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SPECIALITY OF QUADRATIC JORDAN ALGEBRAS

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In this paper we extend to quadratic Jordan algebras certain results due to P. M. Cohn giving conditions under which a Jordan algebra is special, the most important of these being the Shirshov-Cohn Theorem that a Jordan algebra with two generators and no extreme radical is always special. We also prove that the free algebra on two generators x, y modulo polynomial relations $p(x) = 0$, $q(y) = 0$ is special, and by taking a particular $p(x)$ we show that most of the properties of the Peirce decomposition of a Jordan algebra relative to a supplementary family of orthogonal idempotents follow immediately from the analogous properties of Peirce decompositions in associative algebras.

Throughout we will work with algebras over an arbitrary (commutative, associative) ring of scalars Φ. A (unital) *quadratic Jordan algebra* is defined axiomatically in terms of a product $U_x y$ linear in y and quadratic in x [4, p. 1072]. We can introduce a quadratic Jordan structure $\mathfrak{A}^+$ in any unital associative algebra $\mathfrak{A}$ by taking

$$U_x y = xyx .$$

Any (Jordan) subalgebra of such an algebra $\mathfrak{A}^+$ is called a *special Jordan algebra*. A *specialization* of a quadratic Jordan algebra $\mathfrak{J}$ is a homomorphism of $\mathfrak{J}$ into an algebra of the form $\mathfrak{A}^+$.

With any quadratic Jordan algebra $\mathfrak{J}$ we can associate its *special universal envelope*, consisting of a unital associative algebra $su(\mathfrak{J})$ and a (universal) specialization $\sigma_u: \mathfrak{J} \to su(\mathfrak{J})^+$ such that any specialization $\sigma: \mathfrak{J} \to \mathfrak{A}^+$ factors uniquely through an associative homomorphism $su(\sigma)$: $su(\mathfrak{J}) \to \mathfrak{A}$,

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\sigma} & \mathfrak{A}^+ \\ & {\scriptstyle\sigma_u}\searrow \quad \nearrow{\scriptstyle su(\sigma)} & \\ & su(\mathfrak{J}) & \end{array} \tag{1}$$

$su(\mathfrak{J})$ carries a unique involution, the *main involution* π, such that the elements of $\mathfrak{J}^{\sigma_u}$ are symmetric: $x^{\sigma_u \pi} = x^{\sigma_u}$. This association is functorial—if $\varphi: \mathfrak{J} \to \tilde{\mathfrak{J}}$ is a homomorphism of quadratic Jordan algebras there is induced an associative homomorphism $su(\varphi)$ making

$$\begin{array}{ccc} \mathfrak{J} & \xrightarrow{\varphi} & \tilde{\mathfrak{J}} \\ {\scriptstyle\sigma_u}\downarrow & & \downarrow{\scriptstyle\tilde{\sigma}_u} \\ su(\mathfrak{J}) & \xrightarrow[su(\varphi)]{} & su(\tilde{\mathfrak{J}}) \end{array} \tag{2}$$

commutative. An algebra $\mathfrak{J}$ is special if and only if it is imbedded in $su(\mathfrak{J})$ via σ_u.

For any set X we have a *free quadratic Jordan algebra* $FJ(X)$, a *free special Jordan algebra* $FS(X)$, and a *free associative algebra* $F(X)$ on the set X (over the ring Φ). We have $FS(X)$ imbedded in $F(X)$ as the (Jordan) subalgebra of $F(X)^+$ generated by X, and $F(X)$ with this inclusion map serves as special universal envelope for $FS(X)$. When X consists of just two elements $X = \{x, y\}$ we know $FJ(x, y) = FS(x, y)$ by *Shirshov's Theorem*. For all these see [3].

1. Cohn's theorem and criterion. We consider a set $X = \{x_i\}_{i \in I}$ where the indices are linearly ordered. The free associative algebra $F(X)$ carries a *reversal involution*, whose action on a typical monomial is

$$(x_{i_1} \cdots x_{i_n})^* = x_{i_n} \cdots x_{i_1} .$$

The subspace $\mathfrak{H}(F(X), {}^*)$ of *-symmetric elements is a Jordan subalgebra of $F(X)^+$ containing X, hence containing $FS(X)$. Cohn's Theorem measures how far $FS(X)$ is from being all of $\mathfrak{H}(F(X), {}^*)$.

COHN'S THEOREM [1, p. 257; 2, ex. 2 p. 9]. *$\mathfrak{H}(F(X), {}^*)$ is the Jordan subalgebra of $F(X)^+$ generated by 1, X, and all the n-tads*

$$\{x_{i_1} \cdots x_{i_n}\} = x_{i_1} \cdots x_{i_n} + x_{i_n} \cdots x_{i_1}$$

where $n \geqq 4$ and $i_1 < i_2 < \cdots < i_n$.

Proof. Clearly $\mathfrak{H} = \mathfrak{H}(F(X), {}^*)$ contains X and all n-tads. Conversely, to show the subalgebra $\mathfrak{R}$ generated by such elements is all of $\mathfrak{H}$ we must show $\mathfrak{R}$ contains all $\{x_{i_1} \cdots x_{i_n}\} = x_{i_1} \cdots x_{i_n} + x_{i_n} \cdots x_{i_1}$ and all $x_{i_1} \cdots x_{i_n} y x_{i_n} \cdots x_{i_1}$ (where y is either 1 or one of the x_i) since these clearly span $\mathfrak{H}$. Now the $x_{i_1} \cdots x_{i_n} y x_{i_n} \cdots x_{i_1} = U_{x_{i_1}} \cdots U_{x_{i_n}} y$ are generated by X alone, so we need only generate the $\{x_{i_1} \cdots x_{i_n}\}$. We do this by induction on n. The result is trivial for $n = 2, 3$ since $\{x_{i_1} x_{i_2}\} = x_{i_1} \circ x_{i_2}$, $\{x_{i_1} x_{i_2} x_{i_3}\} = U_{x_{i_1}, x_{i_3}} x_{i_2}$ where $x \circ y$ and $U_{x,z} y$ are the linearizations of $x^2 (= U_x 1)$ and $U_x y$. We assume $n \geqq 4$ and that all $\{x_{i_1} \cdots x_{i_m}\}$ for $m < n$ are in $\mathfrak{R}$.

