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ALGEBRAS WITH MINIMAL LEFT IDEALS WHICH ARE HILBERT SPACES

BRUCE A. BARNES

This paper gives a necessary and sufficient condition that certain topological algebras A (normed algebras and algebras which are inner product spaces) be left (right) annihilator algebras. It is assumed that the socle of A is dense in A and that a proper involution * is defined on the socle. Then the necessary and sufficient condition is essentially that the minimal left (right) ideals of A be complete in the norm on A and be a Hilbert space in an equivalent norm.

We prove a useful preliminary result in § 2. In § 3 we deal with the question of when a normed algebra A is a left or right annihilator algebra. In § 4 we consider this same question when A is a topological algebra in a topology defined by an inner product. This section is motivated by the work of P. Saworotnow and B. Yood on such algebras (see, for example, [5] and [9]). In the final section we generalize the well known result of Bonsall and Goldie that B^* annihilator algebras are dual.

Notation and terminology. A is always a complex algebra. S_A denotes the socle of A, when this exists. If E is a subset of A, let $L(E)$ and $R(E)$ denote the left and right annihilator of E respectively ($L(E) = \{u \in A \mid uv = 0 \text{ for all } v \in E\}$). A is a left (right) annihilator algebra if for every proper closed right (left) ideal M of A $L(M) \neq 0 (R(M) \neq 0)$ and $L(A) = 0$ $(R(A) = 0)$. A left (right) ideal M of A is a left (right) annihilator ideal if $M = L(E)$ $(M = R(E))$ for some subset E of A. If A is semi-simple, A is a modular annihilator algebra if A/S_A is a radical algebra; see [8]. Annihilator and dual algebras are defined and discussed in [4, pp.96-107].

An involution * defined on A (or S_A) is proper if $uu^* = 0$ implies $u = 0$. u is a self-adjoint if $u = u^*$. We denote the set of all self-adjoint minimal idempotents of A by H. If * is proper on S_A, then minimal left (right) ideals of A will have the form Ah (hA), $h \in H$, by [4, Lemma 4.10.1, p. 261].

Let $\mathscr{H}$ be a Hilbert space. $\mathscr{B}(\mathscr{H})$ is the algebra of all bounded operators on $\mathscr{H}$, $\mathscr{F}(\mathscr{H})$ is the subalgebra of $\mathscr{B}(\mathscr{H})$ consisting of all operators which have finite dimensional range, and $\mathscr{C}(\mathscr{H})$ is the algebra of compact operators on $\mathscr{H}$. If $T \in \mathscr{B}(\mathscr{H})$, we denote the operator bound of T as $|T|$. Given $u, v \in \mathscr{H}$, we define an operator $(u \mid v)$ on $\mathscr{H}$ by $(u \mid v)(w) = (w, u)v$ for all $w \in \mathscr{H}$. More generally if X is a normed linear space and X' is the normed dual of X, given $x \in X$ and $f \in X'$ we define an operator $(f|x)$ on X by $(f|x)(y) = f(y)x$ for all $y \in X$.

2. Preliminary results. Let $\mathscr{H}$ be a Hilbert space. Assume that B is a subalgebra of $\mathscr{B}(\mathscr{H})$ with $\mathscr{F}(\mathscr{H}) \subset B$. Furthermore, assume that B is a topological linear space with a topology $\mathscr{T}$ such that

(i) The maps $x \rightarrow xy$ and $x \rightarrow yx$ are continuous on B for all $y \in B$;

(ii) $\mathscr{F}(\mathscr{H})$ is dense in B in the topology $\mathscr{T}$;

(iii) If $\{u_n\} \subset \mathscr{H}$ and $u_n \rightarrow 0$ in $\mathscr{H}$, then $(w|u_n) \rightarrow 0$ in $\mathscr{T}$ for any $w \in \mathscr{H}$.

For $\mathscr{K}$ a closed subspace of $\mathscr{H}$, define

$$\mathscr{R}(\mathscr{K}) = \{T \in B \mid T(\mathscr{H}) \subset \mathscr{K}\} .$$

THEOREM 2.1. *Assume B is as given above. Then B is a left annihilator algebra. Also every right annihilator ideal of B is of the form $\mathscr{R}(\mathscr{K})$ for some closed subspace $\mathscr{K}$ of $\mathscr{H}$. If $T \in \overline{TB}$ for all $T \in B$, then every closed right ideal of B is a right annihilator ideal.*

