# OPERATORS THAT COMMUTE WITH A UNILATERAL SHIFT ON AN INVARIANT SUBSPACE

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A co-isometry on a Hilbert space $\mathcal{H}$ is a bounded operator having an isometric adjoint. If $V$ is a co-isometry on $\mathcal{H}$ and $\mathcal{M}$ is an invariant subspace for $V$, then every bounded operator on $\mathcal{M}$ that commutes with $V$ on $\mathcal{M}$ can be extended to an operator on $\mathcal{H}$ that commutes with $V$, and the extension can be made without increasing the norm of the operator. This paper is concerned with unilateral shifts. The questions asked are these: (1) Do shifts enjoy the above property shared by co-isometries and self-adjoint operators? (The answer to this question is "rarely".) (2) Why not? (3) If $S$ is a shift, $\mathcal{M}$ is an invariant subspace for $S$, $S_0$ is the restriction of $S$ to $\mathcal{M}$, and $T$ is a bounded operator on $\mathcal{M}$ satisfying $TS_0 = S_0T$, how tame do $T$ and $\mathcal{M}$ have to be in order that $T$ can be extended (without increasing the norm) to an operator in the commutant of $S$? Extension is possible in a large number of cases.

The result mentioned above for co-isometries is due to Sz.-Nagy and Foias [8]. (An excellent exposition on the problem is found in [3]; see Theorem 4 in particular.) For self-adjoint operators the statement is trivial for the simple reason that every invariant subspace is then reducing and any commuting operator on a subspace can be extended by simply requiring it to be zero on the orthogonal complement of the subspace.

Recall that a unilateral shift $S$ is an isometry having the property that $\bigcap_{n=0}^{\infty} S^n \mathcal{H} = \{0\}$. The Hilbert space dimension of the subspace $(S\mathcal{H})^{\perp}$ is called the multiplicity of $S$. Within the class of partial isometries on $\mathcal{H}$ the unilateral shifts are in a sense as far removed as possible from the co-isometries and the self-adjoint partial isometries. For shifts have no self-adjoint part, and far from being co-isometric if $S$ is a shift $S^{*n}$ goes strongly to zero. (These and other simple properties of shifts may be deduced from problem 118 and the surrounding material in Halmos [5].)

II. We begin with a complex Hilbert space $\mathcal{H}$ (not necessarily separable) and a unilateral shift $S$ on $\mathcal{H}$. It is well known that shifts decompose the underlying Hilbert space in the following way:

$$\mathcal{H} = \bigoplus \sum_{n=0}^{\infty} S^n \mathscr{C} \quad \text{where} \quad \mathscr{C} = (S\mathcal{H})^{\perp},$$

(See for example Halmos [5], problem 118).

We also fix an invariant subspace $\mathscr{M}$ of $S$. By $S_0$ we denote the restriction of $S$ to $\mathscr{M}$, $S_0 = S \,|\, \mathscr{M}$. The commutant of $S$ is the algebra of bounded operators on $\mathscr{H}$ which commute with $S$ and is denoted by $\mathscr{A}_S$.

The invariant subspaces of $S$ are known to the following extent. Every invariant subspace of $S$ is the range of a partial isometry in $\mathscr{A}_S$ whose initial space reduces $S$. (This well known result appears in many forms. The particular form cited here appears in [7], see proof of Theorem 1.) Particularly when a function space model is used these operators are often referred to as inner functions or rigid functions.

Finally we will fix a bounded operator $T$ on $\mathscr{M}$ which commutes with $S_0$. As indicated earlier the problem being considered is that of extending $T$ to an operator on $\mathscr{H}$ lying in $\mathscr{A}_S$ and having norm equal to $\|T\|$.

