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MINIMAL FIRST COUNTABLE HAUSDORFF SPACES

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If $\mathscr{P}$ is a property of topologies, a $\mathscr{P}$-space $(X, \mathscr{T})$ is called a $\mathscr{P}$-minimal space if there exists no $\mathscr{P}$-topology on X properly contained in $\mathscr{T}$. Throughout the following, $\mathscr{H}$ = first countable and Hausdorff and $\mathscr{C}$ = first countable and completely Hausdorff (a space X is called completely Hausdorff if the continuous real valued functions defined on X separate the points of X).

In this paper we give examples of $\mathscr{H}$-minimal $\mathscr{C}$-spaces that are (i) not regular and (ii) regular but neither completely regular nor countably compact.

Two other results obtained are the following. (a) Every locally pseudocompact zero-dimensional $\mathscr{H}$-space can be embedded densely in a pseudocompact zero-dimensional $\mathscr{H}$-space. (b) Let $\mathscr{P} = \mathscr{C}$, completely regular $\mathscr{H}$, or zero-dimensional $\mathscr{H}$, and suppose that X is a $\mathscr{P}$-space such that for every $\mathscr{P}$-space Y and continuous mapping $f: X \to Y$, f is closed. Then X is countably compact.

N will denote the set of natural numbers, and $C(X, Y)$ will denote the family of continuous mappings of X into Y. For definitions, see [4].

1. An embedding theorem and some examples.

Recall that a space $(X, \mathscr{T})$ is said to be *semiregular* if $\{\mathring{\bar{T}} \mid T \in \mathscr{T}\}$ is a base for $\mathscr{T}$. If $(X, \mathscr{T})$ has a property $\mathscr{P}$, then $(X, \mathscr{T})$ is said to be $\mathscr{P}$-*closed* provided that it is a closed subset of every $\mathscr{P}$-space in which it can be embedded.

For many properties $\mathscr{P}$, it is known that $\mathscr{P}$-minimal and $\mathscr{P}$-closed spaces are closely connected. For the case $\mathscr{P} = \mathscr{H}$, the following two results, established in [11], will be used below. An $\mathscr{H}$-space X is $\mathscr{H}$-closed if and only if every countable open filter base on X has nonempty adherence. An $\mathscr{H}$-space is $\mathscr{H}$-minimal if and only if it is semiregular and $\mathscr{H}$-closed.

We shall now describe constructions which can be used to densely embed certain $\mathscr{C}$-spaces in $\mathscr{H}$-minimal ($\mathscr{H}$-closed) $\mathscr{C}$-spaces. As special cases, we shall obtain examples with the properties mentioned in the introduction. First some terminology is needed.

A space X is said to be *locally pseudocompact* (W. W. Comfort) if every point of X has a pseudocompact neighborhood.

A filter base $\mathscr{F}$ is said to be *pseudocompact* if for every $F \in \mathscr{F}$ and $G \in \mathscr{F}$, $F - G$ is pseudocompact. $\mathscr{F}$ is called *zero-dimensional* if the sets belonging to it are open- and-closed.

Notation. (B. Banaschewski). Let $\mathscr{M}$ be a family of open filter bases on a space X. Let $\{p(\mathscr{F}) | \mathscr{F} \in \mathscr{M}\}$ be a new set of distinct points, and let $X(\mathscr{M})$ be the space whose points are the elements of $X \cup \{p(\mathscr{F}) | \mathscr{F} \in \mathscr{M}\}$ and whose topology has as a base sets of the form $V^* = V \cup \{p(\mathscr{F}) | V$ contains some member of $\mathscr{F}\}$, where V is any open subset of X.

THEOREM 1.1. *Let X be an $\mathscr{H}$-space containing a point a such that X-$\{a\}$ is a zero-dimensional locally pseudocompact space. Let $\mathscr{N} = \{\mathscr{F} | \mathscr{F}$ is a free, countable, pseudocompact, zero-dimensional filter base on $X\}$, and denote by $\mathscr{M}$ a maximal subset of $\mathscr{N}$ such that whenever $\mathscr{F}, \mathscr{G} \in \mathscr{M}$ with $\mathscr{F} \neq \mathscr{G}$, then there exist disjoint sets $F \in \mathscr{F}$ and $G \in \mathscr{G}$.*

Then the space $X(\mathscr{M})$ is an $\mathscr{H}$-closed $\mathscr{C}$-space in which X is embedded as a dense subset, and $X(\mathscr{M})$ is $\mathscr{H}$-minimal if and only if X is semiregular.

