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INTERPOLATION IN $C(\Omega)$

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It is known from the work of Bade and Curtis that if $\mathfrak{A}$ is a Banach subalgebra of $C(\Omega)$, Ω a compact Hausdorff space, and if Ω is an F-space in the sense of Gillman and Hendriksen then $\mathfrak{A} = C(\Omega)$. This paper is concerned with the extension of this and similar results to the setting of Grothendieck spaces (G-spaces for short). An important feature of the extension is that emphasis is shifted from the underlying topological structure of Ω to the linear topological character of $C(\Omega)$.

As a corollary we show that if Ω_1 and Ω_2 are infinite compact Hausdorff spaces, then $\Omega_1 \times \Omega_2$ is not a G-space. Consequently if Ω is a G-space then $C(\Omega)$ is not linearly isomorphic to $C(\Omega \times \Omega)$.

If A is a commutative Banach algebra whose spectrum is a totally disconnected G-space, a second corollary of our extension is that the Gelfand homomorphism is onto. This establishes for G-spaces a result due to Seever for N-spaces.

Two definitions of G-space are to be found in the literature.

(A) A Banach space X is a G-space if every weak-* convergent sequence in X^*, the dual of X, is weakly convergent.

(B) A compact Hausdorff space Ω is a G-space if $C(\Omega)$ is a G-space in the sense of (A).

Unless otherwise noted we shall accept (B) as our definition.

It is known from the work of Seever [7] that if Ω is an F-space, i.e., if disjoint open F_σ subsets of Ω have disjoint closures, then Ω is a G-space. A result due to Rudin [3] states that if Ω_1 and Ω_2 are infinite compact Hausdorff spaces then $\Omega_1 \times \Omega_2$ is not an F-space. Corollary 2.6 is an extension of this to G-spaces. Although an example of a G-space which is not an F-space is given in [7], no necessary and sufficient topological characterization of the G property is known.

1. Preliminaries. Let $M(\Omega)$ be the space of regular Borel measures on Ω equipped with the total variation norm. A sequence $\{\mu_n\}$ in $M(\Omega)$ converges for the weak-* topology if for each f in $C(\Omega)$, the space of continuous complex valued functions on Ω, the sequence $\{\mu_n(f)\}$ is convergent. Weak convergence of $\{\mu_n\}$ means convergence of $\{\gamma(\mu_n)\}$ for every γ in $M^*(\Omega)$, the dual of $M(\Omega)$. If Ω is any set $l_1(\Omega)$ will denote the Banach space of point mass measures on Ω with the total variation norm.

A Banach subalgebra (subspace) $\mathfrak{A}$ of $C(\Omega)$ is a subalgebra (subspace) of $C(\Omega)$ under the pointwise operations and is a Banach algebra (space) such that the embedding $\mathfrak{A} \to C(\Omega)$ is continuous. $\mathfrak{A}$ is said to be normal if for each pair F_1, F_2 of disjoint compact subsets of Ω there is an $f \in \mathfrak{A}$ such that $f = 1$ on F_1 and $f = 0$ on F_2. Following [2] we call $\mathfrak{A}$ ε-normal if for each pair F_1, F_2 of disjoint compact subsets of Ω there exists an $f \in \mathfrak{A}$ satisfying

(i) $|f(\omega) - 1| < \varepsilon$, $\omega \in F_1$,
(ii) $|f(\omega)| < \varepsilon$, $\omega \in F_2$.

If Ω_1 and Ω_2 are compact Hausdorff spaces the projective tensor product $V = C(\Omega_1) \widehat{\oplus} C(\Omega_2)$ is the set of all functions of the form

$$\sum_{i=1}^{\infty} f_1(x) g_i(y),\ f_i(x) \in C(\Omega_1)$$

and $g_i(y) \in C(\Omega_2)$ such that $\sum_{i=1}^{\infty} \| f_i \|_\infty \| g_i \|_\infty < \infty$. If $h \in V$ then

$$\| h \|_V = \inf\left\{\sum_{i=1}^{\infty} \| f_i \|_\infty \| g_i \|_\infty : h = \sum_{i=1}^{\infty} f_i g_i\right\} .$$

Two Banach spaces X_1 and X_2 are isomorphic if there is a one-to-one continuous linear map from X_1 onto X_2. If X_2 is a closed subspace of X_1, it is said to be complemented in X_1 if there exists a closed subspace Y of X_1 such that $X_2 + Y = X_1$ and $X_2 \cap Y = \{0\}$. We write $X_1 = X_2 \oplus Y$.