Proof. Assume that N is a closed right ideal of B. Let

$$\mathscr{J} = \{Tu \mid T \in N, u \in \mathscr{H}\} .$$

Assume that $w = T(u) + S(v)$ where $u, v \in \mathscr{H}$ and $T, S \in N$. Assume that $u \neq 0$, and let $\lambda = |u|_2^2$ ($|\cdot|_2$ the norm on $\mathscr{H}$). Then

$$(1/\lambda)S(u|v) \in N$$

and $(T + (1/\lambda)S(u|v))(u) = w$. This proves that $\mathscr{J}$ is a subspace of $\mathscr{H}$. Let $\mathscr{K} = \overline{\mathscr{J}}$. The proof of [4, Lemma 2.8.24, p. 104] implies that $R(L(N)) = \mathscr{R}(\mathscr{K})$. If $v \in \mathscr{K}$, then there exists $\{u_n\} \subset \mathscr{H}$ and $\{T_n\} \subset N$ such that $T_n(u_n) = v_n \rightarrow v$ in $\mathscr{H}$. Then given any $w \in \mathscr{H}$, $(w|v_n) = T_n(w|u_n) \in N$ for all n. By (iii) $(w|v_n) \rightarrow (w|v)$ in the topology $\mathscr{T}$. Thus whenever $v \in \mathscr{K}$ and $w \in \mathscr{H}$, $(w|v) \in N$. Using this result, the proof of [4, Lemma 2.8.26, p. 105] implies that

$$R(L(N)) \cdot B \subset N .$$

Therefore if $L(N) = 0$, $B^2 \subset N$, and it follows that $\mathscr{F}(\mathscr{H}) \subset N$. Then by (ii) $N = B$. This proves that B is a left annihilator algebra.

If $T \in \overline{TB}$ for all $T \in B$, then whenever $T \in R(L(N))$, $T \in \overline{TB} \subset \overline{R(L(N)) \cdot B} \subset N$. Therefore $N = R(L(N))$, so that N is a right annihilator ideal.

A theorem similar to Theorem 2.1 can be proved concerning the left ideals of B. Assume that $\mathscr{F}(\mathscr{H}) \subset B \subset \mathscr{B}(\mathscr{H})$ and that B satisfies (i), (ii), and

(iv) If $\{u_n\} \subset \mathscr{H}$ and $u_n \to 0$ in $\mathscr{H}$, then $(u_n | v) \to 0$ in the topology $\mathscr{T}$ on B for all $v \in \mathscr{H}$

Define $B^* = \{T^* | T \in B\}$. Topologize B^* with the topology

$$\mathscr{T}^* = \{U^* | U \in \mathscr{T}\} .$$

Then $\mathscr{F}(\mathscr{H}) \subset B^* \subset \mathscr{B}(\mathscr{H})$ and B^* satisfies (i) and (ii). But also by (iv) and the fact that $(v|w)^* = (w|v)$ for all $v, w \in \mathscr{H}$, B^* satisfies (iii). Then the conclusions of Theorem 2.1 hold for B^*. Therefore B^* is a left annihilator algebra and every right annihilator ideal is of the form $\{T \in B^* | T(\mathscr{H}) \subset \mathscr{K}\}$ for some closed subspace $\mathscr{K}$ of $\mathscr{H}$. Let N be a proper closed left ideal of B. Then N^* is a proper closed right ideal of B^*. Therefore there exists $T \in B$, $T \neq 0$, such that $T^*N^* = 0$. Then $R(N) \neq 0$. Now assume that N is a left annihilator ideal of B. Then N^* is a right annihilator ideal of B^* which implies that $N^* = \{T \in B^* | T(\mathscr{H}) \subset \mathscr{K}^{\perp}\}$ for $\mathscr{K}$ some closed subspace of $\mathscr{H}$. Then it is not difficult to verify that

$$N = \{T \in B | T(\mathscr{K}) = 0\} .$$

Finally if $T \in \overline{BT}$ for all $T \in B$, then $T \in \overline{TB^*}$ for all $T \in B^*$. This implies that when $T \in \overline{BT}$ for all $T \in B$, then every closed left ideal of B is a left annihilator ideal (by Theorem 2.1 again).

Combining these remarks and Theorem 2.1 we have the following result.

