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DIFFERENTIAL SIMPLICITY AND COMPLETE INTEGRAL CLOSURE

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Let R be an integral domain containing the rational numbers, and let R' denote the complete integral closure of R. It is shown that if R is differentiably simple, then R need not be equal to R', even when R is Noetherian, and then the relationship between R and R' is studied.

Let $\mathscr{D}$ be any set of derivations of R. Seidenberg has shown that the conductor $C = \{x \in R \mid xR' \subset R\}$ is a $\mathscr{D}$-ideal of R, so that when R is $\mathscr{D}$-simple and $C \neq 0$, then $R = R'$. We investigate here the situation when $C = 0$.

The first observation that one must make is that it is no longer true that $R = R'$ when R is differentiably simple, even when R is Noetherian. We show this in Example 2.2 where we construct a 1-dimensional local domain containing the rational numbers which is differentiably simple but not integrally closed. This counterexamples a conjecture of Posner [4, p. 1421] and also answers affirmatively a question of Vasconcelos [6, p. 230].

Thus, it is not a redundant task to study the relationship between a differentiably simple ring R and its complete integral closure. An important tool in this study is the technique of § 3 which associates to any prime ideal P of R containing no D-ideal a rank-1, discrete valuation ring centered on P; by means of this, we show in Theorem 3.2 that over such a prime ideal P of R there lies a unique prime ideal of R'. When R is a Noetherian $\mathscr{D}$-simple ring with $\{P_\alpha\}_{\alpha \in \Lambda}$ as set of minimal prime ideals, Theorem 3.3 asserts that $R' = \bigcap_{\alpha \in \Lambda} \{R_\alpha \mid R_\alpha$ is the valuation ring associated with the minimal prime ideal $P_\alpha\}$; Corollary 3.5 asserts that R' is the largest $\mathscr{D}$-simple overring of R having a prime ideal lying over every minimal prime ideal of R.

1. Preliminaries. Our notation and terminology adhere to that of Zariski-Samuel [7] and [8]. Throughout the paper we use R to denote a commutative ring with 1, K to denote the total quotient ring of R, and A to denote an ideal of R; A is proper if $A \neq R$. A derivation D of R is a map of R into R such that

$$D(a + b) = D(a) + D(b) \quad \text{and} \quad D(ab) = aD(b) + bD(a)$$

for all $a, b \in R$.

Such a derivation can be uniquely extended to K, and we shall also denote the extended derivation by D. D is said to be regular on a subring S of K if $D(S) \subset S$. If $\mathscr{D}$ is a family of derivations of R, A is called a $\mathscr{D}$-ideal if $D(A) \subset A$ for every $D \in \mathscr{D}$; when $\mathscr{D} = \{D\}$, we merely say D-ideal. If R has no $\mathscr{D}$-ideal different from (0) and (1), R is said to be $\mathscr{D}$-simple. We use $D^{(0)}(x)$ to denote x, and for $n \geqq 1$ $D^{(n)}(x)$ to denote $D(D^{(n-1)}(x))$, i.e. the n^{th} derivative of x; by induction one proves Leibnitz's rule:

$$D^{(n)}(ab) = \sum_{i=0}^{n} C_n^i D^{(n-i)}(a) D^{(i)}(b) .$$

We assume henceforth that $\mathscr{D}$ is a family of derivations of R and that $D \in \mathscr{D}$. Let $\varphi: R \to S$ be a homomorphism onto; then

$$D'(\varphi(r)) = \varphi(D(r))$$

defines a derivation D' on S if and only if the kernel I of φ is a D-ideal. Suppose that I is a $\mathscr{D}$-ideal, and write $\mathscr{D}'$ to denote the set of derivations of S thus induced by $\mathscr{D}$; if A is a $\mathscr{D}$-ideal of R, then $\varphi(A)$ is a $\mathscr{D}'$-ideal of S, and conversely if B is a $\mathscr{D}'$-ideal of S, then $\varphi^{-1}(B)$ is a $\mathscr{D}$-ideal of R containing I. Thus, in particular, if A is a maximal proper $\mathscr{D}$-ideal of R, then R/A is $\mathscr{D}'$-simple.

LEMMA 1.1. *Let D be a derivation of R, M a multiplicative system of R, and $h: R \to R_M$ the canonical homomorphism. Then, we can define a derivation on R_M, which we also call D, by*

$$D(h(r)(h(m))^{-1}) = [h(m)h(D(r)) - h(r)h(D(m))](h(m^2))^{-1} .$$

Furthermore, if A is a D-ideal of R, then $h(A)R_M$ is a D-ideal of R_M, and if B is a D-ideal of R_M, then $h^{-1}(B)$ is a D-ideal of R.

