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THE QUOTIENT ALGEBRA OF A FINITE VON NEUMANN ALGEBRA

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We will prove the following: Let M be a finite von Neumann algebra with center Z and A a von Neumann subalgebra of Z. Let Ω be the spectrum space of A and identify A with $C(\Omega)$. Let ε be a σ-weakly continuous linear map of M onto A such that $\varepsilon(x^*x) = \varepsilon(xx^*) \geqq 0$ for every $x \in M$, $\varepsilon(ax) = a\varepsilon(x)$ for every $a \in A$ and $x \in M$, $\varepsilon(1) = 1$ and $\varepsilon(x^*x) \neq 0$ for every nonzero $x \in M$. For each $\omega \in \Omega$, let $\mathfrak{m}_\omega$ denote the set of all $x \in M$ with $\varepsilon(x^*x)(\omega) = 0$. Then $\mathfrak{m}_\omega$ is a closed ideal and the quotient C^*-algebla $M/\mathfrak{m}_\omega$ is a finite von Neumann algebra. Furthermore, if π_ω denote the canonical homomorphism of M onto $M/\mathfrak{m}_\omega$, then $\pi_\omega(N)$ is a von Neumann subalgebra of $M/\mathfrak{m}_\omega$ for every von Neumann subalgebra N containing A.

In [8], [3] and [5] it was shown that the quotient C^*-algebra of a finite von Neumann algebra by any maximal ideal is actually a finite factor. This led us to the algebraic reduction theory for finite von Neumann algebras, which is free from the separability restriction in the direct integral reduction theory. In this paper we will show that the above result still holds for certain ideals, not necessarily maximal. Namely, we will give a straightforward proof for the following.

THEOREM. *Let M be a finite von Neumann algebra with center Z and A a von Neumann subalgebra of Z. Let Ω be spectrum space of A and identify A with $C(\Omega)$. Let ε be a σ-weakly continuous linear map of M onto A such that $\varepsilon(x^*x) = \varepsilon(xx^*) \geqq 0$ for every $x \in M$, $\varepsilon(ax) = a\varepsilon(x)$ for every $a \in A$ and $x \in M$, $\varepsilon(1) = 1$ and $\varepsilon(x^*x) \neq 0$ for every nonzero $x \in M$. For each $\omega \in \Omega$, let $\mathfrak{m}_\omega$ denote the set of all $x \in M$ with $\varepsilon(x^*x)(\omega) = 0$. Then $\mathfrak{m}_\omega$ is a closed ideal and the quotient C^*-algebra $M/\mathfrak{m}_\omega$ is a finite von Neumann algebra. Furthermore, if π_ω denote the canonical homomorphism of M onto $M/\mathfrak{m}_\omega$, then $\pi_\omega(N)$ is a von Neumann subalgebra of $M/\mathfrak{m}_\omega$ for every von Neumann subalgebra N containing A.*

Before going into the proof, we observe that there exists such a map ε if Z is σ-finite. Since M has the $\natural$-operation, it suffices to show that there exists a σ-weakly continuous faithful projection of norm one from Z onto A. If Z is σ-finite, then Z admits a faithful normal state φ. Considering the cyclic representation of Z induced by φ, we may assume that Z acts on a Hilbert space $\mathscr{H}$ containing a vector ξ_0 such that $(x\xi_0 | \xi_0) = \varphi(x)$, $x \in Z$. Let e be the projection of $\mathscr{H}$ onto $[A\xi_0]$. Then e is an abelian projection in A' with central support 1. Note that the center of A' is A itself. Then there exists an isomorphism θ of $eA'e$ onto A such that $\theta(xe) = x$ for every $x \in A$ because A is the center of A'. Put $\varepsilon_Z(x) = \theta(exe)$ for every $x \in Z$. Since e is not orthogonal to any nonzero projection in Z, ε_Z has the required properties. As the composed map of this ε_Z and the $\natural$-operation in M, we get a desired map ε. Hence, the situation in the theorem is always presented for any von Neumann subalgebra A of Z if Z is σ-finite.

The proof of theorem. We will prove the assertion for the subalgebra N which implies immediately the former assertion.

Let $\tau_\omega(x) = \varepsilon(x)(\omega)$, $x \in M$. Then τ_ω is a finite trace of M with the left kernel $\mathfrak{m}_\omega$. Let $\{\pi, \mathscr{H}, \xi_0\}$ be the cyclic representation of M induced by τ_ω. Since π has the kernel $\mathfrak{m}_\omega$, π induces a faithful representation $\tilde{\pi}$ of the C^*-algebra $M/\mathfrak{m}_\omega$. Since $\tilde{\pi} \circ \pi_\omega(N) = \pi(N)$, it suffices to show that $\pi(N)$ is a von Neumann algebra. Since the functional $\tau_\omega(x) = (x\xi_0 | \xi_0)$, $x \in \pi(M)''$, is a faithful trace on the von Neumann algebra $\pi(M)''$, ξ_0 is a cyclic and separating for $\pi(M)''$. Let S_N denote the unit ball of N. Then by Kaplansky's density theorem $\pi(S_N)$ is strongly dense in the unit ball $S_{\tilde{N}}$ of the von Neumann algebra $\tilde{N} = \pi(N)''$ generated by $\pi(N)$. Since the map $x \in \pi(M)'' \to x\xi_0$ is injective, if $\pi(S_N)\xi_0 = S_{\tilde{N}}\xi_0$, then we have $\pi(S_N) = S_{\tilde{N}}$; hence $\tilde{N} = \pi(N)$.