THEOREM 2.1. *If $S$ is the simple shift, i.e., if* $\dim \mathscr{C} = 1$, *then $T$ has an extension in $\mathscr{A}_S$ whose norm is equal to $\|T\|$.*

*Proof.* This theorem will follow from a later result. (See Remark 2.4 below.) The simple shift can be represented as the usual shift on the Hardy space $H^2$ of complex valued functions on the unit circle (Helson [6], chapter 1). It is instructive to sketch a proof in this setting where $\mathscr{M} = BH^2$ with $B$ an inner function in $H^2$. Also $B \in \mathscr{M}$, and $T: B \to Bg$ for some $g$ in $H^2$. The fact that $TS_0 = S_0 T$ allows one to argue that $T: Bf \to Bfg$ for all $f \in H^\infty$, and finally using standard techniques one shows that $g \in H^\infty$, that $T$ is multiplication by $g$ on $\mathscr{M}$, and hence that $T$ has an obvious extension to an operator on $H^2$ which commutes with $S$. The extension does not increase the norm.

EXAMPLE 2.2. *$T$ does not necessarily have a bounded extension which commutes with $S$ if $S$ is a shift of multiplicity two, i.e., if $\dim \mathscr{C} = 2$.*

*Proof.* Here we let $\mathscr{H} = H^2 \oplus H^2$. Vectors in $\mathscr{H}$ will be written as ordered pairs $(f, g)$. Let $\chi$ be the identity function on the unit circle, $\chi(e^{it}) = e^{it}$, and then the shift $S$ of multiplicity two on $\mathscr{H}$ is $S: (f, g) \to (\chi f, \chi g)$.

Let $\mathscr{M}$ be the subspace of $\mathscr{H}$ consisting of all vectors of the form $(f, \chi g)$ where $f, g \in H^2$. Clearly $S \mathscr{M} \subseteq \mathscr{M}$. Define $T$ on $\mathscr{M}$ by $T: (f, g) \to (\bar{\chi} g, 0)$, the bar denoting complex conjugate. It is trivial

to verify that $T$ is a bounded operator mapping $\mathscr{M}$ into $\mathscr{M}$, and that $TS = ST$ on $\mathscr{M}$. But it is equally easy to see that $T$ can have no extension in $\mathscr{A}_S$. For if $T'$ is an extension of $T$ to $\mathscr{H}$, then we must have $T'S: (0, 1) \rightarrow (1, 0)$, whereas everything in the range of $ST'$ must be orthogonal to $(1, 0)$.

It becomes apparent in the discussion which follows that the reason we obtain different answers in the case of the simple shift as opposed to nonsimple shifts is that the simple shift is the only shift having an abelian commutant. Recall that $\mathscr{M} = B\mathscr{H}$ where $B$ is a partial isometry in $\mathscr{A}_S$ and $B^*\mathscr{H}$ reduces $S$. Let $A_T$ be the operator on $\mathscr{H}$ defined by

$$A_T = B^* TB \ .$$

Since $BB^*$ is the orthogonal projection onto $\mathscr{M}$ we have

$$BB^* TBS = TBS = STB = SBB^* TB = BSB^* TB \ ,$$

or $BA_T S = BSA_T$. Now the range of $A_T$ is contained in the range of $B^*$ which is a reducing subspace for $S$. Since $B$ is isometric on the range of $B^*$ we can infer from the last equation that $A_T S = SA_T$. Thus $A_T$ satisfies the three conditions

(i)   $A_T \in A_S$
(ii)  $TB = BA_T$
(iii) $\|A_T\| \leqq \|T\|$.

Clearly an operator $A$ in $\mathscr{A}_S$ is an extension of $T$ if and only if $AB = TB$. Thus it follows that $T$ has an extension in $\mathscr{A}_S$ if and only if there exists an operator $A \in \mathscr{A}_S$ such that $AB = BA_T$, i.e., the problem is now one of solving the operator equation $AB = BA_T$ for $A \in \mathscr{A}_S$. ($B$ and $A_T$ are already in $\mathscr{A}_S$.)

