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AN N-ARC THEOREM FOR PEANO SPACES

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G. T. Whyburn gave an elementary inductive proof of the n-arc theorem for Peano spaces, which had originally been proved by G. Nobeling and K. Menger. In the course of doing this he gave a necessary and sufficient condition for there to be n disjoint arcs joining two disjoint closed sets A and B in a Peano space S. In this paper we split the set A into n disjoint closed subsets $A_1, A_2, \cdots, A_n$ and give a necessary and sufficient condition for there to be n disjoint arcs joining $A_1 \cup A_2 \cup \cdots A_n$ and B in S, exactly one arc meeting each A_i. Our proof uses the inductive technique that Whyburn introduced.

In this paper we present a theorem and a conjecture that arise from [2].

We first recall some definitions from [2]. Let A, B and X be closed subsets of a topological space S. We say that X *broadly separates A and B in S* if $S - X$ is the union of two disjoint open sets (possibly empty) one of which contains $A - X$ and the other of which contains $B - X$. The space S is *n-point strongly connected between A and B* provided no set of less than n points broadly separates A and B in S. An arc ab *joins A and B* if $ab \cap A = \{a\}$ and $ab \cap B = \{b\}$.

The following theorem, in which we have replaced "completeness" by "local compactness," appears in [2]. It is called the *second n-arc theorem* by Menger in [1].

The Second N-Arc Theorem. *Let A and B be disjoint closed subsets of a locally connected, locally compact metric space S. A necessary and sufficient condition that there be n disjoint arcs in S joining A and B is that S be n-point strongly connected between A and B.*

In § 2 we split the closed set A into n disjoint closed subsets A_1, $A_2, \cdots, A_n$. The theorem then gives a necessary and sufficient condition for there to be n disjoint arcs joining A and B, one meeting each A_i.

In § 3 we split A and B into disjoint closed subsets $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$. The conjecture then gives a necessary and sufficient condition for there to be n disjoint arcs joining A and B, one meeting each A_i and one meeting each B_i. (I have given a proof of this conjecture for the case $n = 4$, which is the first case that offers difficulties, but it is not included here.)

It will be noticed that the space S in the theorem and in the conjecture is not actually a Peano space, as the title of the article states, but it becomes one when the property of connectedness is placed on it.

2. The theorem. Let $A_1, A_2, \cdots, A_n$ and B be disjoint closed subsets of a topological space S. We shall say that a subset X of S is a *large point of* S (with respect to $A_1, A_2, \cdots, A_n$) if it is a one-point set or one of the sets A_i. We shall say that S is *n-point strongly connected between* $A_1, A_2, \cdots, A_n$ *and* B provided the union of less than n large points does not broadly separate $A_1 \cup A_2 \cup \cdots \cup A_n$ and B in S.

We shall say that a system of n disjoint arcs in S *joins*

$$A_1, A_2, \cdots, A_n$$

and B if each arc joins $A_1 \cup A_2 \cup \cdots \cup A_n$ and B and each A_i is joined to B by exactly one of the arcs.

THEOREM. *Let $A_1, A_2, \cdots, A_n$ and B be disjoint closed subsets of a locally connected, locally compact metric space S. A necessary and sufficient condition that there be n disjoint arcs in S joining*

$$A_1, A_2, \cdots, A_n$$

to B is that S be n-point strongly connected between $A_1, A_2, \cdots, A_n$ and B.

We need two more definitions for the proof of the theorem. Let $A_1, A_2, \cdots, A_n$ be disjoint closed sets in a topological space S, and let $\beta_1, \beta_2, \cdots, \beta_m$ be disjoint arcs in S. We shall say that A_i is *a zero*, *a single* or *a multiple* with respect to $\beta_1, \beta_2, \cdots, \beta_m$ according as to whether it meets zero, one or more than one of the arcs $\beta_1, \beta_2, \cdots, \beta_m$. A subarc β of some β_i is said to be *a bridge of $\beta_1, \beta_2, \cdots, \beta_m$ spanning $A_1, A_2, \cdots, A_n$* if β joins some A_j to some A_k, for $j \neq k$. Clearly there are only a finite number of bridges in $\beta_1, \beta_2, \cdots, \beta_m$ spanning $A_1, A_2, \cdots, A_n$.

Proof. Using the terminology and notation of the theorem, it is clear that the condition is necessary for the existence of n disjoint arcs joining $A_1, A_2, \cdots, A_n$ to B in S. So we turn to proving that it is sufficient.