Our first task is to show

$$\{x_{i_{\pi(1)}} \cdots x_{i_{\pi(n)}}\} \equiv \pm \{x_{i_1} \cdots x_{i_n}\} \pmod{\mathfrak{R}} \tag{3}$$

for any permutation π. It suffices to do this for the generators $(12 \cdots n)$ and $(1n)$ of the symmetric group S_n. For the transposition $(1n)$ we have

$$\{x_{i_1} \cdots x_{i_n}\} + \{x_{i_n} x_{i_2} \cdots x_{i_{n-1}} x_{i_1}\} = U_{x_{i_1}, x_{i_n}} \{x_{i_2} \cdots x_{i_{n-1}}\} \equiv 0$$

by our induction hypothesis, and for the cycle $(12 \cdots n)$

$$\{x_{i_1} \cdots x_{i_n}\} + \{x_{i_2} \cdots x_{i_n} x_{i_1}\} = x_{i_1} \circ \{x_{i_2} \cdots x_{i_n}\} \equiv 0 .$$

If all the indices are distinct then (3) shows that $\{x_{i_1} \cdots x_{i_n}\}$ is congruent to $\pm$ an n-tad, which belongs to $\mathfrak{K}$ by hypothesis, so $\{x_{i_1} \cdots x_{i_n}\}$ also belongs to $\mathfrak{K}$. If two indices coincide, (3) shows $\{x_{i_1} \cdots x \cdots x \cdots x_{i_n}\} \equiv \pm\{x x_{i_1} \cdots x_{i_n} x\} = U_x\{x_{i_1} \cdots x_{i_n}\} \equiv 0$ by induction. In either case, $\{x_{i_1} \cdots x_{i_n}\} \in \mathfrak{K}$.

Since there are no n-tads for $n \geqq 4$ if there are only three variables, we have the following useful corollary.

COROLLARY. *For $m \leqq 3$, the subalgebra of $F(x_1, \cdots, x_m)^+$ generated by $x_1, \cdots, x_m$ is all of $\mathfrak{H}(F(x_1, \cdots, x_m), {}^*)$.*

The next result gives a criterion for when a homomorphic image of a special Jordan algebra is again special.

COHN'S CRITERION [1, p. 255; 2, p. 10]. *If $\mathfrak{J}$ is a special Jordan algebra and $\mathfrak{K}$ an ideal in $\mathfrak{J}$ then $\mathfrak{J}/\mathfrak{K}$ is special if and only if $\mathfrak{J} \cap \bar{\mathfrak{K}} = \mathfrak{K}$ where $\bar{\mathfrak{K}}$ is the ideal in $su(\mathfrak{J})$ generated by $\mathfrak{K}$.*

Proof. A standard functorial argument shows that the algebra $su(\mathfrak{J}/\mathfrak{K}) = su(\mathfrak{J})/\bar{\mathfrak{K}}$ and the specialization of $\mathfrak{J}/\mathfrak{K}$ induced from $\mathfrak{J} \to su(\mathfrak{J}) \to su(\mathfrak{J})/\bar{\mathfrak{K}}$ by passage to the quotient serve as special universal envelope for $\mathfrak{J}/\mathfrak{K}$ (i.e., satisfy the universal property (1)). The kernel of this specialization is $\mathfrak{J} \cap \bar{\mathfrak{K}}/\mathfrak{K}$, so the specialization is injective (i.e., $\mathfrak{J}/\mathfrak{K}$ is special) if and only if $\mathfrak{J} \cap \bar{\mathfrak{K}} = \mathfrak{K}$.

In particular, for $\mathfrak{J} = FS(X)$ and $su(\mathfrak{J}) = F(X)$ we obtain

COROLLARY. *$FS(X)/\mathfrak{K}$ is special if and only if $\bar{\mathfrak{K}} \cap FS(X) = \mathfrak{K}$ where $\bar{\mathfrak{K}}$ is the associative ideal in $F(X)$ generated by the Jordan ideal $\mathfrak{K}$ in $FS(X)$.*

2. Shirshov-Cohn theorem. The *extreme radical* of a unital quadratic Jordan algebra $\mathfrak{J}$ is the set of elements z such that $U_z = U_{z,x} = 0$ for all x in $\mathfrak{J}$; this always forms an ideal. Since $2z = z \circ 1 = 0$ for such elements, the extreme radical is always zero when $\frac{1}{2} \in \Phi$.

PROPOSITION [1, p. 260]. *If $\mathfrak{K}$ is an ideal in $FS(x, y, z)$ having a set of generators $\{k\}$ such that all tetrads $\{xyzk\}$ belong to $\mathfrak{K}$, and if $FS(x, y, z)/\mathfrak{K}$ has zero extreme radical, then $FS(x, y, z)/\mathfrak{K}$ is special.*

Proof. By the Corollary to Cohn's Criterion $FS(x, y, z)/\mathfrak{K}$ will be special if $\bar{\mathfrak{K}} \cap FS(x, y, z) \subset \mathfrak{K}$. To prove that any $p(x, y, z)$ in $\bar{\mathfrak{K}} \cap$

$FS(x, y, z)$ belongs to $\mathfrak{K}$ it will suffice to show it is in the extreme radical modulo $\mathfrak{K}$,

(i) $U_p r = prp \in \mathfrak{K}$

(ii) $U_{p,q} r = prq + qrp \in \mathfrak{K} \quad (q, r \in FS(x, y, z))$

since we are assuming $FS(x, y, z)/\mathfrak{K}$ has no extreme radical.

It will be enough to prove the stronger results

(i)′ $prp^* \in \mathfrak{K}$

(ii)′ $p + p^* \in \mathfrak{K} \quad (p \in \bar{\mathfrak{K}}, r \in FS(x, y, z))$

since $p = p^*$ if $p \in \bar{\mathfrak{K}} \cap FS(x, y, z)$ and then $prq \in \bar{\mathfrak{K}}$ has $prq + (prq)^* = prq + qrp$.

We tackle (ii)′ first. The proof is the standard one [2, p. 11]. It suffices to consider $p = skt$ for s, t monomials in x, y, z and k a generator of $\mathfrak{K}$, since such elements span $\bar{\mathfrak{K}}$. As $swt + t^*ws^*$ is a symmetric element of the free algebra $F(x, y, z, w)$, by Cohn's Theorem it is a sum of Jordan products of x, y, z, w and the tetred $\{xyzw\}$ where each term in the sum has a factor w or $\{xyzw\}$. But then (applying the homomorphism $F(x, y, z, w) \to F(x, y, z)$ sending $x \to x$, $y \to y, z \to z, w \to k$) we see $p + p^* = skt + t^*ks^*$ is a sum of Jordan products of x, y, z, k and the tetrad $\{xyzk\}$ where each term has a factor $k \in \mathfrak{K}$ or $\{xyzk\} \in \mathfrak{K}$ (by our hypothesis), so $p + p^*$ falls in the ideal $\mathfrak{K}$.

Since (i)′ is not linear in p we must first consider a general $p = \Sigma p_i = \Sigma s_i k_i t_i$. Here $prp^* = \Sigma_i p_i r p_i^* + \Sigma_{i<j}(p_i r p_j^* + p_j r p_i^*)$. By (ii)′ the latter sum is in $\mathfrak{K}$ since the $p_i r p_j^*$ belong to $\bar{\mathfrak{K}}$ if p_i does, so once again we need only consider an individual p_i: to consider prp^* for $p = skt$. Now $prp^* = sktrt^*ks^* = skhks^*$ for

$$h = trt^* \in \mathfrak{H}(F(x, y, z), {}^*) = FS(x, y, z)$$

by the Corollary to Cohn's Theorem. But since $\mathfrak{K}$ is an ideal in $FS(x, y, z)$ this yields $k' = khk = U_k h \in \mathfrak{K}$, and if $s = s_1 \cdots s_m$ where each s_i is an x, y, or z then $sk's^* = U_{s_1} \cdots U_{s_m} k' \in \mathfrak{K}$. Thus $prp^* \in \mathfrak{K}$ in all cases, finishing (i)′ and the Proposition.