THEOREM 2.2. *Assume that* $\mathscr{F}(\mathscr{H}) \subset B \subset \mathscr{B}(\mathscr{H})$ *and that* B *satisfies* (i)-(iv). *Then* B *is an annihilator algebra. If in addition* $T \in \overline{TB}$ *and* $T \in \overline{BT}$ *for all* $T \in B$, *then* B *is dual.*

3. Normed algebras. We assume throughout this section that A is a semi-simple modular annihilator algebra, that there is a proper involution $*$ defined on S_A, and that A is a normed algebra with norm $\|\cdot\|$. Recall that H denotes the set of self-adjoint minimal idempotents of A. When $h \in H$, we define a functional f_h on S_A by the rule $f_h(u)h = huh$. By the proof of [7, Th. 5.2, p. 358] we have that f_h is a positive hermitian functional on S_A. We introduce an inner product on the minimal left ideal Ah by the usual definition, $(uh, vh) = f_h((vh)^*uh)$, $u, v \in A$. We call this inner product the canonical inner product on Ah and denote the corresponding norm by

$|\cdot|_2$. We define a $*$-representation of S_A on the inner product space Ah by $u \to T_u^h$, $u \in S_A$, where $T_u^h(vh) = uvh$ for all $v \in A$. As shown in the proof of [7, Th. 5.2, p. 358], the operators T_u^h are bounded on Ah. Also by [7, Lemma 7.1, p. 358] T_u^h has finite dimensional range on Ah for all $u \in S_A$. In a similar fashion a cannonical inner product can be introduced on the minimal right ideal hA, and a $*$-representation of S_A can be constructed into $\mathscr{B}(hA)$.

Since S_A is a modular annihilator algebra with proper involution $*$, then by [1, (1.3), p. 6] there is a unique norm $|\cdot|$ on S_A with the property that $|uu^*| = |u|^2$ for all $u \in S_A$. We call $|\cdot|$ the operator norm on S_A.

THEOREM 3.1. *Assume that A is a left (right) annihilator algebra in the norm $\|\cdot\|$. Also assume that there exists $K > 0$ such that $K\|u\| \geqq |u|$ for all $u \in S_A$. Then for any $h \in H$, Ah (hA) is a Hilbert space in the cannonical norm $|\cdot|_2$, and $\|\cdot\|$ and $|\cdot|_2$ are equivalent on $Ah\,(hA)$.*

Proof. We consider only the case where A is a left annihilator algebra. Also it is sufficient to prove the theorem when A is primitive. For in the general case given $h \in H$, Ah is a minimal left ideal of some minimal closed two sided ideal M of A. Then M is primitive and by the proof of [4, Th. 2.8.12, p. 99] M is a left annihilator algebra. Therefore assume that A is primitive. We shall show that S_A is a left annihilator algebra. If N is a proper closed right ideal of S_A, then $\bar{N}$, the closure of N in A, is a proper closed right ideal of A. Then $L(\bar{N}) \neq 0$, and therefore there exists a minimal idempotent $e \in L(\bar{N})$. Then $e \in S_A$ and $eN = 0$. Thus S_A is a left annihilator algebra.

Assume $h \in H$. Note that $|uh|^2 = |(uh)^*uh| = |uh|_2^2|h| = |uh|_2^2$ so that $|\cdot|$ and $|\cdot|_2$ coincide on Ah. By hypothesis $K\|u\| \geqq |u|$ for all $u \in S_A$, and therefore $K\|uh\| \geqq |uh|_2$ for all $u \in A$. We prove that $\|\cdot\|$ and $|\cdot|_2$ are equivalent on Ah. Since A is primitive, the representation $u \to T_u^h$ of S_A on Ah is faithful. Let $\mathscr{F} = \{T_u^h \mid u \in S_A\}$. By the proof of [4, Lemma 2.8.20, p. 101] $(f \mid x) \in \mathscr{F}$ whenever f is a continuous linear functional on Ah with respect to $\|\cdot\|$ and $x \in Ah$. It follows that any such functional f must be continuous on Ah with respect to $|\cdot|_2$. Let V be the normed dual of Ah with respect to $\|\cdot\|$, and let B be the unit ball in Ah with respect to $|\cdot|_2$. For any $f \in V$, $\sup_{x \in B} |f(x)| < +\infty$. Then by the Uniform Boundedness Theorem applied to the set B, $\sup_{x \in B} \sup_{\|f\| \leq 1} |f(x)| \leqq J$ for some finite number J. It follows that $\|x\| \leqq J|x|_2$ for all $x \in Ah$. Therefore $\|\cdot\|$ and $|\cdot|_2$ are equivalent on Ah.

It remains to be shown that Ah is a Hilbert space in the norm

$|\cdot|_2$. Since $K\|u\| \geqq |u|$ for all $u \in S_A$, S_A is a left annihilator algebra with respect to $|\cdot|$. Let $\mathscr{H}$ be the Hilbert space completion on Ah. Given $w \in \mathscr{H}$, we define $f(x) = (x, w)$ for $x \in \mathscr{H}$. Choose $uh \in Ah$ such that $|uh|_2 = 1$. $(f\,|\,uh) \in \mathscr{F}$ by the proof of [4, Lemma 2.8.20, p. 101]. Therefore $(f\,|\,uh)^* \in \mathscr{F}$. For any $x \in Ah$,

$$(x, w) = ((f\,|\,uh)x, uh) = (x, (f\,|\,uh)^*uh). \quad \text{Therefore} \quad w = (f\,|\,uh)^*(uh) \in Ah.$$

Thus $\mathscr{H} = Ah$.