If D is a discrete space, $C(D)$ will denote the bounded continuous functions on D. It is well known that $C(D)$ is isometrically isomorphic to $C(\beta D)$ where βD is the Stone-Čech compactification of D. A compact Hausdorff space is totally disconnected if there is a basis for the topology consisting of open and closed neighborhoods.

2. We shall need to recall here a criterion due to Grothendieck [5] for relative weak compactness in $M(\Omega)$. Namely, a bounded sequence $\{\mu_n\}$ in $M(\Omega)$ is relatively weakly compact if and only if for every sequence $\{0_i\}$ of pairwise disjoint Borel sets $\lim_{i\to\infty} \mu_n(0_i) = 0$ uniformly in n. By the Eberlein Smulian theorem this is equivalent to every subsequence of $\{\mu_n\}$ having a weakly convergent subsequence.

LEMMA 2.1. *If Ω is a G-space and K is a closed subspace of Ω, then K is a G-space.*

Proof. Suppose $\{\mu_n\}$ in $M(K)$ is weak-* convergent. One may regard $\{\mu_n\}$ as a weak-* convergent sequence in $M(\Omega)$. It is therefore weakly convergent as a sequence in $M(\Omega)$, and so by the Hahn-Banach Theorem it is a weakly convergent sequence in $M(K)$.

LEMMA 2.2. *Let Ω be a G-space and X a dense Banach subspace such that $X \neq C(\Omega)$. Then for every $M > 0$ there is a measure μ with no atomic part such that* $\|\mu\| \geqq M$ *and* $\sup\{|\mu(f)| : f \in X, \|f\|_X \leqq 1\} \leqq 1$.

Proof. We shall write μ_a for the atomic part of μ and μ_c for the continuous part. By a well known theorem of Banach there is a sequence $\{\mu_n\}$ of measures such that $\|\mu_n\| \geqq n$ and sup

$$\{|\mu_n(f)| : f \in X, \ \|f\|_X \leqq 1\}$$

for each n. Since X is dense in $C(\Omega)$ setting $\nu_n = \mu_n/\|\mu_n\|$ we have $\lim_n \nu_n = 0$ weak-* and hence $\lim_n \nu_n = 0$ weakly since Ω is a G-space. The natural projection $p: M(\Omega) \to l_1(\Omega)$ given by $p\mu = \mu_a$ is continuous and hence weakly continuous. Hence $\lim_n \nu_{n,a} = 0$ weakly. Since in $l_1(\Omega)$ weakly convergent sequences are norm convergent, it follows that $\lim_n \|\nu_{n,a}\| = 0$. Thus for an appropriate sequence of scalars $\{c_n\}$ we have $\lim_n \|c_n \nu_{n,c}\| = \infty$ and

$$\sup\{|c_n \nu_{n,c}(f)| : f \in X, \ \|f\|_X \leqq 1\} \leqq 1$$

for every n.

THEOREM 2.3. *Let Ω be a G-space and let X be a dense Banach subspace of $C(\Omega)$. Then there exists a finite open covering $U_1, \cdots, U_n$ of Ω such that $X|\bar{U}_i = C(\bar{U}_i)$, $1 \leqq i \leqq n$.*

Proof. From the compactness of Ω it suffices to show that each point p of Ω has a neighborhood U_p such that $X|\bar{U}_p = C(\bar{U}_p)$. Suppose this fails for some p, and choose U_1 a neighborhood of p. Let X_1 denote the quotient space of X by all functions in X vanishing on $\bar{U}_1$. Applying Lemmas 2.1 and 2.2 it follows that there is a regular Borel measure μ_1 with no atomic part such that $\|\mu_1\| \geqq 1$, supp $\mu_1 \subseteqq \bar{U}_1$ and such that $|\mu_1(f)| \leqq \|f\|_{X_1} \leqq \|f\|_X$ for every $f \in X$.