Shirshov-Cohn Theorem [1, p. 261; 2, p. 48]. *Any unital quadratic Jordan algebra on two generators without extreme radical is special.*

Proof. By universal properties, any quadratic Jordan algebra $\mathfrak{J}$ on two generators is a homomorphic image of the free quadratic Jordan algebra $FJ(x, y)$ on two generators, hence (by Shirshov's Theorem) of $FS(x, y)$: $\mathfrak{J} \cong FS(x, y)/\mathfrak{K}$ for some ideal $\mathfrak{K}$. We now apply the Proposition; we can forget about tetrads, since we are not concerned with the variable z.

More precisely, let $\{k\}$ be a set of generators for $\mathfrak{K}$, let $\mathfrak{Z}$ be the ideal in $FS(x, y, z)$ generated by z, and let $\mathfrak{L}$ be the ideal generated by z together with the k's. Then $FS(x, y) \cong FS(x, y, z)/\mathfrak{Z}$ and

$$FS(x, y)/\mathfrak{K} \cong (FS(x, y, z)/\mathfrak{Z})/(\mathfrak{L}/\mathfrak{Z}) \cong FS(x, y, z)/\mathfrak{L} .$$

Each $\{xyzk(x, y)\}$ or $\{xyzz\}$ belongs to $\mathfrak{L}$—the latter is $\{xyz^2\} = U_{x,z^2}y$ and the former is a sum of Jordan products of x, y, z each term of which has a factor z, so in fact the tetrads belong to $\mathfrak{Z} \subset \mathfrak{L}$. Since $FS(x, y, z)/\mathfrak{L} \cong \mathfrak{J}$ has no extreme radical, we apply the Proposition to conclude $\mathfrak{J}$ is special.

Note that if $\frac{1}{2} \in \Phi$ then the extreme radical is automatically zero, so in that case we obtain the usual Shirshov-Cohn Theorem that *any* Jordan algebra on two generators is special. A standard example [2, ex. 3 p. 12] shows that this stronger form does not hold in general: if $\mathfrak{K}$ is the ideal spanned by $x^2, x^4, x^5, x^6 \cdots$ in the free algebra

$$FJ(x) = FS(x) = F(x)$$

on a single generator over a field Φ of characteristic 2 then the coset $\bar{x}$ in $FS(x)/\mathfrak{K}$ has $\bar{x}^2 = 0$ but $\bar{x}^3 \neq 0$ so $FS(x)/\mathfrak{K}$ cannot be special. (Of course, $\bar{x}^3$ is in the extreme radical).

An algebra $\mathfrak{J}$ is *power-associative* if each subalgebra $\Phi[z]$ generated by a single element forms an associative algebra under the natural structure induced from $\mathfrak{J}$ [5, p. 293], and *strictly power-associative* if it remains power-associative under all scalar extensions. Power-associativity amounts to the condition that a polynomial relation $p(z) = 0$ implies $zp(z) = 0$. In the previous example it was the failure of this condition which led to trouble. However, the following example shows that imposing power-associativity is not by itself enough to guarantee speciality; the condition is necessary but not sufficient.

EXAMPLE. If $\mathfrak{K}$ is the ideal in $FJ(x, y)$ over a field Φ of characteristic 2 generated by U_xy and all monomials of degree $\geqq 6$, then $\mathfrak{J} = FJ(x, y)/\mathfrak{K}$ is a strictly power-associative algebra generated by two elements which is not special.

Proof. $\mathfrak{J} = FJ(x, y)/\mathfrak{K} = FS(x, y)/\mathfrak{K}$ is not special by Cohn's Criterion since $\bar{\mathfrak{K}} \cap FS(x, y) > \mathfrak{K}$; indeed, $U_xU_yx = xyxyx = xy(U_xy)$ belongs to $\bar{\mathfrak{K}}$ and to $FS(x, y)$, yet not to $\mathfrak{K}$. To see this, recall that the ideal generated by U_xy is spanned by all $M_1 \cdots M_n(U_xy)$ and $M_1 \cdots M_n(U_{U(x)y})m$ for $m \in FS(x, y)$ and $M_i = U_x, U_y, U_{x,y}, V_x, V_y$, or I. The part of the homogeneous ideal $\mathfrak{K}$ of x-degree 3 and y-degree 2 is spanned by $U_{x,y}(U_xy)$, $V_xV_y(U_xy)$, $V_yV_x(U_xy)$, i.e., by

$$x^2yxy + yxyx^2,\ 2xyxyx + x^2yxy + yxyx^2,\ yx^2yx + xyx^2y + x^2yxy + yxyx^2 ,$$

hence by $x^2yxy + yxyx^2$ and $yx^2yx + xyx^2y$ in characteristic 2, so that $xyxyx$ is not in $\mathfrak{K}$.

We will show $\mathfrak{J}$ is power-associative; since any extension $\mathfrak{J}_\Omega$ has the same form over Ω that $\mathfrak{J}$ does over Φ, the same argument will apply to all $\mathfrak{J}_\Omega$, and consequently $\mathfrak{J}$ will be strictly power-associative. We must show that if $p(z) \in \mathfrak{K}$ for some polynomial p then also $zp(z) \in \mathfrak{K}$.

First we get rid of the constant terms. Let $z = \alpha_0 1 + w$ where w contains the homogeneous parts of z of degree $\geqq 1$. Then the degree zero part of $p(z) \in \mathfrak{K}$ is $p(\alpha_0)$, and since $\mathfrak{K}$ is homogeneous and contains only terms of degree $\geqq 3$ we have $p(\alpha_0) = 0$. Thus if $q(\lambda) = p(\lambda + \alpha_0)$ we have $q(0) = p(\alpha_0) = 0$, so q has zero constant term, and

$$p(z) = q(z - \alpha_0 1) = q(w) .$$

Therefore

$$zp(z) = \alpha_0 p(z) + wp(z) = \alpha_0 p(z) + wq(w) ,$$

and it will be enough if $wq(w)$ lies in $\mathfrak{K}$.