Proof. $\ker h = \{x \in R \mid xm = 0 \text{ for some } m \in M\}$ is a D-ideal of R since $0 = D(xm) = xD(m) + mD(x) = xmD(m) + m^2D(x) = m^2D(x)$. Hence D induces a derivation on $R/\ker h$, a derivation which can be then extended to R_M. The remainder of the lemma is straightforward.

LEMMA 1.2. *Let $\mathscr{D}$ be a family of derivations of R, and suppose that R contains the rational numbers. Then, the radical of a $\mathscr{D}$-ideal of R is a $\mathscr{D}$-ideal.*

Proof. See [2, Lemma 1.8, p. 12].

COROLLARY 1.3. *If P is a minimal prime divisor of a $\mathscr{D}$-ideal*

A, and P does not contain an integer $\neq 0$, then P is a $\mathscr{D}$-ideal.

Proof. Localize at P and apply 1.1 and 1.2.

THEOREM 1.4. *Let A be a maximal proper $\mathscr{D}$-ideal of R, then*
(i) *A is primary.*
(ii) *If R/A has characteristic $p \neq 0$, then $\sqrt{A}$ is a maximal ideal.*
(iii) *If R/A has characteristic 0, then A is prime.*

Proof. (i) Suppose $x, y \in R, x \notin A$ and $xy \in A$; then, $\bigcup_{n=0}^{\infty}(A: y^n) \supset A: y > A$. But $\bigcup_{n=0}^{\infty}(A: y^n)$ is a $\mathscr{D}$-ideal; hence, by the maximality of A, $\bigcup_{n=0}^{\infty}(A: y^n) = R$ and there exists n such that $y^n \in A$.

(ii) Let P be a maximal ideal of R containing A. Consider the ideal $B = (A, \{x^p \mid x \in P\}) \subset P$; since R/A has characteristic p, B is a $\mathscr{D}$-ideal; hence, by the maximality of $A, B = A$ and $P = \sqrt{A}$.

(iii) Since R/A has characteristic $0, A$ contains no integer other than 0, hence the prime ideal $P = \sqrt{A}$ contains no integer either, and by 1.3 P is a $\mathscr{D}$-ideal. Then, by the maximality of $A, P = A$.

COROLLARY 1.5. *Let R be of characteristic 0. Then R is $\mathscr{D}$-simple if R contains the rational numbers and has no prime $\mathscr{D}$-ideal different from* (0) *and* (1). *If R is $\mathscr{D}$-simple, then R is a domain.*

One should note that a $\mathscr{D}$-simple ring R always contains a field, namely $F = \{x \in R \mid D(x) = 0 \text{ for all } D \in \mathscr{D}\}$; moreover, if the characteristic of R is $p \neq 0$, 1.4 shows that R is a primary ring and hence is equal to its total quotient ring; so this case will not be of interest in our further considerations, and throughout the remainder of this section we shall be dealing with a $\mathscr{D}$-simple ring of characteristic 0, which is then a domain containing the rational numbers.

DEFINITION 1.6. Let R be a domain with quotient field K. An element $x \in K$ is said to be quasi-integral over R if there exists an element $d \in R, d \neq 0$, such that $dx^n \in R$ for all $n \geqq 1$. The set R' of all elements of K that are quasi-integral over R is a ring, called the complete integral closure of R. R is said to be completely integrally closed if $R = R'$. Note that if R is Noetherian, the concepts of integral dependence and quasi-integral dependence over R for elements of K become the same.

LEMMA 1.7. *Let R be a domain with quotient field K, S a ring* *such that $R \subset S \subset K$, and $\mathscr{D}$ a family of derivations of R regular on S. Then S is $\mathscr{D}$-simple if R is $\mathscr{D}$-simple.*

Proof. If B is any $\mathscr{D}$-ideal of S, then $B \cap R$ is a $\mathscr{D}$-ideal of R, and if B is different from (0) then $B \cap R$ is also different from (0) since $S \subset K$.

THEOREM 1.8. *Let R be a domain of characteristic 0 and R' its complete integral closure. Then R' is $\mathscr{D}$-simple if R is $\mathscr{D}$-simple.*

Proof. By [5, p. 168], any $D \in \mathscr{D}$ is regular on R', hence the theorem follows from 1.7.