From the regularity of μ_1 we may choose open $U_2 \subseteqq U_1$, $p \in U_2$ such that $|\mu_1|(\bar{U}_1 - \bar{U}_2) > 1/2 \ \|\mu_1\|$. Since $X|\bar{U}_2 \neq C(\bar{U}_2)$ we may choose in the same way a μ_2 with no atomic part such that supp $\mu_2 \subseteqq \bar{U}_2$, $\|\mu_2\| \geqq 2$ and $|\mu_2(f)| \leqq \|f\|_X$ for all $f \in X$.

Continuing in this fashion, define inductively a sequence of measures $\{\mu_n\}$ with no atomic parts such that $\|\mu_n\| \geqq n$, $|\mu_n(f)| \leqq \|f\|_X$ for every $f \in X$, supp $\mu_n \subseteqq \bar{U}_n$ and $|\mu_n|(\bar{U}_n - \bar{U}_{n+1}) > 1/2 \ \|\mu_n\|$.

Setting $\nu_n = \mu_n/\|\mu_n\|$ we see $\lim_n \nu_n = 0$ weak-* from the density of X. However, since $|\nu_n|(\bar{U}_n - \bar{U}_{n+1}) > 1/2$ for each n, $\{\nu_n\}$ is not weakly convergent by the Grothendieck criterion. This contradiction establishes the theorem.

REMARK. Theorem 2.3 is the sharpest result in the sense that for every compact Hausdorff space Ω there is a dense Banach subspace X of $C(\Omega)$ such that $X \neq C(\Omega)$. By a result of [8] (corollary 3.2 page 201) there are closed subspaces Y, W of $C(\Omega)$ such that $Y + W$ is dense in $C(\Omega)$ but $Y + W \neq C(\Omega)$; in the terminology of that paper every $C(\Omega)$ contains a quasi-complemented uncomplemented subspace. Setting $X = Y \oplus W$ we have the result.

Our next theorem is an extension to G-spaces of a result of [2]. The work is all done by the following:

LEMMA 2.4. [2] *Let Ω be a compact Hausdorff space, and let $\mathfrak{A}$ be a Banach subalgebra of $C(\Omega)$ such that*

(i) *$\mathfrak{A}$ is ε-normal for some $\varepsilon < 1/2$,*

(ii) *There is an open covering $U_1, \cdots, U_n$ of Ω such that $\mathfrak{A} | \bar{U}_i = C(\bar{U}_i)$, $1 \leqq i \leqq n$.*

Then $\mathfrak{A} = C(\Omega)$.

Combining this with Theorem 2.3 and the remark that density implies ε-normality we obtain:

THEOREM 2.5. *Let Ω be a G-space, and let $\mathfrak{A}$ be a dense Banach subalgebra of $C(\Omega)$. Then $\mathfrak{A} = C(\Omega)$.*

REMARK. As demonstrated in [2] ε-normality for some $\varepsilon < 1/4$ and density of a Banach subspace of $C(\Omega)$ are equivalent in case Ω is an F-space. We do not know if "dense" may be replaced by "ε-normal" in Theorem 2.5.

COROLLARY 2.6. *If Ω_1 and Ω_2 are infinite compact Hausdorff spaces then $\Omega_1 \times \Omega_2$ is not a G-space.*

Proof. We need only take $\mathfrak{A} = C(\Omega_1) \hat{\otimes} C(\Omega_2)$ *and note that* $\mathfrak{A}$ is a dense Banach subalgebra of $C(\Omega_1 \times \Omega_2)$. ($\mathfrak{A}$ *happens to be normal as well.) But it is well known that* $\mathfrak{A} \neq C(\Omega_1 \times \Omega_2)$.

Let X_1 and X_2 be Banach spaces such that X_2 is a continuous linear image of X_1. It is an easy consequence of the Hahn Banach theorem that if X_1 is a G-space in the sense of definition A, then so is X_2. Consequently if Ω is a G-space then $C(\Omega \times \Omega)$ is not even a continuous linear image of $C(\Omega)$. This is contrasted with a result of Milutin [6, p. 42] which states that if Ω_1 and Ω_2 are uncountable compact metric spaces then $C(\Omega_1)$ is isomorphic to $C(\Omega_2)$. In particular for such Ω, $C(\Omega)$ is isomorphic to $C(\Omega \times \Omega)$.