This shows we may assume (after replacing p, z by q, w) that $p(\lambda)$ and z have no constant term:

$$p(\lambda) = \gamma_1\lambda + \cdots + \gamma_n\lambda^n \quad z = z_1 + \cdots + z_m$$

for z_i homogeneous of degree i. We next get rid of the degree one term $z_1 = \alpha x + \beta y$. If $\gamma_1 = \cdots = \gamma_{r-1} = 0$ but $\gamma_r \neq 0$ then the degree r term of $p(z) \in \mathfrak{K}$ is $\gamma_r z_1^r$, so by the homogeneity of $\mathfrak{K}$

$$z_1^r = (\alpha x + \beta y)^r = \alpha^r x^r + \beta^r y^r + \cdots$$

lies in $\mathfrak{K}$. Since all elements of $\mathfrak{K}$ have x-degree $\geqq 2$ and y-degree $\geqq 1$ we see $\alpha^r = \beta^r = 0$. Thus $\alpha = \beta = 0$ and $z_1 = 0$ as desired.

We are reduced to considering $z = z_2 + z_3 + z_4 + z_5$ (modulo terms of degree $\geqq 6$); in this case z^k for $k \geqq 3$ consists entirely of terms of degree $\geqq 6$, so $p(z) \equiv \gamma_1 z + \gamma_2 z^2$ and $zp(z) \equiv \gamma_1 z^2 \bmod \mathfrak{K}$. If $\gamma_1 = 0$ trivially $zp(z) \in \mathfrak{K}$, while if $\gamma_1 \neq 0$ then $\gamma_1 z + \gamma_2 z^2 \equiv \gamma_1 z_2 + \gamma_1 z_3 + (\gamma_1 z_4 + \gamma_2 z_2^2) + (\gamma_1 z_5 + \gamma_2 z_2 \circ z_3) \in \mathfrak{K}$ implies $z_2, z_3 \in \mathfrak{K}$ by homogeneity, so $\gamma_1 z^2 \equiv \gamma_1(z_2^2 + z_2 \circ z_3) \in \mathfrak{K}$. In all cases $zp(z)$ belongs to $\mathfrak{K}$, and $\mathfrak{J}$ is power-associative.

We can improve slightly on the theorem. In dealing with associative algebras $\mathfrak{A}$ with involution $*$ in situations where $\frac{1}{2} \notin \Phi$ it is sometimes more convenient to work with certain "ample" subalgebras of $\mathfrak{H}(\mathfrak{A}, *)$ rather than just with $\mathfrak{H}(\mathfrak{A}, *)$ itself. A subspace $\mathfrak{K}$ of $\mathfrak{H}(\mathfrak{A}, *)$ is *ample* if $\mathfrak{K}$ contains 1 and all aka^* for $a \in \mathfrak{A}$ and $k \in \mathfrak{K}$. (In particular, $\mathfrak{K}$ contains all norms aa^* and traces $a + a^*$, so if $\frac{1}{2} \in \Phi$ then $\mathfrak{K} = \mathfrak{H}$). We will say a Jordan algebra is *reflexive* if $\mathfrak{J}^{\sigma_u}$ is an ample subspace of $\mathfrak{H}(su(\mathfrak{J}), \pi)$ (and *strongly reflexive* if $\mathfrak{J}^{\sigma_u} = \mathfrak{H}(su(\mathfrak{J}), \pi)$).

By the Corollary to Cohn's Theorem $\mathfrak{J} = FJ(x_1, \cdots, x_m)$ is strongly reflexive for $m \leqq 3$, but its homomorphic images may not be. However, they do inherit reflexivity:

THEOREM [2, p. 77] *If $\mathfrak{J}$ is reflexive so is any homomorphic image.*

Proof. Let $\varphi: \mathfrak{J} \to \tilde{\mathfrak{J}}$ be an epimorphism. To see that $\tilde{\mathfrak{J}}^{\tilde{\sigma}_u}$ is ample in $\mathfrak{H}(su(\tilde{\mathfrak{J}}), \tilde{\pi})$ we use (2) to see that (setting $\psi = su(\varphi)$) any $\tilde{a}\tilde{x}\tilde{a}^{\tilde{\pi}}$ for $\tilde{a} = \psi(a) \in su(\tilde{\mathfrak{J}}) = \psi(su(\mathfrak{J}))$, $\tilde{x} = \psi(x) \in \tilde{\mathfrak{J}}^{\tilde{\sigma}_u} = \varphi(\mathfrak{J})^{\tilde{\sigma}_u} = \psi(\mathfrak{J}^{\sigma_u})$ has the form $\psi(a)\psi(x)\psi(a)^{\pi} = \psi(axa^{\pi}) \in \psi(\mathfrak{J}^{\sigma_u}) = \tilde{\mathfrak{J}}^{\tilde{\sigma}_u}$ and hence belongs to $\tilde{\mathfrak{J}}^{\tilde{\sigma}_u}$.

COROLLARY. *Any quadratic Jordan algebra with three or fewer generators is reflexive.*

Since any algebra $\mathfrak{J}$ which is both special and reflexive has $\mathfrak{J} \cong \mathfrak{J}^{\sigma_u}$ ample in $\mathfrak{H}(su(\mathfrak{J}), \pi)$ we have the improved result

SHIRSHOV-COHN THEOREM [2, p. 77]. *Any quadratic Jordan algebra on two generators without extreme radical is isomorphic to an ample subalgebra of $\mathfrak{H}(\mathfrak{A}, {}^*)$ for some associative algebra $\mathfrak{A}$ with involution.*

Again, if $\frac{1}{2} \in \Phi$ the only ample subspace of $\mathfrak{H}(\mathfrak{A}, {}^*)$ is $\mathfrak{H}(\mathfrak{A}, {}^*)$ itself.

3. An example. In this section we consider the free special algebra $FS(x, y, z)$ on three generators, together with three relations $p(x) = 0$, $q(y) = 0$, $r(z) = 0$ where $p(\lambda)$, $q(\lambda)$, $r(\lambda)$ are monic polynomials of degree n, m, l respectively. (We allow any of these to be zero, in which case we take the degree to be ∞).

By singling out powers of x, y, z greater than or equal to n, m, l we can write any monomial in $F(x, y, z)$ uniquely as a word

$$w = a_1 w_1 a_2 w_2 \cdots w_k a_{k+1}$$

where (i) each w_α is an x^i, y^j, or z^k for $i \geqq n, j \geqq m, k \geqq l$; (ii) each a_α is a monomial containing only powers x^i, y^j, z^k for $i < n, j < m, k < l$; (iii) there is no coalescing between the w_α's and the a_α's in the sense that if $w_\alpha = x^i$ then a_α cannot end nor $a_{\alpha+1}$ begin with a factor x (similarly if w_α is y^j or z^k). Since p, q, r are monic it is easy to see (writing $i \geqq n$ as $i = \varepsilon + ne, j \geqq m$ as $j = \eta + mf, k \geqq l$ as $k = \gamma + lg$ for $0 \leqq \varepsilon < m, 0 \leqq \eta < n, 0 \leqq \gamma < l$ and $e, f, g \geqq 1$) that $F(x, y, z)$ has a basis consisting of the

$$m = a_1 m_1 a_2 m_2 \cdots m_k a_{k+1} \tag{4}$$

where the a_α satisfy (ii) and (iii) and the m_α are either $x^\varepsilon p(x)^e$, $y^\eta q(y)^f$, or $z^\gamma r(z)^g$. We say m_α has *weight* $\omega(m_\alpha) = e, f,$ or g and m has *weight* $\omega(m) = \Sigma\omega(m_\alpha)$.