A *hyperinvariant* subspace for $S$ is a subspace which is invariant under every operator which commutes with $S$.

PROPOSITION 2.3.   *If $\mathscr{M}$ is a hyperinvariant subspace of $S$, then $T$ has an extension in $\mathscr{A}_S$ whose norm is $\|T\|$.*

*Proof.* The fact that $\mathscr{M}$ is hyperinvariant guarantees that $B$ can be chosen so as to have the additional property that $B$ commutes with every operator in $\mathscr{A}_S$. (Douglas and Pearcy [2], Theorem 5). Thus $A_T B = BA_T$, and $T$ possesses the desired extension by the remarks above.

REMARK 2.4.   Since every invariant subspace for the simple shift is hyperinvariant, the above proposition contains Theorem 2.1.

There is a relationship between $T$ having an extension in $\mathscr{A}_S$ and a factorization of a familiar type. From the definition of $A_T$ it is clear that range $A_T^* \subseteq$ range $B^*$. Thus by a standard factorization result (Douglas [1]) there exists a bounded operator $D$ on $\mathscr{H}$ such that $A_T = DB$.

PROPOSITION 2.5. *If* $A_T = DB$ *where* $D \in \mathscr{A}_S$, *then* $T$ *has an extension in* $\mathscr{A}_S$.

*Proof.* Suppose $D \in \mathscr{A}_S$ and $A_T = DB$. Then $BA_T = BDB$. Setting $A = BD$ it follows from the remarks made preceeding Proposition 2.3 that $T$ has an extension in $\mathscr{A}_S$.

III. In order that an operator $A$ on $\mathscr{H}$ commute with the shift $S$ it is necessary that every subspace $S^n \mathscr{H} (n \geq 0)$ be invariant under $A$. The proposition below is a slight generalization of this statement. For $n \geq 0$, let $P_n = I - S^n S^{*n}$, the orthogonal projection onto the orthogonal complement of $S^n \mathscr{H}$.

PROPOSITION 3.1. *If* $A \in \mathscr{A}_S$, *then there is a constant* $\alpha$ *such that* $\|P_n A f\| \leq \alpha \|P_n f\|$ *for every* $n \geq 0$ *and every* $f \in \mathscr{H}$. *In fact* $\alpha$ *can be chosen to be* $\|A\|$.

*Proof.* If $n \geq 0$ and $f \in \mathscr{H}$ write $f = S^n g + h$ where $g = S^{*n} f$ and $h = P_n f$. Then since $S^{*n} S^n = I$ and $P_n A^* = P_n A^* P_n$, $\|P_n A f\| = \|P_n A h\| \leq \|A\| \|h\| = \|A\| \|P_n f\|$.

With $T$ defined initially on $\mathscr{M}$ Proposition 3.1 indicates that it is fruitless to look for an extension of $T$ in $\mathscr{A}_S$ unless $T$ initially satisfies a similar condition on $\mathscr{M}$. Henceforth we assume that there exists a constant $\alpha$ such that

$$(*) \qquad\qquad \|P_n T f\| \leq \alpha \|P_n f\|$$

for all $f \in \mathscr{M}$ and $n \geq 0$.

It is easy now to see that in Example 2.2 $T$ could have no extension in $\mathscr{A}_S$ because condition (*) is not satisfied. If in that example we take $f = (0, \chi)$, then $\|P_n f\| = 0$ but $\|P_n T f\| = 1$ when $n = 1$.

Whether condition (*) is sufficient to guarantee that $T$ has an extension in $A_S$ we have been unable to determine (see Remark 3.6). We have been able to show, Example 3.5 below, that such an extension cannot always be made without increasing the norm.

The next theorem indicates the existence of a certain subspace

$\mathscr{W}$ between $\mathscr{M}$ and $\mathscr{H}$ and also invariant under $S$ to which $T$, if $T$ satisfies condition (\*), can always be extended without increasing the norm and so as to commute with $S$. Two corollaries indicate that frequently $\mathscr{W}$ is all of $\mathscr{H}$.