2. Example of a 1-dimensional local ring which is D-simple but not integrally closed. First, in this section, we modify an idea of Akizuki in [1] to construct some 1-dimensional local ring R of arbitrary characteristic such that the integral closure $\bar{R}$ is not a finite R-module.

THEOREM 2.1. *Let k be a field of arbitrary characteristic, Y an indeterminate over $k, \pi = a_1 Y + a_2 Y^3 + \cdots + a_r Y^{2^r-1} + \cdots$ an element of $k[[Y]]$ which is transcendental over $k[Y]$*[1]. *Set*

$$\theta_1 = \pi Y^{-1}, \theta_r = (\theta_{r-1} - a_{r-1}) Y^{-2^{r-1}}$$

for $r \geqq 2$ (alternatively $\theta_r = a_r + a_{r+1} Y^{2^r} + \cdots + a_s Y^{2^s - 2^r} + \cdots$); for $r \geqq 1$, set

$$t_r = (\theta_r - a_r)^2 \quad \textit{and} \quad \pi_r = \pi - (a_1 Y + \cdots + a_r Y^{2^r-1}).$$

Set also $T = k[Y, \pi, t_1, t_2, \cdots, t_r \cdots]$ and $P = (Y, \pi)T$. Note that $T \subset k[[Y]]$ and that $P \subset Yk[[Y]]$. Then,

(i) *For $r > 1, t_{r-1} = Y^{2^r}(a_r^2 + t_r) + 2a_r Y \pi_r$ and P is a maximal ideal of T.*

(ii) *For $r \geqq 1, \pi_r^2 = Y^{2^{r+1}-2}\mathrm{tr}$ and $k(Y, \pi)$ is the quotient field of T.*

(iii) *The ring $R = T_P$ is a 1-dimensional local domain.*

(iv) *The integral closure $\bar{R}$ of R is not a finite R-module.*

Proof. (i) For $r > 1$, we have

$$t_{r-1} = (\theta_{r-1} - a_{r-1})^2 = (Y^{2^{r-1}}\theta_r)^2 = Y^{2^r}(a_r^2 + t_r) + 2a_r Y^{2^r}(\theta_r - a_r)\,.$$

But

$$Y^{2^r}(\theta_r - a_r) = Y[\pi - (a_1 Y + \cdots + a_r Y^{2^r-1})] = Y\pi_r\,,$$

[1] Such an element exists; take for example $\pi = a_1 Y + a_2 Y^3 + \cdots + a_r Y^{2^{r!}-1} + \cdots$ with $a_r \neq 0$ for every $r \geqq 1$.

hence $t_{r-1} = Y^{2^r}(a_r^2 + t_r) + 2a_r Y\pi_r$. Since furthermore $P \subset Yk[[Y]]$, $1 \notin P$, and P is a maximal ideal of T.

(ii)

$$\begin{aligned}\pi_r &= \pi - (a_1 Y + \cdots + a_r Y^{2^r-1}) \\ &= Y^{2^r-1}(a_{r+1}Y^{2^r} + \cdots + a_{r+\ell}Y^{2^{r+\ell}-2^r} + \cdots) \\ &= Y^{2^r-1}(\theta_r - a_r)\ ;\end{aligned}$$

thus $\pi_r^2 = Y^{2^{r+1}-2}t_r$ and $k(Y, \pi)$ is the quotient field of T.

(iii) Let us show that Y belongs to every nonzero prime ideal of R. Since $k(Y, \pi)$ is the quotient field of R it suffices to show that $R[Y^{-1}] = k(Y, \pi)$. Let $\beta \in k[Y, \pi]$; then $\beta = \sum_{i=0}^n s_i\pi^i$ with $s_i \in k[Y]$. For any integer $r \geqq 1$, set $f_r = \sum_{i=0}^n s_i(a_1 Y + \cdots + a_r Y^{2^r-1})^i$; then

$$f_{r+1} = \sum_{i=0}^n s_i(a_1 Y + \cdots + a_r Y^{2^r-1} + a_{r+1}Y^{2^{r+1}-1})^i = f_r + Y^{2^{r+1}-1}h_{r+1}$$

with $h_{r+1} \in k[Y]$, and since $2^{r+1} - 1 > r$, we have $f_r = b_0 + b_1 Y + \cdots + b_r Y^r + Y^{r+1}g_r$ and