THEOREM. *If $\mathfrak{K}$ is the (Jordan) ideal in $FS(x, y, z)$ generated by the elements $p(x)$, $xp(x)$, $q(y)$, $yq(y)$, $r(z)$, $zr(z)$ for some monic $p(\lambda)$, $q(\lambda)$, $r(\lambda)$ then $FS(x, y, z)/\mathfrak{K}$ is special.*

Proof. By the Corollary to Cohn's Criterion it suffices to show $\bar{\mathfrak{K}} \cap FS(x, y, z) \subset \mathfrak{K}$. So suppose $f(x, y, z) \in \bar{\mathfrak{K}}$ is symmetric. It is easy to see that the elements m (as in (4)) of weight $\geqq 1$ form a basis for $\bar{\mathfrak{K}}$ (they are all contained in $\bar{\mathfrak{K}}$, and they span an associative ideal containing p, xp, q, yq, r, zr which are the Jordan generators for $\mathfrak{K}$ and associative generators of $\bar{\mathfrak{K}}$). Since the reverse m^* of an element m again has the form (4), $f(x, y, z)$ is a linear combination of elements $m + m^*$ and of symmetric elements $m = m^*$.

Consider the homomorphism of the free algebra $F(x, y, z, p, q, r)$ on 6 free generators onto $F(x, y, z)$ sending $x \to x$, $y \to y$, $z \to z$, $p \to p(x)$, $q \to q(y)$, $r \to r(z)$. Each $m + m^*$ has a pre-image of the form $n + n^*$ where if m is as in (4) then $n = a_1 n_1 a_2 n_2 \cdots n_k a_{k+1}$ for a_α as before and n_α either $x^\varepsilon p^e$, $y^\eta q^f$, or $z^\gamma r^g$; such $n + n^*$ is symmetric in $F(x, y, z, p, q, r)$, hence by Cohn's Theorem a Jordan product of x, y, z, p, q, r and n-tads $\{x_{i_1} \cdots x_{i_n}\}$ for $4 \leqq n \leqq 6$, where we order the variables $x < p < y < q < z < r$. Applying the homomorphism, $m + m^*$ is a sum of Jordan products of x, y, z, $p(x)$, $q(y)$, $r(z)$ and n-tads. But all the n-tads reduce to Jordan products of x, y, z, $p(x)$, $q(y)$, $r(z)$ together with $xp(x)$, $yq(y)$, $zr(z)$—for example, the 6-tad

$$\{x\, p(x)\, y\, q(y)\, z\, r(z)\} = \{xp(x)\, yq(y)\, zr(z)\}\ .$$

Thus $m + m^*$ is a sum of Jordan products at least one factor of which is a $p(x)$, $q(y)$, $r(z)$ or $xp(x)$, $yq(y)$, $zr(z)$ (since m is of weight $\geqq 1$ and so has at least one factor $p(x)$, $q(y)$, or $r(z)$). This means that $m + m^*$ falls in the Jordan ideal $\mathfrak{K}$.

A similar but more involved argument works for the symmetric $m = m^*$. Consider the homomorphism of the free algebra on 9 generators $F(x, y, z, p, q, r, p', q', r')$ to $F(x, y, z)$ sending $x \to x$, $y \to y$, $z \to z$, $p \to p(x)$, $q \to q(y)$, $r \to r(z)$, $p' \to xp(x)$, $q' \to yq(y)$, $r' \to zr(z)$. We claim $m = m^*$ has a pre-image $n = n^*$ which is symmetric in $F(x, y, z, p, q, r, p', q', r')$. (Once we have this we argue as before; we have to worry about n-tads for $4 \leqq n \leqq 9$ now, where we order the variables $x < p < p' < y < q < q' < z < r < r'$, but again all n-tads reduce to ordinary Jordan products in $FS(x, y, z)$ since $xpp' \to xp(x)^2x$, $xp \to xp(x)$, $pp' \to p(x)xp(x)$ etc.—for example, the 7-tad $\{x\, y\, q\, q'\, z\, r\, r'\}$ reduces to $\{x\, y\, q(y)\, yq(y)\, z\, r(z)\, zr(z)\} = \{x\, yq(y)^2y\, zr(z)^2z\}$—and thus again $m = m^*$ falls in $\mathfrak{R}$). If $m = a_1m_1a_2 \cdots m_ka_{k+1} = m^* = a_{k+1}^*m_k \cdots a_2^*m_1a_1^*$ we have $a_1 = a_{k+1}^*, a_2 = a_k^*, \cdots, a_{k+1} = a_1^*$ and $m_1 = m_k, m_2 = m_{k-1}, \cdots$ by uniqueness of the representation (4). Therefore $n = a_1n_1a_2 \cdots n_ka_{k+1}$ will be a symmetric pre-image of m if the n_α are symmetric pre-images of m_α. So consider $m_\alpha = x^\varepsilon p(x)^e$. Now $x^\varepsilon p^e$ is not symmetric when x, p are free variables, so we must find an alternate representation. If $\varepsilon = 2\varepsilon'$ is even then $x^\varepsilon p(x)^e = x^{\varepsilon'}p(x)^e x^{\varepsilon'}$ has the symmetric pre-image $x^{\varepsilon'}p^e x^{\varepsilon'}$, similarly if $e = 2e'$ is even then $x^\varepsilon p(x)^e = p(x)^{e'}x^\varepsilon p(x)^{e'}$ has pre-image $p^{e'}x^\varepsilon p'$, while if $\varepsilon = 2\varepsilon' + 1$ and $e = 2e' + 1$ are both odd $x^\varepsilon p(x)^e = x^{\varepsilon'}p(x)^{e'}(xp(x))p(x)^{\varepsilon'}x^{\varepsilon'}$ has symmetric pre-image $x^{\varepsilon'}p^{e'}p'p^{\varepsilon'}x^{\varepsilon'}$ (here we need the extra free variables p', q', r'). We also note that since m is of weight $\geqq 1$, n contains at least one factor p, q, r or p', q', r'. As we said above, this is enough to allow us to complete the proof that $m = m^*$ falls in $\mathfrak{R}$.

Since $FJ(x, y) = FS(x, y)$ by Shirshov's Theorem, specializing $z \to 0$ gives

COROLLARY. *If $p(\lambda), q(\lambda)$ are monic polynomials then $FJ(x, y)/\mathfrak{R}$ is special for $\mathfrak{R}$ the ideal generated by $p(x), xp(x), q(y), yq(y)$.*

It is essential (in the general case where $\frac{1}{2} \in \Phi$) that we take $xp(x)$ and $yq(y)$ along with $p(x)$ and $q(y)$. Indeed, in our pathological one-generator example we divided out by x^2 but not x^3, and it was this x^3 that came back to haunt us. However, the Example of § 2 shows that the condition $p(z) \in \mathfrak{R} \Rightarrow zp(z) \in \mathfrak{R}$ is not by itself enough to guarantee speciality.

