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PRINCIPAL IDEALS IN F-ALGEBRAS

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Suppose B is a commutative Banach algebra with unit. Gleason has proved that if I is a finitely generated maximal ideal in B, then there is an open neighborhood U of I in the spectrum of B such that U is homeomorphic in a natural way to an analytic variety and the Gelfand transforms of elements of B are analytic on this variety. In this paper it is shown that this result remains valid for principal ideals in uniform F-algebras with locally compact spectra. From this it follows that if A is an F-algebra of complex valued continuous functions on its spectrum satisfying (1) the spectrum of A is locally compact and has no isolated points, and (2) every closed maximal ideal in A is principal, then the spectrum of A can be given the structure of a Riemann surface in such a way that A can be identified with a closed subalgebra of the algebra of all functions which are analytic on the spectrum of A. Finally an example is given which shows that neither Gleason's result nor the characterization described in the preceding sentence extends to nonuniform algebras.

2. Lemmas and theorems. We assume throughout this paper that A is a uniform commutative F-algebra with unit, and that the spectrum X of A is locally compact and has no isolated points. We identify A and $A^\wedge$ and regard A as a compact open closed subalgebra of $C(X)$ the algebra of all continuous functions on X. Since A is an F-algebra, the space X is hemi-compact. We fix an ascending sequence $\{X_n\}$ of compact A-convex subsets of X such that each compact subset of X is contained in some X_n. (A subset E of X is said to be A-convex if for each x in $X - E$ there is an element f of A such that $|f(x)| > \sup\{|f(y)|: y \in E\}$.) Denote by B_n the completion of the algebra $A \mid X_n$ with respect to the supremum norm on X_n. Then $A = \lim \operatorname{inv} B_n$, $X = \cup X_n$, and $\operatorname{Spec} B_n = X_n$ for each n. Define π_n to be the natural homomorphism of A into B_n. In the case under consideration π_n is the restriction map defined by $f \to f \mid X_n$ for each f in A. Finally for a subset E of X and f in $C(X)$ we let $|f|_E = \sup\{|f(y)|: y \in E\}$.

In [3] Michael defines a strong topological divisor of zero to be an element f of A such that the map $g \to gf$ $(g \in A)$ is not a homeomorphism of A into A.

LEMMA 2.1. *Suppose x is a point of X and f is an element of A such that* $\ker(x) = Af$. *Then f is not a strong topological divisor* *of zero.*

Proof. Suppose $hf = 0$ for some $h \in A$. Since $\text{hull}(f) = \{x\}$, h must be identically zero on $X - \{x\}$. Since $\{x\}$ is not isolated and h is continuous, h must be identically zero on X. Therefore, the map $g \to gf$, $g \in A$ is one-to-one.

Since Af is closed in A, it is an F-algebra. Hence, the map $g \to gf$ is a one-to-one continuous linear map of the F-algebra A onto the F-algebra Af. The open mapping theorem (see [1], p. 55) implies that the map $g \to gf$ is a homeomorphism of A onto Af. Therefore, f is not a strong topological divisor of zero.

For a Banach algebra B we use ∂B to denote the Šilov boundary of B. The next lemma is a generalization of Corollary 3.3.7 in [4].

LEMMA 2.2. *Suppose $f \in A$ and* $\text{hull}(f)$ *is compact. Then f is not a strong topological divisor of zero in A if, and only if, there is an integer n such that for $j \geqq n$,* $\text{hull}(f) \cap \partial B_j = \varnothing$.

Proof. Suppose that for every positive integer n there is a positive integer j such that $j \geqq n$ and $\text{hull}(f) \cap \partial B_j \neq \varnothing$. Then, without loss of generality we can assume that $\text{hull}(f) \cap \partial B_j \neq \varnothing$ for $j = 1, 2, \cdots$. For each integer j and open neighborhood U of $\text{hull}(f)$ we can choose an element $g(j, U)$ of A such that $|g(j, U)|_{U \cap X_j} = 1$ and $|g(j, U)|_{X_j - U} < j^{-1}$.

Order the pairs (j, U) consisting of a positive integer j and an open neighborhood U of $\text{hull}(f)$ by $(j, U) \geqq (i, V)$ if, and only if, $j \geqq i$ and U is contained in V. With this ordering $\{g(j, U)\}$ is a net in A.