Using the previous result, we give an example of a norm on $\mathscr{F}(\mathscr{H})$ in which $\mathscr{F}(\mathscr{H})$ is not a left annihilator algebra. Let $\mathscr{H}$ be an infinite dimensional Hilbert space and denote the norm on $\mathscr{H}$ by $|\cdot|_2$. Let $\|\cdot\|$ be any norm on $\mathscr{H}$ such that $|x|_2 \leqq \|x\|$ for all $x \in \mathscr{H}$, and $|\cdot|_2$ and $\|\cdot\|$ are inequivalent on $\mathscr{H}$. If f is any discontinuous linear functional on $\mathscr{H}$, then $\|x\| = |x|_2 + |f(x)|$ is an example of such a norm. Every functional on $\mathscr{H}$ continuous with respect to $|\cdot|_2$ is continuous with respect to $\|\cdot\|$. It follows that every operator $T \in \mathscr{F}(\mathscr{H})$ is bounded in the norm

$$\|T\| = \sup_{\|x\| \leqq 1} \|Tx\| .$$

We note that there exists $K > 0$ such that $K\|T\| \geqq |T|$ ($|\cdot|$ the operator norm on $\mathscr{F}(\mathscr{H})$) by [4, Th. 2.4.17, p. 69]. Now fix $u \in \mathscr{H}$ such that $|u|_2 = 1$. Let N be the minimal left ideal of $\mathscr{F}(\mathscr{H})$ defined by $N = \{(u\,|\,v) \mid v \in \mathscr{H}\}$. $v \to (u\,|\,v)$ is an isometry of $\mathscr{H}$ in the norm $|\cdot|_2$ onto N in the operator norm since $|(u\,|\,v)| = |u|_2|v|_2 = |v|_2$. To verify that $\mathscr{F}(\mathscr{H})$ is not a left annihilator algebra in the norm $\|\cdot\|$, it is sufficient to prove that the map $v \to (u\,|\,v)$ is a bicontinuous map from $\mathscr{H}$ in the norm $\|\cdot\|$ onto N in the norm $\|\cdot\|$. For then $\|\cdot\|$ and $|\cdot|$ are inequivalent on N, and therefore Theorem 3.1 gives the result.

$$\|(u\,|v)\| = \sup_{\|x\| \leqq 1} \|(u\,|\,v)(x)\| = \sup_{\|x\| \leqq 1} |(x, u)|\|v\| \leqq \|v\| ,$$

and

$$\|(u\,|\,v)\| \geqq \|(u\,|\,v)(u/\|u\|)\| = (1/\|u\|)\|v\| .$$

This completes the example.

Now we prove a converse of Theorem 3.1.

THEOREM 3.2. *Assume that S_A is dense in A. Assume that for every $h \in H$ Ah (hA) is a Hilbert space in the norm $|\cdot|_2$, and that $|\cdot|_2$ and $\|\cdot\|$ are equivalent on Ah (hA). Then A is a left (right)* *annihilator algebra. If in addition* $u \in \overline{uA}$ $(u \in \overline{Au})$ *for all* $u \in A$, *then every closed right (left) ideal of* A *is a right (left) annihilator ideal.*

Proof. We assume that for every $h \in H$ Ah is a Hilbert space in the norm $|\cdot|_2$, and that $|\cdot|_2$ and $\|\cdot\|$ are equivalent on Ah. First suppose that A is primitive. Given $h \in H$, then $u \rightarrow T_u^h$ is a faithful *-representation of S_A on the Hilbert space Ah. Given any $u, v, w \in A$, $T^h_{(uh)(vh)^*}(wh) = (wh, vh)(uh) = (vh \mid uh)(wh)$. Therefore all the operators of the form $(vh \mid uh)$ are in the image of the representation $w \rightarrow T_w^h$. It follows that $\mathscr{F}(Ah)$ is in the image of this representation. By [4, Th. 2.4.17, p. 69] there exists $K > 0$ such that $K\|u\| \geqq |T_u^h|$ for all $u \in S_A$. Then since S_A is dense in A, there is a unique extension of the representation $u \rightarrow T_u^h$ of S_A to a representation $u \rightarrow T_u$ of A onto a subalgebra B of $\mathscr{B}(Ah)$. Therefore

