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ON NONNEGATIVE MATRICES

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The following characterisation of totally indecomposable nonnegative n-square matrices is introduced: A nonnegative n-square matrix is totally indecomposable if and only if it diminishes the number of zeros of every n-dimensional nonnegative vector which is neither positive nor zero. From this characterisation it follows quite easily that:

I. The class of totally indecomposable nonnegative n-square matrices is closed with respect to matrix multiplication.

II. The $(n-1)$-st power of a matrix of that class is positive.

A very short proof of two equivalent versions of the König-Frobenius duality theorem on $(0, 1)$-matrices is supplied at the end.

A matrix is called *nonnegative* or *positive* according as all its elements are nonnegative or positive respectively. An n-square matrix A is said to be *decomposable* if there exists a permutation matrix P such that $PAP^T = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$, where B and D are square matrices; otherwise it is *indecomposable*. A is said to be *partly decomposable* if there exist permutation matrices P, Q such that

$$PAQ = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix}, \text{ where } B \text{ and } D \text{ are square}$$

matrices; otherwise it is *totally indecomposable.*

Whereas the notion of indecomposable matrices first appeared in 1912 in a paper by Frobenius [2] dealing with the spectral properties of nonnegative matrices, totally indecomposable matrices were introduced fairly recently apparently by Marcus and Minc [10]. Their properties have been studied in several papers on inequalities for the permanent function.

In [11] Minc gives the following characterisation of totally indecomposable matrices:

A nonnegative n-square matrix A, $n \geqq 2$, is totally indecomposable if and only if every $(n-1)$-square submatrix of A has a positive permanent.

A well-known theorem states:

THEOREM 1. *If A is an indecomposable nonnegative n-square matrix then*

$$(A+I)^{n-1} > 0 \ [3], [9] .$$

An indecomposable matrix is primitive if its characteristic value of maximum modulus is unique.

Wielandt [15] states (without proof) that for primitive n-square matrices we have

$$A^{n^2-2n+2} > 0 .$$[1]

By using solely the properties of total indecomposability we establish a different characterisation for totally indecomposable matrices from the one given by Minc. Using part of the characterisation we show that if A is a totally indecomposable nonnegative n-square matrix then $A^{n-1} > 0$. This result is best possible as for every n there exist totally indecomposable n-square matrices A for which $A^{n-2} \not> 0$. Theorem 1 then follows as a corollary of the latter result.

We should like to point out that Theorem 2 is by no means essential for the proof of Theorem 3. Two independent proofs of Theorem 3 are given in § 4. It seems justified however to present Theorem 2 on its own merit.

We conclude with a very short proof of two equivalent versions of König's theorem on matrices.

2. Preliminaries. $|S|$ denotes the number of elements of a given set S. Let M_n be the set of all nonnegative n-square matrices, let D_n be the subset of M_n of indecomposable matrices and let T_n be the subset of D_n of totally indecomposable matrices. Let $A \in M_n$ and let p and q be nonempty subsets of $N = \{1, \cdots, n\}$. Then $A[p \mid q]$, $A(p \mid q)$ is the $|p| \times |q|$ submatrix of A consisting precisely of those elements a_{ij} of A for which $i \in p$ and $j \in q$, $i \notin p$ and $j \notin q$ respectively. $A[p \mid q)$ and $A(p \mid q]$ are defined accordingly. We can now formulate equivalent definitions for matrices in D_n and T_n:

D. 1. *$A \in D_n$ if $A[p \mid N-p] \neq 0$ for every nonempty $p \subset N$.*

D. 2. *$A \in T_n$ if $A[p \mid q] \neq 0$ for any nonempty subsets p and q of N such that $|p| + |q| = n$.*

Let us now establish some connections between indecomposable and totally indecomposable matrices.

LEMMA 1. *If $A \in (D_n - T_n)$ then A has a zero on its main diagonal.*[2]

Proof. Since $A \notin T_n$ there exists a zero-submatrix $A[p \mid q]$ with $|p| + |q| = n$; but since $A \in D_n$, $p \cap q \neq \varnothing$, which means that A has

[1] A proof is supplied in [5].

[2] Lemma 1 is part of Lemma 2.3 in [1] but the shortness of our proof seems to justify its presentation.

a zero on its main diagonal.

COROLLARY 1. *If* $A \in D_n$ *then* $A + I \in T_n$.

Proof obvious.

3. **The main results.** Let $A = (a_{ij}) \in M_n$ and let v denote an n-dimensional vector with $a_i(v)$ its ith entry.

