sequence of orthonormal polynomials defined by the recurrence formula

$$\begin{aligned} xP_n(x) &= b_{n-1}P_{n-1}(x) + a_nP_n(x) + b_nP_{n+1}(x) \quad (n \geqq 0)\,, \\ P_{-1}(x) &= 0,\ P_0(x) = 1,\ a_n \text{ real},\ b_n > 0\,. \end{aligned} \tag{1.1}$$

Then it is well known that there is a bounded, nondecreasing function ψ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)d\psi(x) = \delta_{mn}\,,$$

the spectrum of ψ,

$$S(\psi) = \{t \mid \psi(t+\varepsilon) - \psi(t-\varepsilon) > 0 \text{ for all } \varepsilon > 0\}\,,$$

being an infinite set.

If we impose the additional hypothesis

$$\lim_{n\to\infty} b_n = 0\,, \tag{1.2}$$

then the Hamburger moment problem associated with (1.1) is determined (by Carleman's criterion — see [4, p. 59]) and the distribution function ψ is substantially unique (is uniquely determined up to an arbitrary additive constant at all points of continuity).

In [3], D. P. Maki proved that every (finite) sequential limit point of $\{a_n\}_{n=0}^{\infty}$ is a point in $S(\psi)$ and he conjectured that λ is a limit point of $\{a_n\}$ if and only if λ is a point of the derived set, $S(\psi)'$. Maki's conjecture is correct as will be proved below.

If we denote the smallest and largest limit points (in the extended real number system) of $S(\psi)$ by σ and τ, respectively, then it was proved in [2, Th. 7] that under the hypothesis (1.2),

$$\sigma = \liminf_{n\to\infty} a_n, \quad \tau = \limsup_{n\to\infty} a_n . \tag{1.3}$$

It follows that Maki's conjecture will remain valid if we allow infinite limit points also.

2. In the sequel, we will have reference to the J-fraction,

$$\frac{1 \mid}{\mid z - a_0} - \frac{b_0^2 \mid}{\mid z - a_1} - \frac{b_1^2 \mid}{\mid z - a_2} - \frac{b_2^2 \mid}{\mid z - a_2} - \cdots . \tag{2.1}$$

With the hypothesis (1.2), we are dealing with the determinate case so (2.1) converges uniformly on every closed half-plane,

$$\operatorname{Im}(z) \geqq \delta > 0 ,$$

to an analytic function F which is not a rational function. ψ can be obtained from F by the Stieltjes inversion formula (see [5, p. 250]) and this shows that if the analytic continuation of F is regular in a region containing a real open interval (a, b), then ψ is constant on (a, b).

We will denote the n^{th} convergent of (2.1) by $A_n(z)/B_n(z)$ so that $B_n(z)$ is the monic orthogonal polynomial, $B_n(z) = b_0 b_1 \cdots b_{n-1} P_n(z)$.

We also recall that if the Hamburger moment problem is determined, then (see [4, Corollary 2.6])

$$\rho(x) \equiv \left\{ \sum_{n=0}^{\infty} P_n^2(x) \right\}^{-1} \tag{2.2}$$

vanishes at all points of continuity of ψ and equals the jump of ψ at a point of discontinuity.

3. We now state our main result.

THEOREM. *Let* $\lim_{n\to\infty} b_n = 0$. *Then* λ *is a limit point of the sequence* $\{a_n\}_{n=0}^{\infty}$ *if and only if* λ *is a limit point of* $S(\psi)$.

Proof. In view of (1.3), it is sufficient to consider λ finite.

First let λ be a finite limit point of $\{a_n\}$. Then Maki has shown that $\lambda \in S(\psi)$ and also that there is a subsequence $\{P_{n_k}\}$ such that

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} (x - \lambda)^2 P_{n_k}^2(x) d\psi(x) = 0 . \tag{3.1}$$

We will now show that λ cannot be an isolated point of $S(\psi)$.

Assume λ is isolated. Then ψ has a jump at λ so let $J_\lambda > 0$ denote this jump. It follows that there is an $\varepsilon > 0$ such that $S(\psi)$ contains no points in either of the half-open intervals, $[\lambda - \varepsilon, \lambda)$ and $(\lambda, \lambda + \varepsilon]$. Thus, writing $f_k = P_{n_k}$, we have by a modification of a technique used by Maki,

$$\int_{-\infty}^{\infty} f_k^2 d\psi = \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi + f_k^2(\lambda) J_\lambda = 1 ,$$

$$\begin{aligned} \int_{-\infty}^{\infty} (x-\lambda)^2 f_k^2 d\psi &= \int_{-\infty}^{\lambda-\varepsilon} (x-\lambda)^2 f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} (x-\lambda)^2 f_k^2 d\psi \\ &\geqq \varepsilon^2 \left\{ \int_{-\infty}^{\lambda-\varepsilon} f_k^2 d\psi + \int_{\lambda+\varepsilon}^{\infty} f_k^2 d\psi \right\} \\ &= \varepsilon^2 [1 - f_k^2(\lambda) J_\lambda] \geqq 0 . \end{aligned}$$

Therefore, according to (3.1),

$$\lim_{k\to\infty} P_{n_k}^2(\lambda) = J_\lambda^{-1} > 0$$

but this contradicts the fact that $\rho(\lambda) = J_\lambda$ (see (2.2)). Thus $\lambda \in S(\psi)'$.