$$\mathscr{F}(Ah) \subset B \subset \mathscr{B}(Ah) .$$

We consider B normed by $\|\cdot\|$ in the natural way, $\|T_u\| = \|u\|$ for $u \in A$. B clearly has properties (i) and (ii) listed previous to Theorem 2.1. If $|u_n h|_2 \rightarrow 0$, then by hypothesis $\|u_n h\| \rightarrow 0$, and therefore

$$\|(wh \mid u_n h)\| = \|T_{(u_n h)(wh)^*}\| \rightarrow 0$$

for any $w \in A$. This proves that B also satisfies (iii). By Theorem 2.1, B, and hence A, is a left annihilator algebra. If in addition $u \in \overline{uA}$ for all $u \in A$, then again by Theorem 2.1, every closed right ideal of A is a right annihilator ideal. This proves the theorem when A is primitive. In the general case let $\{M_\alpha \mid \alpha \in I\}$ be the set of all minimal closed two sided ideals of A. M_α is primitive for each $\alpha \in I$, and therefore the theorem holds for each M_α. Since A has dense socle, A is the topological sum of the M_α, $\alpha \in I$. Then by the proof of [4, Th. 2.8.29, p. 106], the theorem holds for A.

4. **Algebras which are inner product spaces.** Throughout this section we assume that A is a semi-simple modular annihilator algebra which is an inner product space with inner product $(\cdot, \cdot)$. Also we assume that the maps $x \rightarrow xy$ and $x \rightarrow yx$ are continuous on A for all $y \in A$. An element x has a left (right) adjoint if there exists $w \in A$ such that $(xy, z) = (y, wz)((yx, z) = (x, zw))$ for all $y, z \in A$. If $x \in A$ has a left (right) adjoint, then it is unique. Assume that every element $u \in S_A$ has a left adjoint which we denote by u^*. Suppose that $u^*u = 0$. By [1, (2.2), p. 6] there exists an idempotent $e \in A$ such that $u = ue$. Then $0 = (u^*u, e) = (u, u)$ so that $u = 0$. This verifies that * must be proper on S_A. Similarly if every element in

S_A has a right adjoint, then this adjoint must be proper on S_A. We denote the norm determined on A by the inner product by $\boldsymbol{|}\cdot\boldsymbol{|}_2$.

THEOREM 4.1. *Assume that every element $u \in S_A$ has a left (right) adjoint u^* and that A is a left (right) annihilator algebra. Then for every $h \in H$, $Ah\,(hA)$ is a Hilbert space in the norm $|\cdot|_2$, and $|\cdot|_2$ and $\boldsymbol{|}\cdot\boldsymbol{|}_2$ are equivalent on $Ah\,(hA)$.*

Proof. We prove the "left" part of the theorem only. As in the proof of Theorem 3.1, it is sufficient to prove the theorem when A is primitive. Therefore assume A is primitive. Given $h \in H$,

$$(uh, vh) = ((vh)^*uh, h) = (uh, vh)|h|_2^2$$

for all $u, v \in A$. Therefore $|\cdot|_2$ and $\boldsymbol{|}\cdot\boldsymbol{|}_2$ are equivalent on Ah. $u \to T_u^h$ is a faithful representation of S_A on Ah. Let $\mathscr{F} = \{T_u^h \mid u \in S_A\}$. By the same argument as in the proof of Theorem 3.1, S_A is a left annihilator algebra with respect to $\boldsymbol{|}\cdot\boldsymbol{|}_2$. Then by the proof of [4, Lemma 2.8.20, p. 101] $(f \mid uh) \in \mathscr{F}$ for all $u \in A$ and all functionals f continuous on Ah with respect to $\boldsymbol{|}\cdot\boldsymbol{|}_2$. Then the argument in the last paragraph of the proof of Theorem 3.1 implies that Ah is a Hilbert space in the norm $\boldsymbol{|}\cdot\boldsymbol{|}_2$.

Now we prove a result in the other direction.