Define: $J_k = \{j: a_{kj} = 0\}$, $I_k = \{i: a_{ik} = 0\}$,

$$I_0(v) = \{i: a_i(v) = 0\}\ ,\quad I_+(v) = \{i: a_i(v) > 0\}\ .$$

Let R_n denote the space of n-tuples of real numbers.

Let X_n be the set of all nonnegative vectors in R_n which are neither positive nor zero. We then have the following

THEOREM 2. *A nonnegative n-square matrix A is totally indecomposable if and only if* $|I_0(Ax)| < |I_0(x)|$ *for every* $x \in X_n$.

Proof. Let $A \in T_n$ and $x \in X_n$. A necessary and sufficient condition for $a_{i_0}(Ax) = 0$ for some i_0 is

$$I_+(x) \subseteq J_{i_0}\ . \tag{1}$$

If $I_0(Ax) = \varnothing$, then there is nothing to prove, so we may assume

$$I_0(Ax) \neq \varnothing\ . \tag{2}$$

$x \in X_n$ implies

$$I_+(x) \neq \varnothing\ . \tag{3}$$

(1), (2) and (3) imply that $A[I_0(Ax) | I_+(x)]$ is a zero-submatrix of A. Since $A \in T_n$ by assumption, we have (by D. 2.)

$$|I_0(Ax)| + |I_+(x)| < n = |I_0(x)| + |I_+(x)|$$

and hence $|I_0(Ax)| < |I_0(x)|$ which proves the first part of the theorem. (It is not generally true however that $I_0(Ax) \subseteq I_0(x)$ as it may happen that $a_i(x) > 0$ and $a_i(Ax) = 0$, a situation which differs somewhat from that in the similar case for indecomposable matrices (5.2.2 in [9])).

Let now $A \notin T_n$. Then A contains a zero-submatrix $A[I | J]$ such that $I, J \neq \varnothing$ and $|I| + |J| = n$. Choose now $x \in R_n$ such that

$$I_+(x) = J\ . \tag{4}$$

Then clearly $x \in X_n$. We have $I_0(x) = N - I_+(x) = N - J$, and hence $|I_0(x)| = |I|$. For $i \in I$ we have $J_i \supseteq J$, and hence by (4) $I_+(x) \subseteq J_i$, so that for $i \in I$ according to (1) $a_i(Ax) = 0$ and hence $I_0(Ax) \supseteq I$. Then $|I_0(Ax)| \geqq |I| = |I_0(x)|$. This completes the proof.

X_n in Theorem 2 may of course be replaced by its subset Y_n consisting of the $2^n - 2$ zero-one vectors.

Theorem 2 admits of two simple corollaries which we present as Theorems 3 and 4.

THEOREM 3. *If A is a totally indecomposable nonnegative n-square matrix then*

$$A^{n-1} > 0 .$$

Proof. If for some j_0 we had $|I_{j_0}| \geqq n - 1$ then A would be partly decomposable and hence $|I_{j_0}| \leqq n - 2$ for $j \in N$ and the rest follows.

Theorem 1 follows from Theorem 3 as an immediate consequence of Corollary 1. For $A = I + P$ where P is the n-square permutation matrix with ones in the superdiagonal, so that $a_{ij} = 1$ if $i = j$ or $i = j - 1$, $a_{n1} = 1$ and $a_{ij} = 0$ otherwise, it is easy to show that $A^{n-2} \not> 0$, which shows that our result is best possible.

THEOREM 4. *The product of any finite number of totally indecomposable nonnegative n-square matrices is totally indecomposable.*

Proof. It is clearly sufficient to prove the statement for two matrices. Let therefore $A, B \in T_n$. Choose an arbitrary element x of X_n. We then have

$$|I_0(ABx)| \leqq |I_0(Bx)| < |I_0(x)| \tag{5}$$

by Theorem 2. Since x was arbitrary, (5) applies to all elements of X_n. Again by Theorem 2 it follows that AB is totally indecomposable, which proves the theorem.

4. Independent proofs of Theorem 3. A lemma of Gantmacher [3] states that if $A \in D_n$ and $x \in X_n$, then $I_0[(A + I)x] \subset I_0(x)$.

The following proof of Theorem 3 assuming the lemma has been suggested by London[3]: Let $A \in T_n$. Using the fact that a matrix in T_n possesses a positive diagonal d, put

$$A_1 = \frac{1}{\alpha} P^T(A - \alpha P) = \frac{1}{\alpha} \quad P^T A - I \text{ where } \quad 0 < \alpha < \min a_{ij}(a_{ij} \in d)$$

[3] D. London, oral communication.

and $P = (p_{ij})$ is an n-square permutation matrix such that $p_{ij} = 1$ if and only if $a_{ij} \in d$. Then $A \in T_n$ implies $A_1 \in T_n$.