Proof. $X(\mathscr{M})$ is clearly an $\mathscr{H}$-space. Furthermore, it follows from the hypothesis that each point of $X(\mathscr{M}) - \{a\}$ has a fundamental system of feebly compact open neighborhoods. Thus the characteristic functions of open-and-closed subsets of $X(\mathscr{M})$ separate the points of $X(\mathscr{M})$ and $X(\mathscr{M})$ is a $\mathscr{C}$-space.

Suppose that $\mathscr{F}$ is a countable open filter base on $X(\mathscr{M})$ and no point of X is an adherent point of $\mathscr{F}$. A slight modification of the proof of Lemma 2.17 in [11] shows that there exists a free, countable, pseudocompact, zero-dimensional filter base $\mathscr{G}$ on X which is stronger than the filter base $\mathscr{F} | X$. By the maximality of $\mathscr{M}$, there exists $\mathscr{K} \in \mathscr{M}$ with $G \cap H$ nonempty for all $G \in \mathscr{G}$ and $H \in \mathscr{K}$. Thus $p(\mathscr{K})$ is an adherent point of $\mathscr{F}$.

To check semiregularity, it suffices to observe that if

$$a \in V = \mathrm{Int}_X Cl_X V, \text{ then } V^* = \mathrm{Int}_{X(\mathscr{M})} Cl_{X(\mathscr{M})} V^*.$$

THEOREM 1.2. *Let X and a be as in Theorem* 1.1, *and suppose that $\{V_n | n \in N\}$ is a fundamental system of open neighborhoods for a such that $V_1 = X$ and each $V_n \supset Cl_X V_{n+1}$. Let $\mathscr{M}$ be a maximal family of free, countable, pseudocompact, zero-dimensional filter bases on X such that* (a) *whenever $\mathscr{F}, \mathscr{G} \in \mathscr{M}$ with $\mathscr{F} \neq \mathscr{G}$, then there* *exist disjoint sets* $F \in \mathscr{F}$ *and* $G \in \mathscr{G}$, *and* (b) *for every* $\mathscr{F} \in \mathscr{M}$ *there exists* $n \in N$ *such that* $\cup \mathscr{F} \subset V_n - V_{n+1}$.

Then $X(\mathscr{M})$ *is a regular* $\mathscr{C}$*-space that is* $\mathscr{H}$*-minimal and contains* X *as a dense subspace. If each* V_n *is closed in* X, *then* $X(\mathscr{M})$ *is zero-dimensional.*

Proof. Since $\{p(\mathscr{F})|\mathscr{F} \in \mathscr{M}\} - \{a\}$ is a closed discrete subset of $X(\mathscr{M}) - \{a\}$, it follows from (b) that $Cl_{X(\mathscr{M})} V^*_{n+1} = V^*_{n+1} \cup Cl_X V_{n+1}$. Thus $X(\mathscr{M})$ is regular, and if each V_n is closed in X, then $X(\mathscr{M})$ is zero-dimensional.

The proof that $X(\mathscr{M})$ is feebly compact is similar to the corresponding proof given for Theorem 1.1-one just notes that for some n, $\mathscr{F}|(Cl_X V_n - Cl_X V_{n+1})$ is a filter base, and so $\mathscr{G}$ can be chosen with the property that $\cup \mathscr{G} \subset V_n - V_{n+1}$.

REMARK 1.3. In case the set I of isolated points of X is a dense subset of X, $\mathscr{M}$ can be defined as follows. Let $\mathscr{E}$ be a maximal family of countably infinite subsets of I such that (a) the intersection of any two members of $\mathscr{E}$ is finite, and (b) each member of $\mathscr{E}$ is a closed subset of X (for Theorem 1.2, a closed subset of some $Cl_X(V_n - V_{n+1})$). For each $E \in \mathscr{E}$ let $\mathscr{F}(E)$ be the complements in E of finite subsets of E. Take $\mathscr{M} = \{\mathscr{F}(E)|E \in \mathscr{E}\}$.

REMARK 1.4. For the case $X = N$ and $\mathscr{M}$ infinite, the space $X(\mathscr{M})$ is due to J. Isbell (see [5, 5I]).

REMARK 1.5. In general, the space $X(\mathscr{M})$ is not countably compact and hence not weakly normal, for each $\{p(\mathscr{F})|\mathscr{F} \in \mathscr{M}\} - V^*_n$ is a closed discrete subset of $X(\mathscr{M})$.

COROLLARY 1.6. *Every locally pseudocompact zero-dimensional* $\mathscr{H}$*-space can be embedded densely in a pseudocompact zero-dimensional* $\mathscr{H}$*-space.*

EXAMPLE 1.7. For the following X, the space $X(\mathscr{M})$ is an $\mathscr{H}$-minimal $\mathscr{C}$-space that is not regular.

