PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 3, 1971

SINGULAR PERTURBATIONS OF DIFFERENTIAL EQUATIONS IN ABSTRACT SPACES

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In a recent paper, Kisynski studied the solutions of the abstract Cauchy problem $\varepsilon x^{\cdot\cdot}(t) + x^{\cdot}(t) + Ax(t) = 0$, $x(0) = x_0$ and $x^{\cdot}(0) = x_1$ where $0 \leqq t \leqq T$, $\varepsilon > 0$ is small parameter and A is a nonnegative self-adjoint operator in a Hilbert space H. With the aid of the functional calculus of the operator A, he has showed that as $\varepsilon \to 0$ the solution of this problem converges to the solution of the unperturbed Cauchy problem $x^{\cdot}(t) + Ax(t) = 0$, $x(0) = x_0$. Smoller has proved the same result for equation of higher order.

The purpose of this paper is to study the solution of a similar problem and allowing the operator A to depend on t.

To be precise, we shall show that if the initial data is taken from a suitable dense subset of H, then the solution of the Cauchy problem:

$$\varepsilon x^{\cdot\cdot}(t) + x^{\cdot}(t) + A(t)x(t) = 0, x(0) = x_0, x^{\cdot}(0) = x_1 \tag{1.1}$$

converges to the solution of the unperturbed Cauchy problem

$$x^{\cdot}(t) + A(t)x(t) = 0, x(0) = x_0 \tag{1.2}$$

as $\varepsilon \to 0$ where $0 \leqq t \leqq T$, $\varepsilon > 0$ is a small parameter, $A(t)$ is a continuous semi-group of nonnegative self-adjoint operators in H with infinitesimal generator A.

2. The problem (1.1) when $H = R_1$. Before considering (1.1) in the general case, it is necessary to consider (1.1) in the case when $H = R_1$ (i.e., the real line). Thus we consider the Cauchy problem:

$$\varepsilon u^{\cdot\cdot}(t) + u^{\cdot}(t) + e^{\mu t}u(t) = 0 \,. \quad u(0) = x_0, \ u^{\cdot}(0) = x_1 \tag{2.1}$$

when $t \geqq 0$, $\mu \geqq 0$. $\varepsilon > 0$.

According to theorem (1) in [2], equation (2.1) has two linearly independent solutions:

$$u_1 = \sum_0^{m-1} u_{1j}(t)\varepsilon^j + \varepsilon^m E_0\,, \qquad u_1^{\cdot} = \sum_0^{m-1} u_{1j}^{\cdot}(t)\varepsilon^j + \varepsilon^{m-1}E_1$$

$$u_2 = \sum_0^{m-1} u_{2j}(t)\varepsilon^j e^{-t/\varepsilon} + \varepsilon^m E_0\,, \quad u_2^{\cdot} = \sum_0^{m-1} (d/dt)[u_{2j}(t)e^{-t\varepsilon}]\varepsilon^j + \varepsilon^{m-1}E_1$$

where $u_{ij}(t)$ $(i = 1, 2)$ are C^∞ functions on $[0, T]$ and $u_{i0}(t)$ $(i = 1, 2)$ does not vanish at any point of $[0, T]$ and E_0, E_1 are functions of ε and others, but bounded for small $\varepsilon \geqq 0$.

Hence the general solution of equation (2.1) is $u = c_1u_1 + c_2u_2$. Solving for c_1 and c_2 by using the initial condition we obtain $u = x_0s_{00} + x_1s_{01}$ and $u^\cdot = x_0s_{10} + x_1s_{11}$ where

$$\begin{aligned} s_{00} &= H^{-1}(\varepsilon)[u_2^\cdot(0)u_1(t) - u_1^\cdot(0)u_2(t)] \\ s_{01} &= H^{-1}(\varepsilon)[u_1(0)u_2(t) - u_2(0)u_1(t)] \\ s_{10} &= s_{00}^\cdot = \frac{d}{dt}s_{00} \\ s_{11} &= s_{01}^\cdot = \frac{d}{dt}s_{01} \end{aligned} \tag{2.3}$$

and

$$H(\varepsilon) = u_1(0)u_2^\cdot(0) - u_2(0)u_1^\cdot(0)$$

How taking the limit as $\varepsilon \to 0$, we find that

$$\begin{aligned} s_{00}(t, \varepsilon, \mu) &\longrightarrow x_0u_{10}(t) \\ s_{01}(t, \varepsilon, \mu) &\longrightarrow 0 . \end{aligned} \tag{2.4}$$

Consequently, $u(t, \varepsilon) \to x_0u_{10}(t)$. From equation 15 in [2] we find that $u_{10}(t)$ is the solution of the equation

$$u^\cdot + e^{\mu t}u = 0 \tag{2.5}$$

and this is what we wished to show.