We have $A = \alpha P(A_1 + I)$; since $A_1 \in D_n$ we obtain

$$I_0(Ax) = I_0[P(A_1 + I)x] = I_0[(A_1 + I)x] \subset I_0(x) ,$$

for $x \in X_n$. Then $I_0(A^{n-1}x) = \varnothing$, and $A^{n-1} > 0$.

Another proof has been kindly suggested by the referee of this paper: We show that if A is totally indecomposable, then if $x \in X_n$, then

$$| I_0(Ax) | < | I_0(x) | .$$

The theorem then follows immediately.

Suppose $| I_0(Ay) | \geqq | I_0(y) |$ for some $y \in X_n$.

Put $| I_0(y) | = s$. There are permutation matrices P and Q such that

$$PAy = \begin{bmatrix} 0 \\ u \end{bmatrix} \quad \text{and} \quad Q^T y = \begin{bmatrix} 0 \\ v \end{bmatrix}$$

where u is an $(n - s)$-dimensional nonnegative victor and v is an $(n - s)$-dimensional positive vector: The 0's represent s zero components in each case.

We now write $PAQ = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$ where A_1 is $s \times s$, A_2 is $s \times (n-s)$, A_3 is $(n-s) \times s$ and A_4 is $(n-s) \times (n-s)$. Then $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} 0 \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix}$ and so $A_2 V = 0$. Thus $A_2 = 0$ and hence $A \notin T_n$, a contradiction.

5. **König's Theorem.** Let A be an $m \times n$ matrix. A *covering* of A is a set of *lines* (rows or columns) containing all the positive elements of A. A covering of A is a *minimal covering* of A if there does not exist a covering of A consisting of fewer lines. Let $M(A)$ denote the number of lines in a minimal covering of A. A basis of A is a positive subdiagonal of A of maximal length. $m(A)$ denotes the length of a basis of A. The jth column of A is *essential* to A if $M(A(\varnothing J)) < M(A)$.

We now give the two versions of König's Theorem and their proofs:

K. T. 1. *If A is an $m \times n$ matrix, then $m(A) = M(A)$.*

K. T. 2. *If A is an n-square matrix, then A has k zeros on every diagonal $(k > 0)$ if and only if A contains an $s \times t$ zero-submatrix with $s + t = n + k$.*

This is a generalized version of a theorem of Frobenius. The following theorem appears in [8] (we reproduce it here in a hypothetical form).

E. T.: *If A is an $m \times n$ matrix and* K. T. I. *holds for A, then there exists a minimal covering of A* (called *essential covering*) *containing precisely the essential columns of A* (and may be some rows).

Proof of K. T. 1. $m(A) \leqq M(A)$ holds trivially. The theorem is clearly true for $1 \times n$ matrices for all n. Assume that the theorem is true for all $\mu \times n$ matrices, $\mu < m$ and all n. Let A be an $m \times n$ matrix. Consider $A' = A(\{m\} \mid N]$. A' is an $(m-1) \times n$ matrix so that K. T. 1, holds for A' and hence E. T. holds for A'. Let Q be the essential covering of A'.

Case 1. Q is a covering of A. Then $m(A) \geqq m(A') = M(A') \geqq M(A)$.

Case 2. Q is not a covering of A. Then there exists $j_0 \in N$ for which $a_{mj_0} > 0$ which is not covered by Q and hence the j_0th column is not essential to A'. Then clearly there exists a basis b' of A' without elements in the j_0th column. Then $b = b' \cup \{a_{mj_0}\}$ is a subdiagonal of A and hence $M(A) \leqq M(A') + 1 = m(A') + 1 \leqq m(A)$. This proves K. T. 1.

Proof of K. T. 2. *Necessity*. If A has k zeros on every diagonal then $m(A) \leqq n - k$. By K. T. 1, $M(A) \leqq n - k$. Apply a minimal covering to A. Then there remains an $s \times t$ zero-matrix of A which is not covered, with $s + t \geqq 2n - M(A) \geqq n + k$.

Sufficiency. Let A contain an $s \times t$ zero-submatrix with $s + t = n + k$. Then there are positive elements on at most $2n - (n + k) = n - k$ lines, meaning that there are at least k zero-rows, which proves the sufficiency.

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