Conversely, let $\lambda \in S(\psi)'$ and assume that λ is not a limit point of $\{a_n\}$. Then there is a $\delta > 0$ and an index N_1 such that $|a_n - \lambda| \geqq 2\delta$ for $n \geqq N_1$, hence

$$|z - a_n| \geqq \delta \text{ for } |z - \lambda| \leqq \delta , \qquad n \geqq N_1 .$$

Since by hypothesis, $b_n \to 0$, there is an index N such that

$$\left| \frac{b_n^2}{(z-a_n)(z-a_{n+1})} \right| \leqq \frac{b_n^2}{\delta^2} < \frac{1}{4} \qquad \text{for } n \geqq N$$

and $z \in D = \{w \mid |w - \lambda| \leqq \delta\}$.

It now follows from a Theorem of Worpitzky (see [5, Th. 10.1]) that the J-fraction

$$\frac{b_N^2 \mid}{\mid z - a_{N+1}} - \frac{b_{N+1}^2 \mid}{\mid z - a_{N+2}} - \frac{b_{N+2}^2 \mid}{\mid z - a_{N+3}} - \cdots$$

converges uniformly on D to an analytic function F_N and, from (2.1), we have

$$F(z) = \frac{A_N(z) - A_{N-1}(z) F_N(z)}{B_N(z) - B_{N-1}(z) F_N(z)}$$

for $z \in D$, z not a zero of $B_N - B_{N-1}F_N$.

Since F_N cannot be a rational function, $B_N - B_{N-1}F_N$ can have at most finitely many zeros in D. That is, F has at most finitely many singularities in D, which in turn implies that ψ has at most finitely many points of increase in $[\lambda - \delta, \lambda + \delta]$. Thus we again reach a contradiction.

4. REMARKS. 1. W. R. Allaway has shown (private communication) that when $\{a_n\}$ is bounded, then a theorem of Krein [1, pp. 230–231] can be used to prove Maki's conjecture in the case $S(\psi)'$ is finite.

2. Maki's proof shows that (3.1) holds if for some $\{n_k\}$,

$$b_{n_k-1} \longrightarrow 0,\ b_{n_k} \longrightarrow 0,\ a_{n_k} \longrightarrow \lambda \text{ (finite) } (n \longrightarrow \infty) \tag{4.1}$$

so that (4.1) is sufficient for $\lambda \in S(\psi)'$ (assuming a determined moment problem).

3. Maki showed that if A denotes the self-adjoint operator, $Af = xf$, defined on a dense subset of $L^2(\psi)$, then $\sigma(A) \subset S(\psi)$, where $\sigma(A)$ denotes the spectrum of A.

Consideration of the characteristic function for the singleton set $\{\lambda\}$ shows that λ is an eigenvalue of A if and only if λ is a point of discontinuity of ψ. That is, $P\sigma(A) = D(\psi)$, $C\sigma(A) \subset S(\psi)' \backslash D(\psi)$, where $P\sigma(A)$, $C\sigma(A)$ denote the point and continuous spectra of A and $D(\psi)$ is the set of jump points of ψ.

On the other hand, if $\lambda \in S(\psi)' \backslash D(\psi)$ (λ finite), then there is a measurable function f_n with support in $[\lambda - 1/n, \lambda + 1/n]$ and with $\| f_n \| = 1$. Then

$$\| (A - \lambda) f_n \|^2 \leqq n^{-2} \int_{\lambda-1/n}^{\lambda+1/n} f_n^2 \, d\psi = n^{-2}$$

so that $\lambda \in \sigma(A)$. It follows that $C\sigma(A) = S(\psi)' \backslash D(\psi)$.

REFERENCES

1. N. I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments.* Transl. Math. Monographs, Vol. 2, Amer. Math. Soc., 1962.

2. T. S. Chihara, *Chain sequences and orthogonal polynomials,* Trans. Amer. Math. Soc. **104** (1962), 1-16.

3. D. P. Maki, *A note on recursively defined orthogonal polynomials,* Pacific J. Math. **28** (1969), 611-613.

4. J. Shohat and J. Tamarkin, *The Problem of Moments,* Math. Surveys No. 1, Amer. Math. Soc., 1943, 1950.

5. H. S. Wall, *Analytic Theory of Continued Fractions,* van Nostrand, New York, 1948.

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