If $f \in \mathscr{H}$, let $\rho(f, \mathscr{M}) = \inf\{\|f - g\| : g \in \mathscr{M}\}$.

THEOREM 3.2. *Let $\mathscr{W}$ be the set of all $f \in \mathscr{H}$ such that*

$$\rho(S^n f, \mathscr{M}) \to 0$$

*as $n \to \infty$. Then $\mathscr{W}$ is a (closed) subspace of $\mathscr{H}$ which is invariant under $S$, and if $T$ satisfies condition (\*) on $\mathscr{M}$ then $T$ has an extension to an operator $T'$ on $\mathscr{W}$ satisfying $T'S = ST'$ on $\mathscr{W}$ and $\|T'\| = \|T\|$.*

*Proof.* It is easy to verify that $\mathscr{W}$ is a linear manifold and that $S\mathscr{W} \subseteq \mathscr{W}$. To see that $\mathscr{W}$ is closed, suppose that $f$ is in the closure of $\mathscr{W}$. Then for $g \in \mathscr{W}$,

$$\rho(S^n f, \mathscr{M}) \leq \|S^n f - S^n g\| + \rho(S^n g, \mathscr{M}) .$$

By choosing $g$ sufficiently near to $f$ and $n$ sufficiently large, the two terms on the right can be made as small as desired.

We next describe the manner in which $T$ extends to $\mathscr{W}$. Suppose $f$ is in $\mathscr{W}$. Let $\{g_n\}$ be a sequence in $\mathscr{M}$ such that $\lim \|S^n f - g_n\| = 0$, and set $h_n = S^n f - g_n$. Now if $m \geq n$,

$$\| S^{*n} T g_n - S^{*m} T g_m \| = \| S^{*m} T S^{m-n} g_n - S^{*m} T g_m \|$$

$$\leq \| T \| \| S^{m-n} g_n - g_m \| = \| T \| \| S^{m-n} h_n - h_m \|$$

$$\leq \| T \|(\| h_n \| + \| h_m \|) ,$$

and the last expression goes to zero as $n, m \to \infty$. Thus we have shown that the sequence $\{S^{*n} T g_n\}$ is a Cauchy sequence. To extend $T$ to $\mathscr{W}$, if $f \in \mathscr{W}$ we select a sequence $\{g_n\}$ in $\mathscr{M}$ such that

$$\| S^n f - g_n \| \to 0$$

as $n \to \infty$ and set $T'f = \lim S^{*n} T g_n$. In light of the earlier remarks in this paragraph it is easy to see that the way in which $T'f$ is defined here is independent of the sequence $\{g_n\}$ chosen and coincides with the original operator $T$ in case $f \in \mathscr{M}$. It is also clear that the extension does not increase the norm.

To see that $T'\mathscr{W} \subseteq \mathscr{W}$, we assume $f \in \mathscr{W}$. Let $\{g_n\}$ be a sequence in $\mathscr{M}$ such that $\| S^n f - g_n \| \to 0$. Now making use of the fact that $T$ satisfies condition (\*) we have $\| P_n T g_n \| \leq \alpha \| P_n g_n \|$, and the right-hand side here goes to zero. Furthermore,

$$\rho(S^n T'f, \mathscr{M}) \leqq \| S^n T'f - S^n S^{*n} Tg_n \| + \rho(S^n S^{*n} Tg_n, \mathscr{M}) \ .$$

The first term on the right goes to zero by the definition of $T'f$, and the second goes to zero because $Tg_n \in \mathscr{M}$ and $\| P_n Tg_n \| \to 0$. Thus $T'f \in \mathscr{W}$.