It is also essential that the relations involve only one variable at a time. The situation becomes much more complex when the variables are intermixed. For example, if $\mathfrak{R}$ in $FS(x, y, z)$ is generated by $x^2 - y^2$ then $FS(x, y, z)/\mathfrak{R}$ is not special, but it $\mathfrak{R}$ is generated by $U_xy - x$, $U_xy^2 - 1$ then $F/\mathfrak{R}$ is special. Thus speciality depends very much on the particular relations chosen.

4. Applications to Peirce decompositions. We define the *free Jordan algebra on X with n (supplementary, orthogonal) idempotents* $FJ(X; e_1, \cdots, e_n)$ to be the quotient $FJ(X \cup Y)/\mathfrak{R}$ where $Y = \{y_1, \cdots, y_n\}$ is disjoint from X and $\mathfrak{R}$ is the ideal generated by $1 - \Sigma y_i$, $y_i^2 - y_i$, $U_{y_i}y_j$, $y_i \circ y_j (i \neq j)$. The cosets $e_i = y_i + \mathfrak{R}$ are supplementary orthogonal idempotents in $FJ(X; e_1, \cdots, e_n) = FJ(X \cup Y)/\mathfrak{R}$, and one has the universal property that any map $X \to \mathfrak{J}$ of X into a Jordan algebra $\mathfrak{J}$ with n supplementary orthogonal idempotents $f_1, \cdots, f_n$ extends uniquely to a homomorphism $FJ(X; e_1, \cdots, e_n) \to \mathfrak{J}$ sending $e_i \to f_i$.

Consider the following properties of the Peirce decomposition of an arbitrary Jordan algebra $\mathfrak{J}$ relative to a supplementary family of orthogonal idempotents $e_1, \cdots, e_n$ [2, p. 120–1; 4, p. 1074–5].

(PD 0) $E_{ii} = U_{e_i}$ and $E_{ij} = U_{e_i, e_j} = E_{ji}$ form a supplementary family of orthogonal projections on $\mathfrak{J}$, so $\mathfrak{J} = \bigoplus \mathfrak{J}_{ij}$ for $\mathfrak{J}_{ij} = E_{ij}(\mathfrak{J}) = \mathfrak{J}_{ji}$,

and for elements x_{pq} of the Peirce spaces $\mathfrak{J}_{pq}$ and distinct indices i, j, k, l,

(PD 1) $x_{ii}^2 \in \mathfrak{J}_{ii}$, so $\mathfrak{J}_{ii}^2 \subset \mathfrak{J}_{ii}$

(PD 2) $x_{ij}^2 \in \mathfrak{J}_{ii} + \mathfrak{J}_{jj}$, so $\mathfrak{J}_{ij}^2 \subset \mathfrak{J}_{ii} + \mathfrak{J}_{jj}$

(PD 3) $x_{ii} \circ y_{ij} \in \mathfrak{J}_{ij}$, so $\mathfrak{J}_{ii} \circ \mathfrak{J}_{ij} \subset \mathfrak{J}_{ij}$

(PD 4) $x_{ij} \circ y_{jk} \in \mathfrak{J}_{ik}$, so $\mathfrak{J}_{ij} \circ \mathfrak{J}_{jk} \subset \mathfrak{J}_{ik}$

(PD 5) $x_{pq} \circ y_{rs} = 0$, so $\mathfrak{J}_{pq} \circ \mathfrak{J}_{rs} = 0$ if $\{p, q\} \cap \{r, s\} = \emptyset$

(PD 6) $U_{x_{ii}} y_{ii} \in \mathfrak{J}_{ii}$, so $U_{\mathfrak{J}_{ii}} \mathfrak{J}_{ii} \subset \mathfrak{J}_{ii}$

(PD 7) $U_{x_{ij}} y_{ii} \in \mathfrak{J}_{jj}$, so $U_{\mathfrak{J}_{ij}} \mathfrak{J}_{ii} \subset \mathfrak{J}_{jj}$

(PD 8) $U_{x_{ij}} y_{ij} = x_{ij} \circ U_{e_i}(x_{ij} \circ y_{ij}) - y_{ij} \circ U_{e_j}(x_{ij}^2)$, so $U_{\mathfrak{J}_{ij}} \mathfrak{J}_{ij} \subset \mathfrak{J}_{ij}$

(PD 9) $U_{x_{pq}} y_{rs} = 0$, so $U_{\mathfrak{J}_{pq}} \mathfrak{J}_{rs} = 0$ if $\{r, s\} \not\subset \{p, q\}$

(PD 10) $\{x_{ii} y_{ij} z_{jj}\} = (x_{ii} \circ y_{ij}) \circ z_{jj} = x_{ii} \circ (y_{ij} \circ z_{jj})$, so $\{\mathfrak{J}_{ii} \mathfrak{J}_{ij} \mathfrak{J}_{jj}\} \subset \mathfrak{J}_{ij}$

(PD 11) $\{x_{ii} y_{ij} z_{jk}\} = (x_{ii} \circ y_{ij}) \circ z_{jk} = x_{ii} \circ (y_{ij} \circ z_{jk})$, so $\{\mathfrak{J}_{ii} \mathfrak{J}_{ij} \mathfrak{J}_{jk}\} \subset \mathfrak{J}_{ik}$

(PD 12) $\{x_{ij} y_{jj} z_{jk}\} = (x_{ij} \circ y_{jj}) \circ z_{jk} = x_{ij} \circ (y_{jj} \circ z_{jk})$, so $\{\mathfrak{J}_{ij} \mathfrak{J}_{jj} \mathfrak{J}_{jk}\} \subset \mathfrak{J}_{ik}$

(PD 13) $\{x_{ij} y_{jk} z_{kl}\} = (x_{ij} \circ y_{jk}) \circ z_{kl} = x_{ij} \circ (y_{jk} \circ z_{kl})$, so $\{\mathfrak{J}_{ij} \mathfrak{J}_{jk} \mathfrak{J}_{kl}\} \subset \mathfrak{J}_{il}$

(PD 14) $\{x_{ij} y_{jk} z_{ki}\} = U_{e_i}\{(x_{ij} \circ y_{jk}) \circ z_{ki}\} = U_{e_i}\{x_{ij} \circ (y_{jk} \circ z_{ki})\}$, so $\{\mathfrak{J}_{ij} \mathfrak{J}_{jk} \mathfrak{J}_{ki}\} \subset \mathfrak{J}_{ii}$

(PD 17) $\{x_{ii} y_{ii} z_{ij}\} = x_{ii} \circ (y_{ii} \circ z_{ij})$, so $\{\mathfrak{J}_{ii} \mathfrak{J}_{ii} \mathfrak{J}_{ij}\} \subset \mathfrak{J}_{ij}$