THEOREM 4.2. *Assume that every element $u \in S_A$ has a left (right) adjoint u^*. Assume that A has dense socle in the norm $\boldsymbol{|}\cdot\boldsymbol{|}_2$ and that for every $h \in H$, $Ah\,(hA)$ is a Hilbert space in the norm $\boldsymbol{|}\cdot\boldsymbol{|}_2$. Then A is a left (right) annihilator algebra. If in addition $u \in \overline{uA}$ $(u \in \overline{Au})$ for all $u \in A$, then every closed right (left) ideal of A is a right (left) annihilator ideal.*

Proof. We prove the "left" part of the theorem only. It is sufficient to prove that the theorem holds for each minimal closed two sided ideal M of A. For then by the proof of [4, Th. 2.8.29, p. 106] the result follows for A. Therefore assume that M is a minimal closed two sided ideal of A. Choose $h \in H \cap M$. Then $u \to T_u^h$ is a faithful representation of M on the Hilbert space Ah. T_u^h is a bounded operator on Ah since $u \to ux$ is a continuous map on A. Let $B = \{T_u^h \mid u \in M\}$. We norm B by $|T_u^h|_2 = |u|_2$ for $u \in M$, Given uh and vh, then $T^h_{(uh)(vh)^*} \in B$, and

$$T_{(uh)(vh)^*}(wh) = (wh, vh)uh = (vh \mid uh)(wh)$$

for all $wh \in Ah$. Therefore $\mathscr{F}(Ah) \subset B$. B satisfies properties (i) and (ii) given previous to Theorem 2.1 by hypothesis. Also as noted in the proof of Theorem 4.1, $|uh|_2^2 = |uh|_2^2 |h|_2^2$ for all $u \in A$. Therefore if $|u_n h|_2 \rightarrow 0$, then $|u_n h|_2 \rightarrow 0$, so that for any $v \in A$, $|u_n h(vh)^*|_2 \rightarrow 0$. It follows that $|(vh \mid u_n h)|_2 = |T^h_{(u_n h)(vh)^*}|_2 \rightarrow 0$. Therefore B satisfies (iii). Then Theorem 2.1 applies and this completes the proof.

We apply the previous theorems to right-modular complemented algebras as defined by B. Yood [9, p. 261]. Let A be an algebra with an inner product $(\cdot, \cdot)$. A is a right-modular complemented algebra if

(a) the maps $x \rightarrow xy$ and $x \rightarrow yx$ are continuous for all $y \in A$,

(b) any right or left ideal I for which $I^\perp = \{0\}$ is dense in A (where $I^\perp = \{x \in A \mid (x, y) = 0 \text{ for all } y \in I\}$),

(c) the intersection of the closed modular maximal right ideals of A is $\{0\}$, and $M^\perp$ is a right ideal for each closed modular maximal right ideal M.

We prove the following theorem.

THEOREM 4.3. *Assume that A is a modular annihilator algebra and a right-modular complemented algebra. Then A is an annihilator algebra if and only if every minimal left or right ideal of A is a Hilbert space in the norm determined by the inner product.*

Proof. First note that A is semi-simple by property (c). Since A is a modular annihilator algebra, then by [8, Lemma 3.3, p. 38] every modular maximal right ideal M of A is of the form $(1 - e)A$ where e is a minimal idempotent of A. Then by (a) M is closed. Similarly every modular maximal left ideal of A is closed. Also by [9, Th. 2.1, p. 262] $K^\perp$ is a right (left) ideal for all right (left) ideals K of A.

Assume that every minimal left or right ideal of A is a Hilbert space in the norm determined by the inner product. Given K a minimal right ideal of A, then $N = K^\perp$ is a right ideal. Also $N + K$ is dense by (b). Since K is complete, it follows that $N + K = A$. Therefore N is a modular maximal right ideal of A. By the proof of [8, Th. 4.5, p. 44] every element of $N^\perp = K$ has a left adjoint. Since K was an arbitrary minimal right ideal, then every element in S_A has a left adjoint. A similar proof shows that every element of S_A has a right adjoint. A has dense socle by (b). Therefore by Theorem 4.2, A is an annihilator algebra.

Now assume that A is an annihilator algebra. Take K minimal right ideal of A. Then $N = K^\perp$ is a proper closed right ideal of A. Since A is an annihilator algebra, there exists a modular maximal right ideal M such that $N \subset M$. $K + N$ is a dense right ideal of A

by (b). Assume that $x \in K^{\perp\perp}$. Then there exists $\{x_n\} \subset N$ and $\{y_n\} \subset K$ such that $x_n + y_n \to x$. Then

$$|x_n|_2 = |(x_n + y_n - x, x_n/|x_n|_2)| \leqq |x_n + y_n - x|_2 \to 0 .$$

Therefore $y_n \to x$ and since K is closed, $x \in K$. It follows that $K = K^{\perp\perp}$. Now $K^\perp \subset M$, and therefore $M^\perp \subset K$. Since $M^\perp$ is a nonzero right ideal of A, $M^\perp = K$. Then every element in K has a left adjoint by the proof of [8, Th. 4.5, p. 44]. It follows that every element in S_A has a left adjoint, and by a similar proof every element in S_A has a right adjoint. Then Theorem 4.1 implies that every minimal left or right ideal of A is a Hilbert space in the norm determined by the inner product.