Let $T = \{0\} \cup \{1/n \in N\}$, with the usual topology, choose a point a not in the product space $N \times T$, and let $X = \{a\} \cup (N \times T)$, topologized as follows: every open subset of $N \times T$ is open in X; a neighborhood of a is any set of the form $V_n = \{a\} \cup \{(x, y) \in X|x \geqq n$ and $1/y$ is an even integer}, $n \in N$. (X is homeomorphic to $E - \{b\}$, where E is as in [13, p. 268].)

One can take $\mathscr{M}$ to be a maximal family of infinite subsets of $X - Cl\,V_1$ such that the following hold:

(i) For all $M, M' \in \mathscr{M}$, $M \neq M'$ implies $M \cap M'$ is finite;

(ii) For all $M \in \mathscr{M}$ and $n \in N$, $M \cap (\{n\} \times T)$ is finite.

EXAMPLE 1.8. For the following X, the space $X(\mathscr{M})$ (of Theorem 1.2) is an $\mathscr{H}$-minimal $\mathscr{C}$-space that is regular but not completely regular.

Let Y be the set of ordinal numbers less than the first uncountable ordinal, with the order topology, let M be the set of limit ordinals in Y, and denote $Y - M$ by I. Let $Z = I \times \{0\} \cup Y \times N$, topologized as follows: $Y \times N$ has the product topology, and $Y \times N$ is open in Z; a neighborhood of a point $(i, 0) \in Z$ is any subset of Z that contains $(i, 0)$ and all but finitely many elements of $\{i\} \times N$. Let L and R denote the product spaces $Z \times \{1\}$ and $Z \times \{2\}$, and set $U = L \cup R$, with the weak topology generated by $\{L, R\}$. Let S be the relation on U defined by the rule: $(x, i, j) S (y, k, n)$ if (a) $x = y$, $i = k$, and $j = n$, or (b) $x = y \in M$ and $i = k$. Denote the quotient space U/S by T. We shall continue to use the symbols (x, i, j) for the points of T.

On the product space $T \times N$ define $(t, n)\, W (t', n')$ if (a) $t = t'$ and $n = n'$, or (b) $t = (x, 0, j)$, $t' = (x, 0, j')$, and $n' - n = j - j' = 1$ or $n - n' = j' - j = 1$. Let V be the quotient space $(T \times N)/W$. Choose a new point a and let $X = V \cup \{a\}$, topologized as follows: every open subset of V is open in X; a neighborhood of a is any set of the form $V_n = \{a\} \cup \{(t, m) \in V | m \geqq n\}$, $n \in N$.

It is not difficult to see that X is a first countable regular space whose isolated points are dense, and $X - \{a\}$ is zero-dimensional and locally compact. X is not completely regular, because for every $f \in C(X)$ there exists $m \in Y$ such that f is constant on

$$\{(x, 0, j, n) | x \geqq m, j = 1 \text{ or } j = 2, \text{ and } n \in N\} .$$

Thus V_2, for example, contains no zero set neighborhood of a.

REMARK 1.9. The construction above is a modification of Tychonoff's regular but not completely regular space [12].

In [7] F. B. Jones has constructed a $\mathscr{C}$-space that is not completely regular but that is a Moore space. His space cannot be used here, however, because it is neither locally pseudocompact nor zero-dimensional.

In the literature there are many less messy examples of $\mathscr{C}$-closed or $\mathscr{H}$-minimal spaces that are not regular; however, the author does not know of any $\mathscr{C}$-minimal space appearing elsewhere that is not regular (or completely regular).

REMARK 1.10. If one glues together (as in [2]) two copies of the space in Example 1.8, then one gets an example of a regular $\mathscr{H}$-minimal space that is not completely Hausdorff.

2. $\mathscr{C}$-minimal spaces and closed mappings. If $\mathscr{P}$ denotes any one of the usual separation properties, it is known that every $\mathscr{P}$-minimal completely Hausdorff space is compact (e.g., see [6]). Moreover C. T. Scarborough [9] has observed that a completely Hausdorff-minimal space is compact.

One might then expect $\mathscr{C}$-minimal spaces to be well behaved, to be, say, at least countably compact. Of course, Isbell's example or Mrówka's [8] (or ours) shows that this is not the case. The following characterization theorems may, therefore, be of interest.