$$f_{r+1} = b_0 + b_1 Y + \cdots + b_r Y^r + b_{r+1}Y^{r+1} + Y^{r+2}g_{r+1}$$

with $b_0, \cdots, b_r, b_{r+1} \in k$ and $g_r, g_{r+1} \in k[Y]$. Now, since

$$\pi = \pi_r + (a_1 Y + \cdots + a_r Y^{2^r-1}), \qquad \beta = \sum_{i=0}^n s_i\pi^i = \pi_r\delta_r + f_r$$

with $\delta_r \in T$. Hence, there exists $b_0, b_1, \cdots, b_r, \cdots \in k$, $\delta_1, \cdots, \delta_r, \cdots \in T$ and $g_1, \cdots, g_r, \cdots \in k[Y]$ such that

$$(*) \qquad \beta = \sum_{j=0}^r b_j Y^j + \pi_r\delta_r + Y^{r+1}g_r\ .$$

Note that $\pi_r \in P$ and therefore that π_r is a nonunit in R.

If $b_0 \neq 0$, with $r = 1$, the relation (*) gives that $\beta = b_0 + (b_1 Y + \pi_1\delta_1 + Y^2 g_1)$ is a unit in R and thus that $\beta^{-1} \in R \subset R[Y^{-1}]$.

If $b_0 = b_1 = \cdots = b_{r-1} = 0$ and $b_r \neq 0$, the relation (*) gives $\beta = Y^r(b_r + Yg_r) + \pi_r\delta_r$ where $w_r = b_r + Yg_r$ is a unit in R; then

$$\beta(Y^r w_r - \pi_r\delta_r) = Y^{2r}w_r^2 - \pi_r^2\delta_r^2 = Y^{2r}(w_r^2 - Y^{2^{r+1}-2r-2}t_r\delta_r^2)$$

where $w_r^2 - Y^{2^{r+1}-2r-2}t_r\delta_r^2$ is a unit in R, so that $\beta^{-1} \in R[Y^{-1}]$.

If $b_r = 0$ for every $r \geqq 0$, then by the relation (*) we have

$$\beta \in \bigcap_{r=1}^\infty (\pi_r, Y^{r+1})T \subset \bigcap_{r=1}^\infty Y^{r+1}k[[Y]] = (0)\ .$$

Thus, if $\beta \in k[Y, \pi]$, either $\beta^{-1} \in R[Y^{-1}]$ or $\beta = 0$. If $\eta \in k(Y, \pi)$, then $\eta = \nu\lambda^{-1}$ with $\nu, \lambda \in k[Y, \pi], \lambda \neq 0$, so that $\eta \in R[Y^{-1}]$; hence $R[Y^{-1}] = k(Y, \pi)$.

Now,

$$\pi^2 = (Y\theta_1)^2 = [a_1 Y + (\theta_1 - a_1) Y]^2 = (t_1 - a_1^2) Y^2 + 2a_1 Y\pi$$

so that $Y^{-1} \in R[\pi^{-1}]$, $k(Y, \pi) = R[Y^{-1}] \subset R[\pi^{-1}]$, and π belongs also to every nonzero prime ideal of R. Thus $PR = (Y, \pi)R$, which is the unique maximal ideal of R and which is contained in every nonzero prime ideal of R, is the only nonzero prime ideal of R. As furthermore PR is finitely generated, R is a 1-dimensional local ring.

(iv) First, let us show that $\theta_1 = \pi Y^{-1} \notin T$. Suppose that $\theta_1 \in T = k[Y, \pi, t_1, \cdots, t_r, \cdots]$; then $\theta_1 = f(\pi, t_1, \cdots, t_\ell)$ where f is a polynomial in $\ell + 1$ indeterminates over $k[Y]$. For $r < \ell$, by (i), t_r can be expressed as a linear combination of 1, t_ℓ and π with coefficients in $k[Y]$, hence $\theta_1 = f(\pi, t_1, \cdots, t_\ell) = F(\pi, t_\ell) = F(Y\theta_1, (\theta_\ell - a_\ell)^2)$ where F is a polynomial in two indeterminates over $k[Y]$. Furthermore, by definition $\theta_{r-1} = Y^{2^{r-1}}\theta_r + a_{r-1}$, hence $\theta_1 = Y^{2^\ell-2}\theta_\ell + \beta_\ell$ with $\beta_\ell \in k[Y]$ and we have

$$(**) \qquad Y^{2^\ell-2}\theta_\ell = G(Y^{2^\ell-1}\theta_\ell, (\theta_\ell - a_\ell)^2)$$

where G is a polynomial in two indeterminates over $k[Y]$; but π being transcendental over $k[Y]$, θ_ℓ is transcendental over $k[Y]$ also, and the relation (**) has to be an identity, which is absurd. Thus, $\theta_1 \notin T$.