These notions may be of use in solving complementation problems. Suppose that X_2 is a complemented subspace of X_1. Then if X_1 is a G-space in the sense of definition A, so is X_2. For example, if D

denotes an infinite discrete space, $C(\beta D \times \beta D)$ may be viewed in a natural way as a closed subspace of $C(D \times D)$. Since $\beta(D \times D)$ is a G-space, by the above remarks $C(\beta D \times \beta D)$ has no complement in $C(D \times D)$.

COROLLARY 2.7. [cf. [7] corollary 2 p. 278]. *Let A be a commutative Banach algebra whose spectrum Ω is a totally disconnected G-space. Then the Gelfand homomorphism is onto.*

Proof. By the Šilov idempotent theorem the image of A in $C(\Omega)$ contains the characteristic functions of open closed sets. Hence A is a dense Banach subalgebra of $C(\Omega)$ and the theorem applies.

REMARK. An interesting fact suggested by the proof of Theorem 2.3 is that if Ω is a G-space then no normal subalgebra A of $C(\Omega)$, closed in the uniform norm, is such that $C(\Omega)/A$ has countable (infinite) dimension. To see this suppose to the contrary that $C(\Omega)/A$ has countable dimension. Recall that if A is a normal subalgebra of $C(\Omega)$ such that every point p of Ω has a neighborhood U_p such that $A|\bar{U}_p = C(\bar{U}_p)$ then $A = C(\Omega)$. Thus there is a point $p \varepsilon \Omega$ such that for every neighborhood U_p of p, $A|\bar{U}_p \neq C(\bar{U}_p)$. Since A contains the constant functions, by a result of Glicksberg [4 p. 421] we may choose $\mu_1 \in A^\perp$, $||\mu_1|| = 1$ such that $|\mu_1|$ $(\bar{U}_1^*) > \delta > 0$ where $\bar{U}_1^*$ is a closed deleted neighborhood of p. By regularity of μ_1 we may choose a neighborhood U_2 of p such that $\bar{U}_2 \subsetneqq U_1$ and $|\mu_1|$ $(\bar{U}_2^*) < \delta/2$. Again we may choose $\mu_2 \in A^\perp$, $||\mu_2|| = 1$, such that $|\mu_2|(\bar{U}_2{}^*) > \delta > 0$. Continuing in this fashion we get a sequence of measures $\{\mu_n\} \in A^\perp$, $||\mu_n|| = 1$, and a nested sequence of neighborhoods of p, $\{U_n\}$, $\bar{U}_{n+1} \subsetneqq U_n$ such that $|\mu_n|(\bar{U}_n - \bar{U}_{n+1}) > \delta/2$ for each n. By Grothendieck'e criterion no subsequence of $\{\mu_n\}$ is weakly convergent. Since $C(\Omega)/A$ is separable, the unit ball in $A^\perp$ is weak-* sequentially compact. Thus a subsequence of $\{\mu_n\}$ may be found which is weak-* convergent and hence weakly convergent. This contradiction completes the proof.

In [7] the following theorem is proved.

THEOREM 2.8. *If Ω is an F-space, and if X is a normal Banach subspace of $C(\Omega)$,* then $X = C(\Omega)$.

Question [7]. In Theorem 2.8 can "F" be replaced by "G"? In the terminology of that paper is every G-space an N-space? The following may be of help in giving an answer.

THEOREM 2.9. *Let X be a G-space in the sense of definition A, and let Y be a closed subspace such that X/Y is separable. Then Y*

is a G-space.

Proof. Let $\{y_n^*\}$ denote a sequence in Y^*. It suffices to show that if $\lim_n y_n^* = 0$ weak-* then $\{y_n^*\}$ has a subsequence $\{y_{n_k}^*\}$ such that $\lim_k y_{n_k}^* = 0$ weakly. Let x_n^* be any normpreserving extension of y_n^* to all of X. Since X/Y is separable, a sequence $\{w_n\}$ in X may be found such that $sp\{w_n\} + Y$ is dense in X. By a diagonal argument a subsequence $\{x_{n_k}^*\}$ of $\{x_n^*\}$ may be found such that $\{x_{n_k}^*\}$ converges on each member of $\{w_n\}$ and hence on $sp\{w_n\} + Y$. Since $\{\|x_{n_k}^*\|\}$ is bounded, $\{x_{n_k}^*\}$ is weak* convergent in X and hence weakly convergent. Thus $\lim_k y_{n_k}^* = 0$ weakly.

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