(PD 18) $\{x_{ij} y_{ji} z_{ik}\} = x_{ij} \circ (y_{ji} \circ z_{ik})$, so $\{\mathfrak{J}_{ij} \mathfrak{J}_{ji} \mathfrak{J}_{ik}\} \subset \mathfrak{J}_{ik}$

(PD 19) $\{x_{pq} y_{rs} z_{tv}\} = 0$, so $\{\mathfrak{J}_{pq} \mathfrak{J}_{rs} \mathfrak{J}_{tv}\} = 0$ unless the indices may be linked

(PD 20) $U_{x_{ij}} e_i = U_{e_j} x_{ij}^2$

(PD 21) $e_i \circ y_{ij} = y_{ij}$, $x_{ii}^2 \circ y_{ij} = x_{ii} \circ (x_{ii} \circ y_{ij})$, $U_{x_{ii}} z_{ii} \circ y_{ij} = x_{ii} \circ (z_{ii} \circ (x_{ii} \circ y_{ij}))$ so that $V_{e_i} = I$, $V x_{ii}^2 = V_{x_{ii}^2}$, $V_{U(x_{ii}) z_{ii}} = V_{x_{ii}} V_{z_{ii}} V_{x_{ii}}$ on $\mathfrak{J}_{ij}$.

It is an easy matter to verify these for special Jordan algebras, since if $\mathfrak{A} = \Sigma_{i,j} \mathfrak{A}_{ij}$ is the Peirce decomposition of the associative algebra $\mathfrak{A}$ then $\mathfrak{J} = \Sigma_{i \leq j} \mathfrak{J}_{ij}$ for $\mathfrak{J}_{ij} = \mathfrak{A}_{ij} + \mathfrak{A}_{ji}$ is the Peirce decomposition of the Jordan algebra $\mathfrak{J} = \mathfrak{A}^+$.

We claim that if these relations hold in $\tilde{\mathfrak{J}} = FJ(\tilde{x}; \tilde{e}_1, \cdots, \tilde{e}_n)$ (taking $X = \{\tilde{x}\}$ to consist of one element) they hold in any $\mathfrak{J}$. (This is why there are two "missing" relations

(PD 15) $\{x_{ij} y_{jj} z_{ji}\} = U_{e_i}\{(x_{ij} \circ y_{jj}) \circ z_{ji}\} = U_{e_i}\{x_{ij} \circ (y_{jj} \circ z_{ji})\}$, so $\{\mathfrak{J}_{ij} \mathfrak{J}_{jj} \mathfrak{J}_{ji}\} \subset \mathfrak{J}_{ii}$

(PD 16) $\{x_{ii} y_{ij} z_{ij}\} = U_{e_i}\{(x_{ii} \circ y_{ij}) \circ z_{ji}\}$ so $\{\mathfrak{J}_{ii} \mathfrak{J}_{ij} \mathfrak{J}_{ji}\} \subset \mathfrak{J}_{ii}$;

these do not seem to follow from $\tilde{\mathfrak{J}}$, and must be verified directly).

The reason for this is that for any collection of elements x_{ij} from *distinct* Peirce spaces $\mathfrak{J}_{ij}$ there is an element $x = \Sigma x_{ij}$ having the x_{ij} as its Peirce ij-compoments; there is a homomorphism $\tilde{\mathfrak{J}} \to \mathfrak{J}$ sending $\tilde{x} \to x$ and $\tilde{e}_i \to e_i$, so the Peirce components $\tilde{x}_{ij}$ of $\tilde{x}$ map into the Peirce components x_{ij} of x. Hence any relation holding among the $\tilde{x}_{ij}$ will also hold for the x_{ij}. That is, any relation involving elements from *distinct* Peirce spaces will hold in $\mathfrak{J}$ if it holds in $\tilde{\mathfrak{J}}$. This immediately applies to (PD 1–5), (PD 7), (PD 9–14), (PD 19–20), and the first two parts of (PD 21). The same argument works for (PD 0): if $\tilde{I} = \Sigma \tilde{E}_{ij}$, $\tilde{E}_{ij}^2 = \tilde{E}_{ij}$, $\tilde{E}_{pq}\tilde{E}_{rs} = 0$ on $\tilde{x}$ then $I = \Sigma E_{ij}$, $E_{ij}^2 = E_{ij}$, $E_{pq}E_{rs} = 0$ an any x, so the E_{ij} are supplementary orthogonal idempotents).

The remaining formulas can be derived from the previous ones by various stratagems. For (PD 17–18) we use the relation

$$\{abb\} = a \circ b^2 \quad \{abc\} + \{acb\} = a \circ (b \circ c)$$

valid in any Jordan algebra. In (PD 18) $\{x_{ij}y_{ji}z_{ik}\} = x_{ij} \circ (y_{ji} \circ z_{ik}) - \{x_{ij}z_{ik}y_{ji}\} = x_{ij} \circ (y_{ji} \circ z_{ik})$ since $U_{\mathfrak{J}_{ij}}\mathfrak{J}_{ik} = 0$ by (PD 9), and similarly in (PD 17) since $U_{\mathfrak{J}_{ii}}\mathfrak{J}_{ij} = 0$. (This argument also shows either one of (PD 15), (PD 16) implies the other).

For (PD 6), (PD 8), and the last part of (PD 21) we use

$$\partial_y\{x^3\}|_x = U_x y + U_{x,y}x = U_x y + \{xxy\} = U_x y + x^2 \circ y .$$

Now the relations

(PD 6)′ $U_{x_{ii}}x_{ii} \in \mathfrak{J}_{ii}$
(PD 8)′ $U_{x_{ij}}x_{ij} = x_{ij} \circ U_{e_i}(x_{ij}^2)$
(PD 21)′ $V_{U(x_{ii})x_{ii}} = V_{x_{ii}}^3$ on $\mathfrak{J}_{ij}$

will be inherited from $\tilde{\mathfrak{J}}$, and this remains true over any scalar extension Ω of Φ, so we can linearize to get

$$U_{x_{ii}}y_{ii} + x_{ii}^2 \circ y_{ii} \in \mathfrak{J}_{ii}$$
$$U_{x_{ij}}y_{ij} + x_{ij}^2 \circ y_{ij} = y_{ij} \circ U_{e_i}(x_{ij}^2) + x_{ij} \circ U_{e_i}(x_{ij} \circ y_{ij})$$
$$V_{U(x_{ii})z_{ii}} + V_{x_{ii}^2 \circ z_{ii}} = V_{x_{ii}}V_{z_{ii}}V_{x_{ii}} + V_{x_{ii}}^2 V_{z_{ii}} + V_{z_{ii}}V_{x_{ii}}^2 .$$

The first of these implies (PD 6) via (PD 1), the second implies (PD 8) via (PD 2), and the third implies (PD 21) since we already know $V_{x_{ii}^2} = V_{x_{ii}}^2$ and so $V_{x_{ii} \circ y_{ii}} = V_{x_{ii}}V_{y_{ii}} + V_{y_{ii}}V_{x_{ii}}$.