By the arcwise connectivity theorem, the condition is sufficient for $n = 1$. So we assume its sufficiency for each positive integer $< n$ and prove its sufficiency for n by induction.

By the second n-arc theorem there are n disjoint arcs $\beta_1, \beta_2, \cdots, \beta_n$ in S joining $A_1 \cup A_2 \cup \cdots \cup A_n$ and B. Let p be the number of singles of $A_1, A_2, \cdots, A_n$ with respect to $\beta_1, \beta_2, \cdots, \beta_n$. We shall suppose that $p < n$ and show how to construct a second system of n disjoint arcs joining $A_1 \cup A_2 \cup \cdots \cup A_n$ and B with respect to which the number of singles is $p + 1$. The process can be repeated $n - p$ times to obtain the desired system of arcs joining $A_1, A_2, \cdots, A_n$ and B.

Let $A_1, A_2, \cdots, A_p$ be the singles, $A_{p+1}, A_{p+2}, \cdots, A_q$ the zeros and $A_{q+1}, A_{q+2}, \cdots, A_n$ the multiples of $A_1, A_2, \cdots, A_n$ with respect to $\beta_1, \beta_2, \cdots, \beta_n$. Since $p < n$ there is at least one zero and at least one multiple here. We shall construct a system of n disjoint arcs joining $A_1 \cup A_2 \cup \cdots \cup A_n$ and B with respect to which $A_1, A_2, \cdots, A_{p+1}$ are singles. To this end we consider the locally connected, locally compact space $S - A_{p+2} \cup A_{p+3} \cup \cdots \cup A_n$. Since it is $(p + 1)$-point strongly connected between $A_1, A_2, \cdots, A_{p+1}$ and B and $p + 1 \leqq q < n$, it follows from the inductive hypothesis that it contains $p + 1$ disjoint arcs $\alpha_1, \alpha_2, \cdots, \alpha_{p+1}$ joining $A_1, A_2, \cdots, A_{p+1}$ and B. We suppose, further, that α_r meets A_r for $r \leqq p + 1$.

We now use an inductive technique that is familiar from [2]. We relabel $\beta_1, \beta_2, \cdots, \beta_n$ so that β_r meets A_r for $r \leqq p$, and we start by defining $\alpha_r = \alpha_r^0 \cap A_r$ for $r \leqq p + 1$ and $\beta_r^0 = \beta_r$ for $r \leqq p$. Now we suppose that we have defined systems of arcs $\alpha_1^m, \alpha_2^m, \cdots, \alpha_{p+1}^m$ (possibly degenerate) and $\beta_1^m, \beta_2^m, \cdots, \beta_p^m$ such that (a) $\alpha_r \cap A_r \subset \alpha_r^m \subset \alpha_r$ and α_r^m does not meet $B \cup \beta_{p+1} \cup \beta_{p+2} \cup \cdots \cup \beta_n$, (b) $\beta_s \cap B \subset \beta_s^m \subset \beta_s$, (c) if A_r, β_s^m meet then α_r^m is degenerate, (d) if α_r^m, β_s^m meet then they meet in a common end point, (e) exactly one of the sets

$$\alpha_1^m \cup A_1, \alpha_2^m \cup A_2, \cdots, \alpha_{p+1}^m \cup A_{p+1}$$

fails to meet $\beta_1^m \cup \beta_2^m \cup \cdots \cup \beta_p^m$, (f) if b_m is the number of bridges of $\beta_1^m, \beta_2^m, \cdots, \beta_p^m$ that span

$$\alpha_1 \cup A_1, \alpha_2 \cup A_2, \cdots, \alpha_{p+1} \cup A_{p+1},$$

then $b_m < b_{m-1}$ for $m \geqq 1$. We now show how the induction may be continued to the next stage and how it leads, after at most a finite number of stages, to the construction of n disjoint arcs joining

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

to B with respect to which $A_1, A_2, \cdots, A_{p+1}$ are singles.