Fix a positive integer k and an $\varepsilon > 0$. Choose an open neighborhood V of $\text{hull}(f)$ such that $|f|_V < \varepsilon$. Then for $j \geqq \max(k, \varepsilon^{-1}|f|_{X_k})$ and U contained in V we have, $|g(j, U)f|_{X_k} < \varepsilon$. Therefore $\lim_{(j,U)} |g(j, U)f|_{X_k} = 0$. Since k was arbitrary we have that $g(j, U)f$ converges to zero in A.

Since $\text{hull}(f)$ is compact and X is locally compact there is an open neighborhood W of $\text{hull}(f)$ which is pre-compact. Since W is pre-compact there is an integer n such that W is contained in X_j for $j \geqq n$. Then for $(j, U) \geqq (1, W)$ we have $1 \leqq |g(j, U)|_W \leqq |g(j, U)|_{X_n}$. Hence, $\{g(j, U)\}$ does not converge to zero in A. Therefore f is a strong topological divisor of zero in A.

Suppose there is an integer n such that for $j \geqq n$ we have $\text{hull}(f) \cap \partial B_j = \varnothing$. Assume $\{g_i\}$ is a sequence in A such that $\lim_i (g_i f) = 0$. Fix a positive integer k greater than n. Let $\delta =$

$\min\{|f(x)|: x \in \partial B_k\}$. Since hull $(f) \cap \partial B_k = \varnothing$ we have that δ is greater than zero. Now $\delta|g_i|_{X_k} = \delta|g_i|_{\partial B_k} \leqq |g_i f|_{X_k}$. Combining this estimate on $|g_i|_{X_k}$ with $\lim_i |g_i f|_{X_k} = 0$ we have that $\lim_i |g_i|_{X_k} = 0$. Since the only restriction on k was that k be greater than n we have that $\{g_i\}$ converges to zero in A. Therefore f is not a strong topological divisor of zero.

LEMMA 2.3. *If $x \in X$ and $f \in A$ such that* ker $(x) = Af$, *then there is an integer n such that x is not an element of ∂B_j for $j \geqq n$.*

Proof. Lemma 2.1 implies that f is not a strong topological divisor of zero in A. Since $Af =$ ker (x) we have that hull $(f) = \{x\}$. Lemma 2.2 implies there is an integer n such that $x \notin \partial B_j$ for $j \geqq n$.

Recall that π_n is the natural projection of A into B_n.

LEMMA 2.4. *If $x \in X$ and $f \in A$ such that* ker $(x) = Af$, *then there is an integer n such that $B_j\pi_j(f)$ is a maximal ideal in B_j for $j \geqq n$.*

Proof. Lemma 2.3 implies that there is an integer n such that x is not contained in ∂B_j for $j \geqq n$. We may assume, without loss of generality, that x is contained in X_j for $j \geqq n$.

Fix $j \geqq n$. Since hull $(f) \cap \partial B_j = \varnothing$, we have that $B_j\pi_j(f)$ is closed in B_j. Since Af is a closed maximal ideal in A we have that $A = Af + \mathbf{C}$. Hence $\pi_j(A)\pi_j(f) + \mathbf{C}$ is dense in B_j. Therefore $B_j\pi_j(f) + \mathbf{C}$ is dense in B_j. Since $B_j\pi_j(f)$ is closed in B_j we have that $B_j = B_j\pi_j(f) + \mathbf{C}$. This proves that $B_j\pi_j(f)$ is a maximal ideal in B_j.

The following lemma appeared in [5]. Consequently we will merely sketch a proof.

LEMMA 2.5. *If x is a point of X and x is isolated in each X_n which contains it, then x is isolated in X.*

Proof. Suppose x is isolated in each X_n which contains it. We use Šilov's idempotent theorem on the Banach algebras B_n and the fact that the only idempotent in the radical of a Banach algebra is zero to obtain an idempotent e in A such that $e(x) = 1$ and $e = 0$ elsewhere on X.