3. Estimates for the Functions $s_{ij}(t, \varepsilon, \mu)$. In this section we would like to find estimates for the functions $s_{ij}(t, \varepsilon, \mu)$ $(i, j = 0, 1)$. We may do so by solving for $u_{ij}(t)$ $(i = 1, 2; j = 0, 1, \cdots, m - 1)$ from equation 15 in [2]. Since this would be rather tedious we will take the simpler approach of estimating $u_i(t, \varepsilon, \mu)$ and $u_i^\cdot(t, \varepsilon, \mu)$ $(i = 1, 2)$. Multiplying (2.1) by $u^\cdot$ and integrating between 0 and t we obtain:

$$\frac{\varepsilon u^{\cdot 2}}{2} + \int_0^t u^{\cdot 2} + \frac{u^2}{2}e^{\mu t} - \frac{1}{2}\mu\int_0^t u^2 e^{\mu t} = c .$$

Consequently

$$u^2 \leqq 2\,|\,c\,| + \mu\int_0^t u^2 e^{\mu t}dt .$$

Now using Bellman's lemma, we obtain

(3.1) $$u^2 \leqq 2/c/e^{e^{\mu t}} .$$

For estimating $u^{\boldsymbol{\cdot}}(t)$, we multiply equation (2.1) by $e^{-\mu t}u^{\boldsymbol{\cdot}}$, integrating between 0 and t and using Bellman's lemma we obtain:

(3.2) $$u^{\boldsymbol{\cdot}2}(t) \leqq 2\varepsilon^{-1}/c/e^{2\mu t} .$$

In [2] page 323 we proved that for all small $\varepsilon \geqq 0$ $H(\varepsilon) \neq 0$, therefore we see that (2.3), (3.1), and (3.2) yield,

(3.3) $$|s_{00}| \leqq K(\varepsilon) \exp\left(\frac{e^{\mu t}}{2}\right)$$

$K(\varepsilon)$ is a bounded function in ε, and

(3.4) $$|s_{01}| \leqq \bar{K}(\varepsilon) \exp(e^{e^{\mu t}/2})$$

$\bar{K}(\varepsilon)$ is a bounded function in ε.

To obtain an estimate for s_{ij} $(i, j = 1, 2)$ we write equation (2.1) in amatrix form as:

$$U^{\boldsymbol{\cdot}} = AU$$

when

$$A = \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^1 \exp(\mu t) & -\bar{\varepsilon}^1 \end{pmatrix} .$$

Hence

$$U = \exp\left[\int A(s)ds\right] = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}$$

and from the equation

(3.5) $$(d/dt)\begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix} = \begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^1 \exp(\mu t) & -\bar{\varepsilon}^1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -\bar{\varepsilon}^1 \exp(\mu t) & -\bar{\varepsilon}^1 \end{pmatrix}\begin{pmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{pmatrix}$$

we obtain

(3.6) $$s_{10} = -s_{01}\varepsilon^{-1} \exp(\mu t)$$

(3.7) $$s_{11} = s_{00} - \varepsilon^{-1}s_{01} .$$

4. The problem (1.1) in abstract Hilbert space. We shall now consider the problem (1.1) in any Hilbert space H with norm $\|\cdot\|$.

Since $\{A(t)\}$ is a semi-group of a nonnegative selfadjoint operator in H, with infinitesimal generator A, there is a resolution of the identity E_μ such that $A(t)$ has the spectral representation:

$$A(t) = \int_0^\infty e^{\mu t} dE_\mu \,.$$

We shall next use the functional calculus of the operator $A(t)$. For fixed $\varepsilon > 0$, $t \geqq 0$, we define the operator S_{ij} on H by

$$S_{ij}(t, \varepsilon) = \int_0^\infty s_{ij}(t, \varepsilon, \mu) dE_\mu \qquad (i, j = 0, 1) \tag{4.1}$$

where the $s_{ij}(t, \varepsilon, \mu)$ are defined by (2.3). If we let D denote the dense domain of the operator $e^{A^2(t)}$ for all t, then our estimates (3.2) through (3.7) imply that D is contained in the domain of $S_{ij}(t, \varepsilon)$ for every $i, j = 0, 1$.

For x_0 and x_1 in D, we write

$$x_\varepsilon(t) = S_{00}(t, \varepsilon)x_0 + S_{01}(t, \varepsilon)x_1 \tag{4.2}$$

and we see that $x_\varepsilon(t)$ is in the domain of $A(t)$ for every $\varepsilon > 0$. We now state the main theorem.