Finally we show that $T'S = ST'$ on $\mathscr{W}$. If $f \in \mathscr{W}$, let $\{g_n\}$ be a sequence in $\mathscr{M}$ such that $\| S^n f - g_n \| \to 0$. Then

$$\| T'Sf - ST'f \| \leqq \limsup \| S^{*n} TSg_n - SS^{*n} Tg_n \|$$
$$= \limsup \| S^{*(n-1)} Tg_n - SS^{*n} Tg_n \|$$
$$\leqq \limsup \| S^{(n-1)} S^{*(n-1)} Tg_n - Tg_n \| + \limsup \| Tg_n - S^n S^{*n} Tg_n \|$$
$$\leqq \alpha \limsup \| P_{n-1} g_n \| + \alpha \limsup \| P_n g_n \| = 0 \ .$$

Frequently the subspace $\mathscr{W}$ of Theorem 3.2 will be all of $\mathscr{H}$. The two corollaries below give examples of this occurrence.

COROLLARY 3.3.    *If* $\dim \mathscr{M}^\perp < \infty$, *and if* $T$ *satisfies* (*), *then* $T$ *has an extension in* $\mathscr{A}_S$ *whose norm is* $\| T \|$.

*Proof.*    Let $\mathscr{W}$ be the subspace of Theorem 3.2.    Assume that $x$ is an eigenvector for the operator on $\mathscr{W}^\perp$ obtained by compressing $S$ to $\mathscr{W}^\perp$, the operator $(I - P)S | \mathscr{W}^\perp$ where $P$ is the orthogonal projection of $\mathscr{H}$ onto $\mathscr{W}$. Let $\lambda$ be the corresponding eigenvalue, so $|\lambda| \leqq 1$ and $Sx = y + \lambda x$ where $y = PSx$.

Then $S^2 x = Sy + \lambda Sx = (Sy + \lambda y) + \lambda^2 x$.    In general

$$S^n x = y_n + \lambda^n x$$

where $y_n \in \mathscr{W}$.    Now if $|\lambda| = 1$ then $\| Sx \|^2 = \| y \|^2 + \| x \|^2$, implying that $y = 0$ since $S$ is a contraction.    But this would imply that $\lambda$ is an eigenvalue of $S$, and since $S$ is a shift $S$ has no eigenvalues.

Thus $|\lambda| < 1$, and $\lambda^n x \to 0$ as $n \to \infty$, implying that $x \in \mathscr{W}$. This too is a contradiction and we have shown that in fact $(I - P)S | \mathscr{W}^\perp$ can have no eigenvalues and hence since $\mathscr{W}^\perp$ is finite demensional we must have $\dim \mathscr{W}^\perp = 0$.    The proof is now complete in light of Theorem 3.2.

There is a special type of invariant subspace for nonsimple shifts which is encountered frequently in the literature.    Such subspaces are the ones which, in the Hardy space model (Helson [6], chapter 6), correspond to operator valued analytic functions on the unit disk assuming unitary values on the boundary.    For a general invariant subspace the corresponding rigid function (see Halmos, [4]) can be required only to assume partially isometric values.

There is an equivalent abstract formulation of the condition that an invariant subspace correspond to a unitary valued function.    First of all it is evident that the minimal unitary extension of a unilateral

shift is a bilateral shift of the same multiplicity. If we continue to let $S$ and $\mathscr{H}$ denote respectively a unilateral shift and the space on which it acts and now let $U$ and $\mathscr{K}$ denote respectively the minimal unitary extension of $S$ and the space $\mathscr{K}$ on which $U$ acts, then for each subspace $\mathscr{M}$ of $\mathscr{H}$ invariant under $S$ it is clear that $\mathscr{M}$ is invariant under $U$ as well. It can be shown without great difficulty that in the Hardy space model $\mathscr{M}$ corresponds to a unitary function if and only if the smallest reducing subspace for $U$ containing $\mathscr{M}$ is $\mathscr{K}$ itself.