THEOREM 2.6. *Suppose that A is a commutative uniform F-algebra with unit and that the spectrum X of A is locally compact.*

If x is a nonisolated point of X and $\ker(x) = Af$ *for some $f \in A$, then there is an open subset U of X such that $x \in U$, f maps U homeomorphically onto an open disc Δ in $\boldsymbol{C}$, and gf^{-1} is analytic on Δ for each $g \in A$.*

Proof. Lemma 2.4 allows us to choose an integer n_1, such that $B_j\pi_j(f)$ is a maximal ideal in B_j for $j \geqq n_1$. Since x is not isolated we can use Lemma 2.5 to obtain an integer n_2 such that x is not isolated in X_j for $j \geqq n_2$. Since X is locally compact x has a precompact open neighborhood W. There is an integer n_3 such that W is contained in X_j for $j \geqq n_3$.

Fix an integer k such that $k \geqq \max(n_1, n_2, n_3)$. Then $B_k\pi_k(f)$ is a nonisolated maximal ideal in B_k. Gleason proved in [2] that there is an open neighborhood V_1 of x in X_k and a disc Δ' about the origin in $\boldsymbol{C}$ such that $\pi_k(f)$ maps V_1 homeomorphically onto Δ', and $g\pi_k(f)^{-1}$ is analytic on Δ' for each $g \in B_k$. Set $V_2 = V_1 \cap W$ where W is the open set defined in the previous paragraph. Let Δ be an open disc centered at the origin of $\boldsymbol{C}$ such that Δ is contained in $f(V_2)$. Set $U = f^{-1}(\Delta) \cap V_2$. Then U is an open neighborhood of x in X, f maps U homeomorphically onto Δ, and gf^{-1} is analytic on Δ for each $g \in A$.

COROLLARY 2.7. *Suppose that A is a commutative uniform F-algebra with unit, and that the spectrum X of A is locally compact and has no isolated points. If every closed maximal ideal in A is principal, then X can be given the structure of a Riemann surface in such a way that A is topologically isomorphic to a closed subalgebra of* $\mathrm{Hol}(X)$.

Proof. It follows from Theorem 2.6 that for each point x of X we can choose $f_x \in A$ and an open set U_x containing x such that f_x maps U_x topologically onto the open unit disc in $\boldsymbol{C}$ and gf_x^{-1} is analytic on the open unit disc for each $g \in A$. Let x and y be elements of X such that $U_x \cap U_y \neq \emptyset$. Then $f_x \circ f_y^{-1}$ is an analytic map of $f_y(U_x \cap U_y)$ onto $f_x(U_x \cap U_y)$. Therefore the set $\{f_x : x \in X\}$ is a set of local coordinates for X with respect to which $A \subset \mathrm{Hol}(X)$.

EXAMPLE. We give an example of a nonuniform F-algebra A such that (1) every maximal ideal in A is principal, and (2) A has no analytic structure. Thus the restriction to uniform algebras in Theorem 2.6 and Corollary 2.7 is essential. We note that the example shows that Gleason's result does not extend to nonuniform F-algebras.

Let $A = C^\infty(R)$ with seminorms $\{\|\cdot\|_n\}$ defined by $\|f\|_n = \sum_{i=0}^n (1/i!) \max\{|f^{(i)}(x)| : x \in [-n, n]\}$. With respect to these seminorms A is an F-algebra. We list below some of the properties of A.

(1) A is (singly) generated by the function f defined by $f(x) = x$ for each $x \in R$.

(2) A is semisimple.

(3) Spec $A = R$; hence, Spec A is locally compact and connected, but Spec A contains no discs.

(4) Every closed maximal ideal in A is principal.

Properties 1, 2, and 3 are clear. There are several ways to prove property 4. The most elementary is to show that for any g in A such that $g(0) = 0$ the function h defined by

$$h(x) = \begin{cases} x^{-1}g(x), & \text{if} \quad x \neq 0 \\ g^{(1)}(0), & \text{if} \quad x = 0 \end{cases}$$

is in A. This can be accomplished by an induction argument using l'Hospital's rule and the mean value theorem. Once we have shown h is in A we have $g = hf$. This shows that the maximal ideal consisting of all functions which vanish at zero is Af. Clearly, a similar argument establishes that all closed maximal ideals are principal.

3. Remarks. We have proved that Gleason's result extends to principal ideals in uniform F-algebras with locally compact spectra. We have also produced an example to show that it does not extend even to principal ideals in nonuniform F-algebras. The obvious question is: does Gleason's result remain valid for finitely generated ideals in uniform commutative F-algebras with unit? At the present time we do not know the answer to this question.

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