Now, let R^* be the completion of R with the (PR)-adic topology; $\{\pi_r\}_{r\geq 0}$ is a Cauchy sequence in R. Suppose that $\pi_r \in P^2R$ for some $r \geqq 1$; since P^2 is a primary ideal of T, we have $\pi_r \in P^2R \cap T = P^2 \subset YT$, and $\pi = \pi_r + (a_1 Y + \cdots + a_r Y^{2^r-1}) \in YT$ which is absurd since $\theta_1 \notin T$. Thus, for every $r \geqq 0$, $\pi_r \notin P^2R$ and $\beta = \lim_r \pi_r$ is $\neq 0$. However, we also have $\beta^2 = \lim_r \pi_r^2 = \lim_r Y^{2^{r+1}-2}t_r = 0$; hence R^* has a nonzero nilpotent element and $\bar{R}$ is not a finite R-module [1, p. 330].

EXAMPLE 2.2. Let Q be the rational numbers, $(X_1, \cdots, X_r, \cdots)$ a set of indeterminates over Q and $k = Q(X_1, \cdots, X_r, \cdots)$. Let

$$\pi = b_1 X_1 Y + \cdots + b_r X_r Y^{2^r-1} + \cdots$$

be transcendental over $k[Y]$ with $b_i \in Q - \{0\}$ for every $i \geqq 1$[2]. Construct the rings $T = k[Y, \pi, t_1, \cdots, t_r, \cdots]$ and $R = T_P$ as in 2.1. On the quotient field $k(Y, \pi) = Q(X_1, \cdots, X_r, \cdots; Y, \pi)$ define a derivation D by

$$\begin{aligned} D(q) &= 0 \quad \text{for every} \quad q \in Q \\ D(Y) &= 1 \\ D(\pi) &= 3b_2 X_2 Y^2 + b_1 X_1 \\ D(X_1) &= 0 \end{aligned}$$

[2] There exists such a π since k is countable.

$$\begin{aligned} D(X_2) &= -7b_3b_2^{-1}X_3Y^3 \\ &\vdots \\ D(X_i) &= -(2^{i+1}-1)b_{i+1}b_i^{-1}X_{i+1}Y^{2^{i+1}-2^{i}-1} \\ &\vdots \end{aligned}$$

Then,

(i) D is regular on R

(ii) R is a 1-dimensional local D-simple ring which is not integrally closed.

Proof. (i) Since $R = T_P$, it suffices to show that $D(T) \subset R$. By definition of D we already have $D(k) \subset R$, $D(Y) \in R$ and $D(\pi) \in R$; hence it remains to show that $D(t_r) \in R$ for every $r \geqq 1$. Differentiating $\pi_r^2 = Y^{2^{r+1}-2}t_r$, we get $2\pi_r D(\pi_r) = Y^{2^{r+1}-2}D(t_r) + (2^{r+1}-2)Y^{2^{r+1}-3}t_r$; but $t_r \in YR$ by 2.1, hence $D(t_r) \in R$ if and only if $\pi_r D(\pi_r) \in Y^{2^{r+1}-2}R$. Let us show that in fact we have $D(\pi_r) \in Y^{2^{r+1}-2}R$. From $\pi_1 = \pi - b_1X_1Y$ we get $D(\pi_1) = D(\pi) - b_1X_1 = 3b_2X_2Y^2$; by induction, if we suppose that $D(\pi_{r-1}) = (2^r-1)b_rX_rY^{2^r-2}$ and if we differentiate the relation $\pi_r = \pi_{r-1} - b_rX_rY^{2^r-1}$, we get $D(\pi_r) = (2^{r+1}-1)b_{r+1}X_{r+1}Y^{2^{r+1}-2} \in Y^{2^{r+1}-2}R$. Hence D is regular on R.

(ii) The only prime ideal of R which is not (0) or (1) is $PR = (Y, \pi)R$; it is not a D-ideal since $D(Y) = 1$; thus by 1.5, R is D-simple. Furthermore by 2.1. R is a 1-dimensional local, not integrally closed, domain.