COROLLARY 3.4. *If the smallest reducing subspace for $U$ which contains $\mathscr{M}$ is $\mathscr{K}$ (where $U$ and $\mathscr{K}$ are as in the preceeding paragraph) then every operator $T$ on $\mathscr{M}$ satisfying (\*) has an extension in $\mathscr{A}_s$ whose norm is $\|T\|$.*

*Proof.* Recall that $\mathscr{M} = B\mathscr{H}$ where $B$ is a partial isometry in $\mathscr{A}_s$. From the folklore of the field we know that $B$ has a unique extension to an operator on $\mathscr{K}$, call it $B'$, which commutes with $U$. (This also can be deduced from the lifting theorem of Sz-Nagy and Foias, Theorem 4 of [3].) Now the range of $B'$ reduces $U$ and contains $\mathscr{M}$. Hence by assumption $B'\mathscr{K} = \mathscr{K}$.

Let $f \in \mathscr{H}$. Since the subspaces $U^{*n}\mathscr{H}$, $n \geq 0$, span $\mathscr{K}$, for each $\varepsilon > 0$ there is an integer $n \geq 0$ and a $g \in U^{*n}\mathscr{H}$ such that

$$\|B'g - f\| < \varepsilon .$$

We have $U^n B'g = B'U^n g \in B\mathscr{H} = \mathscr{M}$, and $\|S^n f - U^n B'g\| < \varepsilon$. Thus we have shown that $\mathscr{W} = \mathscr{H}$ in Theorem 3.2 and therefore that $T$ has the desired extension.

Our final task will be to show that in general condition (\*) on $T$ and $\mathscr{M}$ is not sufficient to guarantee an extension in $\mathscr{A}_s$ with norm equal to $\|T\|$. Because the condition is sufficient in the rather inclusive instances already considered, it is not surprising that some care must be exercised in constructing the following example.

EXAMPLE 3.5. We take $S$ to be a shift of multiplicity 7 on $\mathscr{H}$. Let $\{e_i\}_{i=1}^7$ be an orthonormal basis for $(S\mathscr{H})^\perp$. We take the subspace $\mathscr{M}$ of $\mathscr{H}$ to be the smallest invariant subspace for $S$ containing the following vectors:

$$u_1 = e_1 + Se_2, \ u_2 = e_3 + Se_4, \ u_3 = e_5 + Se_6, \ u_4 = e_5 + Se_7 .$$

The operator $T$ is defined on a dense linear manifold in $\mathscr{M}$ by requiring that

$$Tu_1 = u_3, \; Tu_2 = u_4, \; \text{ and } \; Tu_3 = Tu_4 = 0$$

and by requiring that $TS = ST$. (The linear manifold referred to is the linear span of the vectors $P(S)u_k$, $k = 1, 2, 3, 4$, where $P(S)$ is a polynomial in $S$.)

Some elementary calculations show that $T$ is in fact bounded on this linear manifold, and that moreover $\|T\| \leq \sqrt{3}/\sqrt{2}$. Furthermore it can be shown that $T$ on $\mathcal{M}$ satisfies condition (*) where the constant $\alpha$ can be taken to be $\sqrt{2}$.

Finally one shows that any extension of $T$ to $\mathcal{H}$ which is to commute with $S$ on $\mathcal{H}$ must map $e_1 + e_3$ to $2e_5$, and must hence have norm not less than $\sqrt{2}$. Thus $T$ cannot be extended to an operator which commutes with $S$ on $\mathcal{H}$ without increasing the norm.

REMARK 3.6. It is peculiar in the above example that we could show only that any extension of $T$ to an operator in $\mathcal{A}_S$ must have norm not less than $\alpha$ where $\alpha$ is the constant in (*). This leads naturally to the following conjecture.

CONJECTURE. If $T$ on $\mathcal{M}$ satisfies (*) then $T$ has an extension in $\mathcal{A}_S$ having norm less than or equal to $\alpha$.

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