THEOREM. *Let $x_\varepsilon(t)$ be defined as in* (4.2) *when x_0, x_1 are in D. Then $x_\varepsilon(t)$ is the unique solution of the Cauchy problem* (1.1) *and $x_\varepsilon(t)$ converges to the solution of* (1.2) *as $\varepsilon \to 0$.*

To prove this theorem we first prove the following lemmas:

LEMMA 1. *For $x \in D$, $(d/dt)S_{ij}(t, \varepsilon)x$ exists and*

$$(d/dt)S_{ij}(t, \varepsilon)x = \int_0^\infty (d/dt)s_{ij}(t, \varepsilon, \mu)dE_\mu x \qquad (i, j = 0, 1) \,. \tag{4.3}$$

Proof. We shall prove the lemma for $i = j = 0$. Since the proofs for the other cases are similar, they will be omitted. For $x \in D$ and $t \geqq 0$ fixed, we have:

$$\begin{aligned} &\left\| \frac{S_{00}(t + \Delta t, \varepsilon) - S_{00}(t)}{\Delta t} \times - S_{10}(t, \varepsilon)x \right\|^2 \\ &= \int_0^\infty \left[\frac{s_{00}(t + \Delta t, \varepsilon, \mu) - s_{00}(t, \varepsilon, \mu)}{\Delta t} - s_{10}(t, \varepsilon, \mu) \right]^2 d \,\| E_\mu x \|^2 \\ &= \int_0^\infty [s_{10}(t', \varepsilon, \mu) - s_{10}(t. \varepsilon. \mu)]^2 d \,\| E_\mu x \|^2 \,, \end{aligned}$$

where $t \leqq t' \leqq t + \Delta t$, using the theorem of the mean and (2.3).

Now there is a T such that $t + \Delta t \leqq T$ for all Δt sufficiently small, so that if we use (3.3) through (3.7) we see that

$$\begin{aligned} | s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu) | &\leqq | s_{10}(t', \varepsilon, \mu) | + | s_{10}(t, \varepsilon, \mu) | \\ &\leqq \varepsilon^{-1} e^{\mu T} K(\varepsilon) e^{(1/2)e^{\mu T}} \leqq N(\varepsilon, T) e^{e^{\mu T}} \end{aligned}$$

where $N(\varepsilon, T)$ is a constant depending on T and ε only. Therefore the function $| s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu) |^2$ is summable with respect to the measure $d \| E_\mu x \|^2$ if Δt is sufficiently small. Furthermore,

$$\lim_{\Delta t \to 0} [s_{10}(t', \varepsilon, \mu) - s_{11}(t, \varepsilon, \mu)]^2 = 0 .$$

So that the Lebseque dominated convergence theorem yields:

$$\lim_{\Delta t \to 0} \int_0^\infty [s_{10}(t', \varepsilon, \mu) - s_{10}(t, \varepsilon, \mu)]^2 d \| E_\mu x \|^2 = 0 .$$

This completes the proof of the lemma.

LEMMA 2. *For $x \in D$ and $t \geqq 0$, we have*

$$\lim_{\varepsilon \to 0} \left\| S_{00}(t, \varepsilon)x - \exp\left(-\int A(s)ds\right)x \right\| = 0 \tag{4.4}$$

$$\lim_{\varepsilon \to 0} \| S_{01}(t, \varepsilon)x \| = 0 . \tag{4.5}$$

Proof.

$$\begin{aligned} &\left\| S_{00}(t, \varepsilon)x - \exp\left(-\int A(s)ds\right) \times \right\|^2 \\ &= \int_0^\infty \left| \left(s_{00}(t, \varepsilon, \mu) - \exp\left(-\int^t e^{\mu s} ds\right)\right) \right|^2 d \| E_\mu x \|^2 . \end{aligned}$$

From (3.3) we see that $\left[s_{00}(t, \varepsilon, \mu) - \exp\left(-\int^t e^{\mu s} ds\right)\right]^2$ is summable with respect to the measure $d \| E_\mu x \|^2$ and, as we have seen in (2.4) and (2.5), the integrand converges pointwise to zero. We apply the Lebesgue dominated convergence theorem to conclude that the integral likewise converges to zero as $\varepsilon \to 0$. This proves (4.4). Relation (4.5) follows from (2.4) and (2.5) likewise.

LEMMA 3. *Let B be a bounded operator in H. If $x^{\cdot}(t) + Bx(t) = 0$, $0 \leqq t \leqq 0$, and $x(0) = 0$, then $x(t) \equiv 0$.*

The proof of the above lemma is in [3] and therefore will be omitted.

The proof of the theorem. That $x_\varepsilon(t)$ defined by (4.2) is a solution of (1.1) follows at once from Lemma 1 by direct verification. The uniqueness of $x_\varepsilon(t)$ follows from Lemma 3 just as in [1]. Finally, since $\exp\left(-\int^t A(s)ds\right)x_0$ is the solution of (1.2) Lemma 2 shows that.

$$\lim_{\varepsilon \to 0} \left\| x_\varepsilon(t) - \exp\left(-\int^t A(s)ds\right)x_0 \right\| = 0 .$$

This completes the proof of the theorem.

References

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Received December 2, 1969.

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