3. On the complete integral closure of a $\mathscr{D}$-simple ring. We have seen in the preliminaries that a $\mathscr{D}$-simple ring of characteristic $p \neq 0$ is equal to it total quotient ring. In this section we are concerned with rings of characteristic 0. Henceforth, R will denote a ring containing the integers.

THEOREM 3.1. *Let R be a ring, D a derivation on R, P a prime ideal of R containing no D-ideal other than* (0). *Define v: $R\backslash\{0\} \to$ {nonnegative integers} by $v(x) = n$ if $D^{(i)}(x) \in P$ for $i = 0, \cdots, n-1$ and $D^{(n)}(x) \notin P$. Then,*

(i) *R is domain.*

(ii) *v is rank-1-discrete valuation whose valuation ring R_v contains R and whose maximal ideal M_v lies over P.*

(iii) *D is regular on R_v and R_v is D-simple.*

Proof. (i) If n is any integer, $D(n) = 0$ and nR is a D-ideal of R; hence 0 is the only integer contained in P. Now, (0) is a D-ideal, hence by 1.3 any minimal prime divisor Q of (0) is a D-ideal also; then, by the hypothesis made on P, we have $(0) = Q$ and R is a domain.

(ii) Let x and y be two nonzero elements of R, and let $v(x) = n$, $v(y) = m$, $n \leqq m$. For every i such that $0 \leqq i \leqq n-1$, both $D^{(i)}(x)$ and $D^{(i)}(y)$ belong to P, hence $D^{(i)}(x+y) \in P$ and

$$v(x+y) \geqq n = \inf\{v(x), v(y)\} .$$

Let k be such that $0 \leqq k \leqq n+m-1$. For $0 \leqq i \leqq \inf\{k, n-1\}$ we have $D^{(i)}(x) \in P$, hence also $C_k^i D^{(i)}(x) D^{(k-i)}(y) \in P$; for $n \leqq k$ and $n \leqq i \leqq k$ we have $0 \leqq k-i \leqq k-n \leqq m-1$, hence $D^{(k-i)}(y) \in P$ and $C_k^i D^{(i)}(x) D^{(k-i)}(y) \in P$; thus

$$D^{(k)}(xy) = \sum_{i=0}^{k} C_k^i D^{(i)}(x) D^{(k-i)}(y) \in P .$$

Now,

$$D^{(n+m)}(xy) = \sum_{i=0}^{n+m} C_{n+m}^i D^{(i)}(x) D^{(n+m-i)}(y); \sum_{i=0}^{n-1} C_{n+m}^i D^{(i)}(x) D^{(n+m-i)}(y)$$
$$+ \sum_{i=n+1}^{n+m} C_{n+m}^i D^{(i)}(x) D^{(n+m-1)}(y) \in P$$

whereas $C_{n+m}^n D^{(n)}(x) D^{(m)}(y) \notin P$ since C_{n+m}^n, $D^{(n)}(x)$, $D^{(m)}(y) \notin P$; thus

$$D^{(n+m)}(xy) \notin P , \qquad v(xy) = n + m = v(x) + v(y)$$

and v is a valuation, rank-1-discrete since its value group is the group of integers. Furthermore, we obviously have $R \subset R_v$ and $M_v \cap R = P$.

(iii) Let ab^{-1} be any element of R_v with $a, b \in R$, $b \neq 0$, $v(a) \geqq v(b)$; then $D(ab^{-1}) = [bD(a) - aD(b)]b^{-2}$. If $v(a) > v(b)$, then $v(D(a)) = v(a) - 1 \geqq v(b)$ and $v(D(b)) \geqq v(b) - 1$ so that

$$v(bD(a) - aD(b)) \geqq \inf\{v(b) + v(D(a)), v(a) + v(D(b))\} \geqq 2v(b)$$

and $D(ab^{-1}) \in R_v$. If $v(a) = v(b) = 0$, then $v(bD(a) - aD(b)) \geqq 0 = 2v(b)$ and $D(ab^{-1}) \in R_v$. If $v(a) = v(b) = n > 0$, then $v(bD(a)) = v(aD(b)) = 2n - 1$, so that $D^{(k)}(bD(a) - aD(b)) \in P$ for every $k \leqq 2n - 2$; furthermore we have