Therefore, we shall prove that $\pi(S_N)\xi_0$ is complete. Let $\{x_n\}$ be a sequence in S_N such that

$$\lim_{n,m\to\infty} \| \pi(x_n)\xi_0 - \pi(x_m)\xi_0 \| = 0 .$$

Considering a subsequence of $\{x_n\}$, we may assume that

$$\| \pi(x_n)\xi_0 - \pi(x_{n+1})\xi \| < 2^{-n} , \qquad n = 1, 2, \cdots .$$

In other words,

$$\varepsilon((x_n - x_{n+1})^* (x_n - x_{n+1}))(\omega) < 4^{-n} , \qquad n = 1, 2, \cdots .$$

Let $\{U_n\}$ be a decreasing sequence of neighborhoods of ω in Ω such that

$$\varepsilon((x_n - x_{n+1})^* (x_n - x_{n+1}))(\sigma) < 4^{-n}$$

for every $\sigma \in U_n$, $n = 1, 2, \cdots$. For each $n = 1, 2, \cdots$, let e_n be the projection of A corresponding to the closure of U_n. Then $e_n(\omega) = 1$

for $n = 1, 2, \cdots$. Putting $y_1 = x_1$ and $y_n = e_n x_n + (1 - e_n) y_{n-1}$ for $n = 2, 3, \cdots$ by induction,

$$\varepsilon((y_n - y_{n+1})^* (y_n - y_{n+1})) < 4^{-n} ;$$

$$\pi(y_n)\xi_0 = \pi(x_n)\xi_0 , \qquad n = 1, 2, \cdots .$$

Now, for any normal state φ of A, put $\tau_\varphi(x) = \varphi \circ \varepsilon(x)$, $x \in N$. Then τ_φ is a normal finite trace of N with the support $s(\varphi) \in A$, where $s(\varphi)$ means the support of φ in A. By the inequality:

$$\tau_\varphi((y_n - y_{n+1})^* (y_n - y_{n+1})) = \varphi \circ \varepsilon((y_n - y_{n+1})^* (y_n - y_{n+1})) < 4^{-n} ,$$

$n = 1, 2, \cdots$, $\{y_n s(\varphi)\}$ converges σ-strongly to $y_\varphi \in S_N$ because the σ-strong topology in $S_N \cap Ns(\varphi)$ is induced by the metric d defined by $d(x, y) = \tau_\varphi((x - y)^* (x - y))^{1/2}$, $x, y \in S_N \cap Ns(\varphi)$. Let $\{\varphi_i\}_{i \in I}$ be a maximal family of normal states of A with orthogonal supports. Then $\sum_{i \in I} s(\varphi_i) = 1$. Let $y = \sum_{i \in I} y_{\varphi_i} \in S_N$. Since $\{y_n s(\varphi_i)\}$ converges σ-strongly to $s(\varphi_i) y = y_{\varphi_i}$ for each $i \in I$, $\{y_n\}$ converges σ-strongly to y. Now we have, by the triangular inequality,

$$\begin{aligned}\varepsilon((y_n - y_{n+p})^* (y_n - y_{n+p}))^{1/2} &\leqq \sum_{k=n}^{n+p-1} \varepsilon((y_k - y_{k+1})^* (y_k - y_{k+1}))^{1/2} \\ &\leqq \sum_{k=n}^{n+p-1} 2^{-k} \leqq 2^{-n+1}\end{aligned}$$

for n, $P = 1, 2, \cdots$. Hence we have

$$\varepsilon((y_n - y)^* (y_n - y))^{1/2} = \lim_{p \to \infty} \varepsilon((y_n - y_{n+p})^* (y_n - y_{n+p}))^{1/2} \leqq 2^{-n+1} ,$$

so that

$$|| \pi(y_n)\xi_0 - \pi(y)\xi_0 || = \varepsilon((y_n - y)^* (y_n - y))(\omega)^{1/2} \leqq 2^{-n+1} ;$$

hence

$$\lim_{n \to \infty} \pi(y_n)\xi_0 = \pi(y)\xi_0 .$$

Therefore, the given Cauchy sequence $\{\pi(x_n)\xi_0\}$ in $\pi(S_N)\xi_0$ converges to $\pi(y)\xi_0 \in \pi(S_N)\xi_0$. Hence $\pi(S_N)\xi_0$ is complete, hence closed in $\mathscr{H}$. This completes the proof.