Thus the task of verifying Peirce relations for an arbitrary Jordan algebra $\mathfrak{J}$ reduces to verifying them for the free Jordan algebra $\tilde{\mathfrak{J}}$ on one generator with idempotents. The whole point of this reduction is that $\tilde{\mathfrak{J}}$ is *special*, and we already remarked that the relations were easily verified in any special algebra.

THEOREM. *The free Jordan algebra $FJ(x; e_1, \cdots, e_n)$ on one generator with n supplementary orthogonal idempotents is special.*

To show $FJ(x; e_1, \cdots, e_n) = FJ_\Phi(x; e_1, \cdots, e_n)$ is special it will be enough if it is imbedded in a special algebra $FJ_\Phi(x; e_1, \cdots, e_n)_\Omega = FJ_\Omega(x; e_1, \cdots, e_n)$. We choose Ω as follows. Consider the polynomial ring $\Phi[\lambda_1, \cdots, \lambda_n]$. The element $\mu = \prod_{i<j} (\lambda_i - \lambda_j)$ is homogeneous in the λ's and the coefficient of $\lambda_1^{n-1}\lambda_2^{n-2} \cdots \lambda_{n-1}^1$ in μ is 1, so μ is not a zero divisor in $\Phi[\lambda_1, \cdots, \lambda_n]$. This guarantees Φ is imbedded in $\Omega = \Phi[\lambda_1, \cdots, \lambda_n][1/\mu]$; the important thing about Ω is that each $\lambda_i - \lambda_j$ is invertible in Ω. Since μ is not a zero-divisor in

$$FJ_\Phi(X; e_1, \cdots, e_n) \otimes \Phi[\lambda_1, \cdots, \lambda_n] ,$$

$FJ_\Phi(X; e_1, \cdots, e_n)$ is imbedded in $FJ_\Phi(X; e_1, \cdots, e_n)_\Omega = FJ_\Omega(X; e_1, \cdots, e_n)$.

PROPOSITION. *For any X, $FJ_\Omega(X; e_1, \cdots, e_n) \cong FJ_\Omega(X, y)/\mathfrak{K}$ where $\mathfrak{K}$ is the ideal generated by $p(y) = \prod (y - \lambda_i 1)$ and $yp(y)$.*

Proof. Consider the polynomials $p(\lambda) = \prod (\lambda - \lambda_i)$ and $p_i(\lambda) = \prod_{j \neq i} (\lambda - \lambda_j)/\prod_{j \neq i} (\lambda_i - \lambda_j)$ in Ω. We have $p_i(\lambda_i) = 1$, $p_i(\lambda_j) = 0$ if $j \neq i$. Therefore $1 - \sum p_i(\lambda)$ is of degree $\leqq n - 1$ yet has n roots $\lambda_1, \cdots, \lambda_n$, so it must be identically zero, and similarly for $\lambda = \sum \lambda_i p_i(\lambda)$:

$$\sum p_i(\lambda) = 1, \ \sum \lambda_i p_i(\lambda) = \lambda .$$

(We always assume $n > 1$ since for $n = 1$ $FJ(X; e_1) = FJ(X; 1) = FJ(X)$ has only the trivial idempotent $e_1 = 1$). Also

$$U_{p_i(\lambda)} p_j(\lambda) = p_i(\lambda)^2 p_j(\lambda), \ p_i(\lambda) \circ p_j(\lambda) = 2p_i(\lambda)p_j(\lambda),$$
$$p_i(\lambda)^2 - p_i(\lambda) = p_i(\lambda)^2 - \sum p_i(\lambda)p_j(\lambda) = \sum_{j \neq i} p_i(\lambda)p_j(\lambda)$$

are all divisible by $p(\lambda)$ and belong to the (Jordan) ideal generated by $p(\lambda)$ and $\lambda p(\lambda)$.

These conditions imply that the elements $\tilde{e}_i = p_i(y)$ in $FJ_\Omega(X, y)$ satisfy $\sum \tilde{e}_i = 1$, $\sum \lambda_i \tilde{e}_i = y$, $U_{\tilde{e}_i} \tilde{e}_j \in \mathfrak{K}$, $\tilde{e}_i \circ \tilde{e}_j \in \mathfrak{K}$, $\tilde{e}_i^2 - \tilde{e}_i \in \mathfrak{K}$, so the cosets $e_i = \tilde{e}_i + \mathfrak{K}$ in $FJ_\Omega(X, y)/\mathfrak{K}$ form a supplementary family of orthogonal idempotents. (Note $p_i(y)$ is defined since we are allowed to divide by $\lambda_i - \lambda_j$ in Ω). We show $FJ_\Omega(X, y)/\mathfrak{K}$ is isomorphic to $FJ_\Omega(X; e_1, \cdots, e_n)$ by showing it has the universal property of the latter. Given any map φ of X into a Jordan algebra $\mathfrak{J}$ with idempotents $f_1, \cdots, f_n$ we have a homomorphism $FJ_\Omega(X, y) \to \mathfrak{J}$ sending $x \to \varphi(x)$, $y \to \sum \lambda_j f_j$. Then $\tilde{e}_i = p_i(y)$ is mapped into

$$p_i(\sum \lambda_j f_j) = \sum p_i(\lambda_j) f_j = f_i ,$$

$p(y)$ into $p(\sum \lambda_j f_j) = \sum p(\lambda_j) f_j = 0$, and $yp(y)$ into $\sum \lambda_j p(\lambda_j) f_j = 0$. Since $p(y)$ and $yp(y)$ generate $\mathfrak{K}$ we have an induced homomorphism

$$FJ_\Omega(X, y)/\mathfrak{K} \longrightarrow \mathfrak{J}$$

sending $e_i \to f_i$. The uniqueness follows since $FJ_\Omega(X, y)/\mathfrak{K}$ is generated over Ω by X and the e_i (because $\sum \lambda_i e_i = y$).

Applying the Proposition when $X = \{x\}$, we have

$$FJ_\Omega(x; e_1, \cdots, e_n) \cong FJ_\Omega(x, y)/\mathfrak{K}$$

where $\mathfrak{K}$ is generated by $p(y)$ and $yp(y)$. By the Corollary to the Theorem of the previous Section (with $q(\lambda) = 0$), $FJ_\Omega(x, y)/\mathfrak{K}$ is special. Therefore $FJ(x; e_1, \cdots, e_n) \subset FJ_\Omega(x; e_1, \cdots, e_n)$ is special too, completing the proof of the theorem.

The algebra $FJ(x, y; e_1, \cdots, e_n)$ on two generators is no longer special, since it has the exceptional algebra $\mathfrak{H}(\mathfrak{C}_3)$ as a homomorphic image ($\mathfrak{C}$ a Cayley algebra); indeed, the exceptional algebra can be generated by two elements x, y and the idempotents e_1, e_2, e_3 [2, ex. 1 p. 51].

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