$$D^{(2n-1)}(bD(a)) = \sum_{i=0}^{2n-1} C_{2n-1}^i D^{(i)}(b) D^{(2n-i)}(a) = \alpha_1 + C_{2n-1}^n D^{(n)}(b) D^{(n)}(a)$$

with $\alpha_1 \in P$, and similarly $D^{(2n-1)}(aD(b)) = \alpha_2 + C_{2n-1}^n D^{(n)}(a) D^{(n)}(b)$ with $\alpha_2 \in P$, so that $D^{(2n-1)}(bD(a) - aD(b)) = \alpha_1 - \alpha_2 \in P$; hence, $v(bD(a) - aD(b)) \geqq 2n$ and $D(ab^{-1}) \in R_v$. Thus D is regular on R_v. Moreover, R_v is D-simple since if $A \neq (0)$ were a D-ideal of R_v, then $A \cap R \neq (0)$ would be a D-ideal of R contained in P, which would be absurd.

THEOREM 3.2. *Let R be a domain with quotient field K, S a ring such that $R \subset S \subset K$ and D a derivation of R regular on S.*

Let P be a prime ideal of R such that R_P is D-simple. Then,

(i) *There is at most one prime ideal Q of S lying over P, Q being a minimal prime ideal when P is.*

(ii) *If S is the complete integral closure R' of R there is exactly one prime ideal P' of R' lying over P.*

Proof. (i) Let Q be a prime ideal of S such that $Q \cap R = P$. Being regular on S, D is also regular on S_Q, and S_Q is D-simple since $S_Q \supset R_P$. Define v: $R \backslash \{0\} \rightarrow$ {nonnegative integers} by $v(x) = n$ if

$$D^{(0)}(x), \cdots, D^{(n-1)}(x) \in P \quad \text{and} \quad D^{(n)}(x) \notin P\,,$$

and w: $S \backslash \{0\} \rightarrow$ {nonnegative integers} by

$$w(y) = m \text{ if } D^{(0)}(y), \cdots, D^{(m-1)}(y) \in Q$$

and $D^{(m)}(y) \notin Q$. By 3.1, v and w extend to valuations of K; furthermore, for $x \in R$ we have $D^{(k)}(x) \in P$ if and only if $D^{(k)}(x) \in Q$ since $Q \cap R = P$; hence $v = w$, and $Q = M_V \cap S$ where M_V is the maximal ideal of the valuation ring R_v of v.

If P is a minimal prime ideal of R, suppose that Q' is a prime ideal of S such that $0 < Q' \subset Q$. We have $0 < Q' \cap R \subset Q \cap R = P$ and $Q' \cap R = P$ by the minimality of P; then $Q' = Q$ since Q is the only prime ideal of S lying over P.

(ii) By [5, p. 168] every derivation of R is regular on R'. Being a rank-1 valuation ring, R_v is completely integrally closed and contains R'. Then, $P' = M_V \cap R'$ is a prime ideal of R' lying over P; of course, by (i), P' is unique.

THEOREM 3.3. *Let R be a Noetherian $\mathscr{D}$-simple ring and $\bar{R}$ its integral closure. Let $\{P_\alpha\}_{\alpha \in A}$ be the set all the minimal prime ideals of R. Then,*

(i) *For every $\alpha \in A$, there exists $D \in \mathscr{D}$ such that R_{P_α} is D-simple, and there exists a unique prime ideal $\bar{P}_\alpha$ of $\bar{R}$ lying over P_α.*

(ii) *$\{\bar{P}_\alpha\}_{\alpha \in A}$ is the set of all the minimal prime ideals of $\bar{R}$.*

(iii) *Let $D \in \mathscr{D}$ such that $D(P_\alpha) \not\subset P_\alpha$, w_α the valuation associated by 3.1, and R_α its valuation ring. Then $R_\alpha = \bar{R}_{\bar{P}_\alpha}$ (hence, any two derivations D and D' such that $D(P_\alpha) \not\subset P_\alpha$ and $D'(P_\alpha) \not\subset P_\alpha$ give rise to the same valuation w_α).*

(iv) $\bar{R} = \bigcap_{\alpha \in A} R_\alpha$.

Proof. (i) Being $\mathscr{D}$-simple, R is a domain containing the rational numbers, and for any $\alpha \in A$, there exists $D \in \mathscr{D}$ such that $D(P_\alpha) \not\subset P_\alpha$, and by 1.3, R_{P_α} is D-simple. Then, by 3.2, there exists a unique prime ideal $\bar{P}_\alpha$ of $\bar{R}$ lying over P_α.