By [7], we should remind that if M is a von Neumann algebra of type II_1 and if ω is not an isolated point of Ω then $M/\mathfrak{m}_\omega$ does not admit nontrivial representation on a *separable* Hilbert space even if M does have faithful normal representation on a *separable* Hilbert space.

Suppose now A is σ-finite and ω is not an isolated point of Ω.

Suppose that any nonzero projection $e \in N$ majorizes a projection $f \in N$ such that $\varepsilon(f) = \varepsilon(e - f)$. Then we claim that the von Neumann algebra $\pi_\omega(N)$ does not admit a faithful separable normal representation.

Let $\{e_n\}$ be a decreasing sequence of projections in A converging σ-strongly to zero such that $e_n(\omega) = 1$ for $n = 1, 2, \cdots$. Such a sequence does exist by the nonisolatedness of ω and the σ-finiteness of A. Let $f_n = e_n - e_{n+1}$ for $n = 1, 2, \cdots$. By the assumption for N, there exists orthogonal projections $p^n_{1,1}$ and $p^n_{1,2}$ in N such that $f_n = p^n_{1,1} + p^n_{1,2}$ and $\varepsilon(p^n_{1,1}) = \varepsilon(p^n_{1,2}) = \frac{1}{2} f_n$. Suppose we have found projections $\{p^n_{i,j}: i = 1, \cdots, k, j = 1, 2, \cdots, 2^i\}$ such that

(1) for fixed i, $\{p^n_{i,j}: j = 1, \cdots, 2^i\}$ are orthogonal;

(2) $p^n_{i-1,j} = p^n_{i,2j-1} + p^n_{i,2j}$;

(3) $\varepsilon(p^n_{i,j}) = 2^{-i} f_n$.

By the assumption for N, we can find orthogonal projections $\{p^n_{i+1,j}: j = 1, 2, \cdots, 2^{i+1}\}$ such that

$$p^n_{i,j} = p^n_{i+1,2j-1} + p^n_{i+1,2j} ;$$
$$\varepsilon(p^n_{i+1,j}) = 2^{-(i+1)} f_n , \qquad j = 1, 2, \cdots, 2^{i+1} .$$

For each integer i, put

$$u_{n,i} = \sum_{j=1}^{2^n} (-1)^j p^n_{i,,j} .$$

Then we have $u^2_{n,k} = f_n$ and for different i_1 and i_2, $u_{n,i_1} u_{n,i_2}$ is the difference of two orthogonal projections p and q such that $\varepsilon(p) = \varepsilon(q) = \frac{1}{2} f_n$; hence $\varepsilon(u_{n,i_1} u_{n,i_2}) = 0$ if $i_1 \neq i_2$.

To each real number s we associate a sequence $\{i_{s,n}\}$ of integers such that

$$\lim_{n \to \infty} \frac{i_{s,n}}{2^n} = s .$$

If $s \neq t$, there is an n_0 such that $i_{s,n} \neq i_{t,n}$ for every $n \geqq n_0$. Put

$$u_s = \sum_{n=1}^{\infty} u_{n,i_{s,n}} .$$

Then we have $\varepsilon(u_s u_t)(1 - e_{n_0}) = \varepsilon(u_s u_t)$. Therefore we have

$$\tau_\omega(u_s^2) = 1, \tau_\omega(u_s u_t) = 0 \quad \text{if} \quad s \neq t .$$

Therefore $\{\pi(u_s)\xi_0\}$ is a continuum of orthogonal vectors in $[\pi(N)\xi_0]$. Therefore, the standard representation of the von Neumann algebra $\pi_\omega(N)$ is not separable. Thus $\pi_\omega(N)$ does not admit a faithful normal separable representation.

Now, let A and B be two abelian von Neumann algebras with no minimal projections. Let C be the tensor product $A \otimes B$ of A and B. Then A and B are regarded as subalgebras of C. If B admits a faithful normal state ψ, then there exists a faithful normal projection ε of norm one of C onto A defined by

$$\langle \varepsilon(x), \varphi) \rangle = \langle x, \varphi \otimes \psi \rangle$$

for every $\varphi \in A_*$. This map has the property:

$$\varepsilon(a \otimes b) = \varphi(b)a, a \in A, b \in B .$$

If A is σ-finite, then $C/\mathfrak{m}_\omega$ is an abelian von Neumann algebra, with no separable faithful normal representation. It is easily seen that the map π_ω is σ-weakly continuous on B; hence $\pi_\omega(B)$ is a proper von Neumann subalgebra of $C/\mathfrak{m}_\omega$ if B has a faithful separable normal representation. Therefore, *the pathology that the component algebras are much larger than the synthetic algebra does occur even in the abelian case.*

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