We proceed by denoting by $\alpha_t^m \cup A_t$ the set, given in (e), which does not meet $\beta_1^m \cup \beta_2^m \cup \cdots \cup \beta_p^m$. We let x be the first point of α_t in the direction $\alpha_t \cap A_t, \alpha_t \cap B$ that belongs to the union of the three sets $\beta_1^m \cup \beta_2^m \cup \cdots \cup \beta_p^m$, $\beta_{p+1} \cup \beta_{p+2} \cup \cdots \cup \beta_n$ and

$$B - \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n .$$

We consider separately the three mutually exclusive cases (1)

$$x \in \beta_1^m \cup \beta_2^m \cup \cdots \cup \beta_p^m ,$$

(2) $x \in \beta_{p+1} \cup \beta_{p+2} \cup \cdots \cup \beta_n$ and (3) $x \in B - \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$.

We first consider case (1) and let $x \in \beta_u^m$. We define $\alpha_r^{m+1} = \alpha_r^m$ for $r \neq t$, $r \leqq p + 1$, and α_t^{m+1} as the subarc of α_t whose endpoints are $a_t \cap A_t$, x. We define $\beta_s^{m+1} = \beta_s^m$ for $s \neq u$, $s \leqq p$, and β_u^{m+1} as the subarc of β_u^m whose endpoints are $\beta_u \cap B$, x. It is easily seen that (a)—(d) of the inductive hypotheses are preserved. In order to verify that (e) is preserved, we notice that it follows from (a)—(d) that each β_s^m meets at most one $\alpha_r^m \cup A_r$. Thus it follows from (e) that the relation $(\alpha_r^m \cup A_r) \cap \beta_s^m \neq \emptyset$ establishes a one-to-one correspondence between the collections $\beta_1^m, \beta_2^m, \cdots, \beta_p^m$ and

$$\alpha_1^m \cup A_1, \alpha_2^m \cup A_2, \cdots, \alpha_{t-1}^m \cup A_{t-1}, \alpha_{t+1}^m \cup A_{t+1}, \cdots, \alpha_{p+1}^m \cup A_{p+1} .$$

If we now let $\alpha_v^m \cup A_v$ be the set that correspond to β_u^m under this relation, it is clear that by (d) $\alpha_v^{m+1} \cup A_v$ does not meet

$$\beta_1^{m+1} \cup \beta_2^{m+1} \cup \cdots \cup \beta_p^{m+1} ,$$

and that it is the only set among $\alpha_1^{m+1} \cup A_1, \alpha_2^{m+1} \cup A_2, \cdots, \alpha_{p+1}^{m+1} \cup A_{p+1}$ with this property. It is clear that (f) is also preserved, since

$$(\beta_u^m - \beta_u^{m+1}) \cup \{x\}$$

is an arc that joins $\alpha_v^m \cup A_v$ and $\alpha_t^m \cup A_t$, and so it contains at least one bridge of $\beta_1^m, \beta_2^m, \cdots, \beta_p^m$ spanning $\alpha_1 \cup A_1, \alpha_2 \cup A_2, \cdots, \alpha_{p+1} \cup A_{p+1}$ that is not contained in $\beta_1^{m+1} \cup \beta_2^{m+1} \cup \cdots \cup \beta_p^{m+1}$; i.e., $b_{m+1} < b_m$.

Thus in case (1) the inductive hypotheses are preserved. We notice that it follows from (f) that case (1) can occur for only a finite number of values of m, since b_0 is finite. Thus case (2) or case (3) must eventually occur. We complete the proof of the theorem by showing that in either of these cases we can readily obtain a system of n disjoint arcs joining $A_1 \cup A_2 \cup \cdots \cup A_n$ and B with respect to which $A_1, A_2, \cdots, A_{p+1}$ are singles.

We shall only deal with case (2), as case (3) is practically identical to it. Thus we let $x \in \beta_w$, $p + 1 \leqq w \leqq n$. We define α as the subarc of α_t whose endpoints are $a_t \cap A_t$, x and β as the subarc of β_w whose endpoints are $\beta_w \cap B$, x. We first notice that it follows from (a)—(d) that if $\alpha_r^m \cup A_r$, β_s^m meet, then $\alpha_r^m \cup \beta_s^m$ is an arc joining A_r, B. Since a one-to-one correspondence is established between the collections

$$\alpha_1^m \cup A_1, \alpha_2^m \cup A_2, \cdots, \alpha_{t-1}^m \cup A_{t-1}, \alpha_{t+1}^m \cup A_{t+1}, \cdots, \alpha_{p+1}^m \cup A_{p+1}$$

and $\beta_1^m, \beta_2^m, \cdots, \beta_p^m$ by the relation $(\alpha_r^m \cup A_r) \cap \beta_s^m \neq \emptyset$ it follows that the union of