DEFINITION. (H. E. Hayes) An open filter base $\mathscr{F}$ on a space X is said to be *completely Hausdorff* provided that for every $x \in X$, if x is not an adherent point of $\mathscr{F}$, then there exist $f \in C(X)$ and $F \in \mathscr{F}$ such that $f(F) = 0$ and $f(x) = 1$.

Using usual techniques, one can prove the following.

THEOREM 2.1. *Let X be a $\mathscr{C}$-space. The following are equivalent.*
(i) *X is $\mathscr{C}$-closed.*
(ii) *Every countable completely Hausdorff filter base on X has an adherent potnt.*
(iii) *For every $\mathscr{C}$-space Y and $f \in C(X, Y)$, $f(X)$ is $\mathscr{C}$-closed.*

In order to obtain a $\mathscr{C}$-analogue of Theorem 2.4 of [11], we need a second definition.

DEFINITION. An open filter base $\mathscr{F}$ on a space X is said to be *almost completely Hausdorff* if there exists $p \in X$ so that for every $x \in X - \{p\}$, if x is not an adherent point of $\mathscr{F}$, then there exist $f \in C(X)$ and $F \in \mathscr{F}$ such that $f(F) = 0$ and $f(x) = 1$.

THEOREM 2.2. *Let X be a $\mathscr{C}$-space. The following are equivalent.*

(i) *X is $\mathscr{C}$-minimal.*

(ii) *Every countable completely Hausdorff filter base on X that has a unique adherent point is convergent.*

(iii) *X is semiregular, and every countable almost completely Hausdorff filter base on X has an adherent point.*

The proof is somewhat similar to the proofs needed for Theorems 2.4 and 2.9 in [11].

The next result, to be contrasted with (iii) of Theorem 2.1, is a partial converse to the following well-known theorem: If X is a countably compact space, Y is an $\mathscr{H}$-space (or a space of the type E_1 studied in [1]), and $f \in C(X, Y)$, then f is closed.

We shall call an open filter base $\mathscr{F}$ on X *completely regular* if for each $F \in \mathscr{F}$ there exist $G \in \mathscr{F}$ and $f \in C(X, [0, 1])$ such that f vanishes on G and equals 1 on $X - F$.

THEOREM 2.3. *Let $\mathscr{P}$ denote either completely Hausdorff, completely regular, or zero-dimensional, and suppose that X is a $\mathscr{P}$-space which is also an $\mathscr{H}$-space. The following are equivalent.*

(i) *X is countably compact.*

(ii) *For every $\mathscr{H}$-space Y and $f \in C(X, Y)$, f is closed.*

(iii) *For every $\mathscr{P}$-space Y that is an $\mathscr{H}$-space and $f \in C(X, Y)$, f is closed.*

(iv) *For every closed subset C of X and every countable $\mathscr{P}$-filter base $\mathscr{F}$ on X, if $\mathscr{F}|C$ is a filter base and if $\cap \mathscr{F} = \cap\{\bar{F} | F \in \mathscr{F}\}$, then there is a point $c \in C$ which is in $\cap \mathscr{F}$.*

Proof. (i) $\Rightarrow$ (ii) is known. (ii) $\Rightarrow$ (iii) is obvious. A proof not too different from one in [3] shows that (iii) $\Leftrightarrow$ (iv). We shall prove that (iv) $\Rightarrow$ (i) for the case $\mathscr{P}$ = completely Hausdorff.

Let us suppose then that X is a $\mathscr{C}$-space which contains a countably infinite closed discrete subset C.

Consider a point $c \in C$. Since X is completely Hausdorff and $C - \{c\}$ is countable, there exists $f \in C(X)$ for which $f(c) \notin f(C - \{c\})$. Since $C - \{c\}$ is a closed subset of X and f is closed, we can choose $g \in C((-\infty, \infty))$ with $g(f(c)) = 1$ and $g(f(C - \{c\})) = 0$. Set $h_c = g \circ f$.

Let $\mathscr{F}$ be the family of all finite intersections of

$$\{h_c^{-1}(-1/n, 1/n) | n \in N \text{ and } c \in C\} .$$

Then it is easy to see that $\mathscr{F}$ is a countable completely regular (and hence completely Hausdorff) filter base on X, that $\cap \mathscr{F} = \cap\{\bar{F} \mid F \in \mathscr{F}\}$, and that $\mathscr{F} \mid C$ is a filter base. On the other hand, one also has $C \cap \cap \mathscr{F} = \phi$. This contradicts (iv).

REMARK 2.4. There exists an $\mathscr{H}$-space X that is not countably compact but which has the property: for every Hausdorff space Y and $f \in C(X, Y)$, f is closed. See [3] and [14].

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