(ii) That every P_α is a minimal prime ideal of $\bar{R}$ is given by 3.2. Now, let $\bar{P}$ be a minimal prime ideal of $\bar{R}$, and let $P = \bar{P} \cap R$; let M be a minimal prime ideal of R contained in P; by [3, (10.8), p. 30] there exists a prime ideal $\bar{M}$ of $\bar{R}$ lying over M; since $\bar{P}$ is the only prime ideal of $\bar{R}$ lying over P, we have $\bar{M} \subset \bar{P}$ by [3, (10.9), p. 30], hence $\bar{M} = \bar{P}$, and $P = \bar{P} \cap R = M$ is a minimal prime ideal of R.

(iii) Since R is Noetherian, $\bar{R}$ is a Krull ring [3, (33.10), p. 118], and $\bar{R}_{\bar{P}_\alpha}$ is a rank-1-discrete valuation ring. As furthermore $\bar{R}_{\bar{P}_\alpha} \subset R_\alpha$ we get $\bar{R}_{\bar{P}_\alpha} = R_\alpha$.

(iv) $\bar{R}$ is a Krull ring and $\{\bar{P}_\alpha\}_{\alpha \in \Lambda}$ is the set of all the minimal prime ideals of $\bar{R}$; thus $\bar{R} = \bigcap_{\alpha \in \Lambda} \bar{R}_{\bar{P}_\alpha} = \bigcap_{\alpha \in \Lambda} R_\alpha$.

COROLLARY 3.4. *Let R be a Noetherian $\mathscr{D}$-simple ring with quotient field K. Let S be a ring such that $R \subset S \subset K$ and such that every $D \in \mathscr{D}$ is regular on S. Then, the following statements are equivalent:*

(i) *For every minimal prime ideal P of R there exists a (unique) prime ideal Q of S lying over P.*

(ii) *S is integral over R.*

(iii) *For every prime ideal M of R there exists a (unique) prime ideal N of S lying over M.*

Proof. That (ii) $\Rightarrow$ (iii) is a consequence of [3, (10.7), p. 30] and 3.2; that (iii) $\Rightarrow$ (i) is obvious. Now, let $\{P_\alpha\}_{\alpha \in \Lambda}$ be the set of the minimal prime ideals of R, $\{w_\alpha\}_{\alpha \in \Lambda}$ the associated valuations and $\{R_\alpha\}_{\alpha \in \Lambda}$ the valuation rings of the w_α's. For any $\alpha \in \Lambda$, let $D \in \mathscr{D}$ be such that $D(P_\alpha) \not\subset P_\alpha$, and let Q_α be a prime ideal of S lying over P_α; S_{Q_α} is D-simple, the valuation associated to Q_α is equal to w_α and $S \subset R_\alpha$. Hence, $S \subset \bar{R} = \bigcap_{\alpha \in \Lambda} R_\alpha$.

COROLLARY 3.5. *Let R be a Noetherian $\mathscr{D}$-simple ring with quotient field K, and $\bar{R}$ its integral closure. Then,*

(i) *$\bar{R}$ is the largest $\mathscr{D}$-simple overring of R in K having a prime ideal lying over every prime ideal of R.*

(ii) *$\bar{R}$ is the largest $\mathscr{D}$-simple overring of R in K having a prime ideal lying over every minimal prime ideal of R.*

Proof. Apply 3.4.

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REFERENCES

1. Y. Akizuki, *Einige Bemerkungen uber primare Integritatsbereiche mit Teilerkettensatz*, Proc. Physico-Math Soc. Japan **17** (1935), 327-336.

2. I. Kaplansky, *An Introduction to Differential Algebra*, Actualités Scientifiques et industrielles, Herman, 1957.

3. M. Nagata, *Local Rings*, John Wiley and Sons, New York, 1962.

4. E. C. Posner, *Integral closure of rings of solutions of linear differential equations*, Pacific J. Math. **12** (1962), 1417-1422.

5. A. Seidenberg, *Derivations and integral closure*, Pacific J. Math. **16** (1966), 167-173.

6. W. V. Vasconcelos, *Derivations of commutative Noetherian rings*, Math. Zeit **112** (1969), 229-233.

7. O. Zariski and P. Samuel, *Commutative Algebra*, V. 1, Van Nostrand Co., Princeton, 1958.

8. ———, *Commutative Algebra*, V. 2, Van Nostrand Co., Princeton, 1960.

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