5. Algebras dual in the operator norm.

A well known theorem of Bonsall and Goldie states that an annihilator B^*-algebra is dual. This was generalized by B. Yood who proved that any modular annihilator B^*-algebra is dual; see [8, Th. 4.1, p. 42]. In this section we generalize this result still further. We assume throughout that A is a modular annihilator algebra with an involution * and a norm $|\cdot|$ with the property that $|u^*u| = |u|^2$ for all $u \in A$ (such a norm always exists on A when A is a normed algebra and * is proper by [7, Th. 5.2, p. 358]). We call $|\cdot|$ the operator norm on A.

THEOREM 5.1. *Assume that A has the properties given above. Then if every minimal left ideal of A is complete in the operator norm, A is dual.*

We prove three lemmas.

LEMMA 5.2. *If every minimal left ideal of A is complete in the operator norm, then there is an isometric *-representation $u \to T_u$ of A onto a subalgebra B of the compact operators on a Hilbert space $\mathscr{H}$ with the following properties:*

(1) *$\mathscr{H}$ is the Hilbert space direct sum of a set of closed subspaces $\mathscr{H}_\alpha$, $\alpha \in I$ where I is some index set.*

(2) *If $T \in B$, then T is reduced by each $\mathscr{H}_\alpha$, $\alpha \in I$* (i.e.,

$$T(\mathscr{H}_\alpha) \subset \mathscr{H}_\alpha \text{ and } T(\mathscr{H}_\alpha{}^\perp) \subset \mathscr{H}_\alpha{}^\perp$$

all $\alpha \in I$).

(3) *If $T \in \mathscr{F}(\mathscr{H})$ and T is reduced by $\mathscr{H}_\alpha$ for all $\alpha \in I$, then $T \in B$.*

(4) *$B \cap \mathscr{F}(\mathscr{H})$ is dense in B.*

Proof. Let $\{M_\alpha | \alpha \in I\}$ be the set of minimal two sided ideals of

A, I some index set. For each $\alpha \in I$, choose an element $h_\alpha \in H \cap M_\alpha$. Let $\mathscr{H}_\alpha = Ah_\alpha$. Ah_α is an inner product space in the cannonical inner product. Also $|uh_\alpha|^2 = |(uh_\alpha)^*(uh_\alpha)| = |uh_\alpha|_2^2$. Therefore $|\cdot|_2$ coincides with $|\cdot|$ on Ah_α. Therefore $\mathscr{H}_\alpha$ is a Hilbert space. Let $\mathscr{H}$ be the Hilbert space direct sum of the $\mathscr{H}_\alpha$, $\alpha \in I$. For each α we have a $*$-representation $u \rightarrow T_u^{h_\alpha}$ of A on $Ah_\alpha = \mathscr{H}_\alpha$. $|T_u^{h_\alpha}| \leqq |u|$ for all $u \in A$, $\alpha \in I$. Then we define $u \rightarrow T_u$ a representation of A on $\mathscr{H}$ in the usual fashion, $T_u(\sum_{\alpha \in I} v_\alpha h_\alpha) = \sum_{\alpha \in I} T_u^{h_\alpha}(v_\alpha h_\alpha)$. $u \rightarrow T_u$ is a faithful $*$-representation of A onto a subalgebra B of $\mathscr{B}(\mathscr{H})$. By [1, (1.3), p. 6] $|u| = |T_u|$ for all $u \in A$. T_u has finite dimensional range for all $u \in S_A$ by [7, Lemma 5.1, p. 358]. Also the socle of A is dense in A by the proof of [2, Lemma 2.6, p. 287]. It follows that $\mathscr{F}(\mathscr{H}) \cap B$ must be dense in B and that $B \subset \mathscr{C}(\mathscr{H})$. It remains to prove (3). By Theorem 3.2 A is a left annihilator algebra, and by the proof of that theorem $\mathscr{F}(\mathscr{H}_\alpha) \subset \{T_u^{h_\alpha} \mid u \in M_\alpha\}$. Assume that $T \in \mathscr{F}(\mathscr{H})$, $T(\mathscr{H}_\alpha) \subset \mathscr{H}_\alpha$, and $T(\mathscr{H}_\alpha^\perp) \subset \mathscr{H}_\alpha^\perp$ for all $\alpha \in I$. Then $T(\mathscr{H}_\alpha) = 0$ for all but a finite number of $\alpha \in I$, $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then there exists $u_k \in M_{\alpha_k}$, $1 \leqq k \leqq n$, such that $T_{u_k}^{h_{\alpha_k}}(x) = T(x)$ for all $x \in \mathscr{H}_\alpha$. Let $u = u_1 + \cdots + u_n$. Then $T_u(x) = T(x)$ for all $x \in \mathscr{H}$. This proves (3).