$$\alpha_1^m, \alpha_2^m, \cdots, \alpha_{t-1}^m, \alpha_{t+1}^m, \cdots, \alpha_{p+1}^m, \beta_1^m, \beta_2^m, \cdots, \beta_p^m$$

may be expressed as a union of p disjoint arcs joining

$$A_1, A_2, \cdots, A_{t-1}, A_{t+1}, \cdots, A_{p+1}$$

and B. Furthermore, by (a), (b) these arcs are disjoint from the arcs $\beta_{p+1}, \beta_{p+2}, \cdots, \beta_{w-1}, \beta_{w+1}, \cdots, \beta_n, \alpha, \beta$. Thus the union of

$$\alpha_1^m, \alpha_2^m, \cdots, \alpha_{t-1}^m, \alpha_{t+1}^m, \cdots \alpha_{p+1}^m, \beta_1^m, \beta_2^m, \cdots, \beta_p^m,$$
$$\beta_{p+1}, \beta_{p+2}, \cdots, \beta_{w-1}, \beta_{w+1}, \cdots, \beta_n\ ,$$

α, β may be expressed as a union of n disjoint arcs joining

$$A_1 \cup A_2 \cup \cdots \cup A_n$$

and B with respect to which $A_1, A_2, \cdots, A_{p+1}$ are singles. This completes the proof of the theorem.

3. The conjecture. Let $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots B_n$ be disjoint closed subsets of a topological space S. We shall say that a subset X of S is a *large point of* S (with respect to $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$) if it is a one-point set, a set A_i, or a set B_i. We shall say that S is *n-point strongly connected between* $A_1, A_2, \cdots, A_n$ *and* $B_1, B_2, \cdots, B_n$ provided the union of less than n large points does not broadly separate $A_1 \cup A_2 \cup \cdots \cup A_n$ and $B_1 \cup B_2 \cup \cdots \cup B_n$ in S.

We shall say that a system of n disjoint arcs in S *joins*

$$A_1, A_2, \cdots, A_n \quad \textit{and} \quad B_1, B_2, \cdots, B_n$$

if each arc joins $A_1 \cup A_2 \cup \cdots \cup A_n$ and $B_1 \cup B_2 \cup \cdots \cup B_n$, and each A_i meets just one arc, and each B_i meets just one arc.

Conjecture. *Let $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ be disjoint closed subsets of a locally connected, locally compact metric space S. A necessary and sufficient condition that there be n disjoint arcs in S joining $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ is that S be n-point strongly connected between $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$.*

The necessity of the condition is again trivial, so it is the sufficiency of the condition that is interesting.

The conjecture is clearly true if the sets

$$A_1, A_2, \cdots, A_n \quad \text{and} \quad B_1, B_2, \cdots, B_n$$

are compact. For in this case the quotient space Q obtained by identifying a pair of points if they belong to a common A_i or a common B_j is locally compact, locally connected and metrizable. If π is the natural projection from S onto Q, it is clear that Q is n-point strongly connected between

$$\pi(A_1) \cup \pi(A_2) \cup \cdots \cup \pi(A_n) \quad \text{and} \quad \pi(B_1) \cup \pi(B_2) \cup \cdots \cup \pi(B_n) .$$

Consequently it follows from the second n-arc theorem that there are n disjoint arcs in Q joining

$$\pi(A_1) \cup \pi(A_2) \cup \cdots \cup \pi(A_n) \quad \text{and} \quad \pi(B_1) \cup \pi(B_2) \cup \cdots \cup \pi(B_n) .$$

The π-inverse of each of these arcs contains a connected closed set which meets both $A_1 \cup A_2 \cup \cdots \cup A_n$ and $B_1 \cup B_2 \cup \cdots \cup B_n$, from which it easily follows that there are n-disjoint arcs in S joining $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$.

When some of the sets $A_1, A_2, \cdots, A_n$ or $B_1, B_2, \cdots, B_n$ fail to be compact, the above argument does not suffice as the quotient space Q is not in general metrizable.

There ought to be a combinatorial proof of this conjecture along the lines of the proof in § 2, which would work equally well whether some of the sets $A_1, A_2, \cdots, A_n$ or $B_1, B_2, \cdots, B_n$ fail to be compact or not. Such a proof has been given for the case $n = 4$, as was remarked in paragraph § 1.

REFERENCES

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