LEMMA 5.3. *Let B be as in Lemma 5.2. Then $T \in \overline{TB}$ and $T \in \overline{BT}$ for all $T \in B$.*

Proof. Assume that $T \in B$. Then T^*T is a compact operator on the Hilbert space $\mathscr{H}$. Let $\{\lambda_k\}$ be the sequence of distinct nonzero eigenvalues of T^*T. Let $\{E_k\}$ be the sequence of projections onto the corresponding eigenspaces. For all $\alpha \in I$ denote by F_α the projection onto the subspace $\mathscr{H}_\alpha$. By hypothesis $F_\alpha T^*T = T^*TF_\alpha$ for all $\alpha \in I$. It follows that $F_\alpha E_k = E_k F_\alpha$ for all $\alpha \in I$ and all k. By (3) of Lemma 5.2 $E_k \in B$ for all k. Then $|T - \sum_{k=1}^N TE_k|^2 = |(T - \sum_{k=1}^N TE_k)^*(T - \sum_{k=1}^N TE_k)| = |T^*T - \sum_{k=1}^N \lambda_k E_k|$. Since $T^*T = \sum_{k=1}^{+\infty} \lambda_k E_k$ by the Spectral Theorem for compact operators, then $T(\sum_{k=1}^N E_k) \rightarrow T$ as $N \rightarrow +\infty$. This proves $T \in \overline{TB}$. A similar argument using TT^* in place of T^*T shows that $T \in \overline{BT}$.

LEMMA 5.4. *Assume that $\mathscr{K}$ is a Hilbert space. Then $\mathscr{F}(\mathscr{K})$ is dual in the operator norm.*

Proof. Assume that M is a closed right ideal of $\mathscr{F}(\mathscr{K})$, and let $N = M + L(M)^*$. N is a right ideal of $\mathscr{F}(\mathscr{K})$. Let

$$\mathscr{J} = \{Tu \mid T \in N, u \in \mathscr{K}\} .$$

As in the proof of Theorem 2.1, $\mathscr{J}$ is a subspace of $\mathscr{K}$. If $w \perp \mathscr{J}$, then for every $u \in \mathscr{K}$ and $T \in N$, $(w \mid w)T(u) = (Tu, w)w = 0$. Therefore $(w \mid w)N = 0$. But then $(w \mid w)M = 0$ and $L(M)(w \mid w) = 0$. Therefore $|w|_2^2(w \mid w) = (w \mid w)^2 = 0$ so that $w = 0$. This proves that $\mathscr{J}$ is dense in $\mathscr{K}$. Assume $v, w \in \mathscr{K}$. Choose $\{u_n\} \subset \mathscr{K}$ and $\{T_n\} \subset N$ such that $T_n(u_n) = v_n \to v$. Then $T_n(w \mid u_n) = (w \mid v_n) \to (w \mid v)$ so that $(w \mid v) \in N$. Therefore $\mathscr{F}(\mathscr{K}) = N$. Take $T \in R(L(M))$. $T = T_1 + T_2$ where $T_1 \in M$ and $T_2 \in L(M)^*$. Then $T_2^* T = 0$ and $T_2^* T_1 = 0$. Thus $T_2^* T_2 = 0$ which implies $T_2 = 0$. It follows that $R(L(M)) = M$. If M is a closed left ideal of $\mathscr{F}(\mathscr{K})$, then $L(R(M)) = M$ by taking involutions. Therefore $\mathscr{F}(\mathscr{K})$ is dual.

Now we complete the proof of Theorem 5.1. By Lemma 5.2. it is enough to prove that an algebra B with the properties listed in that lemma is dual. Let F_α be the projection of $\mathscr{H}$ onto $\mathscr{H}_\alpha$ for all $\alpha \in I$. Set $S_\alpha = \{T \in \mathscr{F}(\mathscr{H}) \mid TF_\alpha = F_\alpha T = T\}$. By Lemma 5.2 $S_\alpha \subset B$. Furthermore $\mathscr{F}(\mathscr{H}_\alpha)$ is isometrically isomorphic to S_α. Therefore S_α is dual by Lemma 5.4. Also S_α is a two sided ideal of B for each $\alpha \in I$, and B is the topological sum of the S_α, $\alpha \in I$. By Lemma 5.3 $T \in \overline{TB}$ and $T \in \overline{BT}$ for all $T \in B$. Then it follows from the proof of [4, Th. 2.8.29, p. 106] that B is dual.

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