PACIFIC JOURNAL OF MATHEMATICS
Vol. 36, No. 3, 1971

EXPANSIVE AUTOMORPHISMS OF BANACH SPACES, II

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An automorphism of a complex Banach space is shown to be uniformly expansive if and only if its approximate point spectrum is disjoint from the unit circle.

The problem of giving a spectral characterization of the property that an operator be uniformly expansive was investigated in [2], but the theorem stated above was obtained only for automorphisms of a Hilbert space. The proof given in this note is both more general and more transparent than the special version. We also note some topological properties of the various classes of expansive operators in the space of all invertible operators.

1. Uniformly expansive automorphisms.

If T is an automorphism (a bounded, invertible, linear operator) on a complex Banach space X denote its spectrum by $\Lambda(T)$, its compression spectrum by $\Gamma(T)$, its approximate point spectrum by $\Pi(T)$, and its point spectrum by $\Pi_0(T)$. Denote the unit circle $\{\lambda: |\lambda| = 1\}$ in the complex plane by C. The automorphism T is said to be

(1) expansive if for each $x \in X$ with $||x|| = 1$ there exists some non-zero integer n with $||T^n x|| \geqq 2$;

(2) uniformly expansive if there exists some positive integer n such that if $x \in X$ with $||x|| = 1$ then either $||T^n x|| \geqq 2$ or $||T^{-n} x|| \geqq 2$;

(3) hyperbolic if there exists a splitting $X = X_s \oplus X_u$, $T = T_s \oplus T_u$, where X_s and X_u are closed T-invariant linear subspaces of X, $T_s = T|X_s$ is a proper contraction, and $T_u = T|X_u$ is a proper dilation.

A discussion of these classes of automorphisms may be found in [2].

It is well-known [2, Lemma 1] that an automorphism T is hyperbolic if and only if $\Lambda(T) \cap C = \varnothing$. The principal result weakens both conditions.

THEOREM 1. *Let T be an automorphism of a complex Banach space X. Then T is uniformly expansive if and only if $\Pi(T) \cap C = \phi$.*

The proof requires the Banach space version of an interesting numerical lemma.

LEMMA 1. *Given any complex numbers* $c_1, \cdots, c_s$ *there exists* $\lambda \in C$ *such that* $\sum_{j=1}^{s} \lambda^j c_j \geq 0$.

Proof. [2, Lemma 2]

LEMMA 2. *Given any complex numbers* $c_{-r}, \cdots, c_s$ *with* $c_0 \neq 0$ *there exists* $\lambda \in C$ *such that* $|\sum_{j=-r}^{s} \lambda^j c_j| \geq |c_0|$.

Proof. We may assume that $c_0 > 0$: otherwise set $d_j = (\bar{c}_0/|c_0|)\, c_j$ and proceed. Let $f(\lambda) = \sum_{j=1}^{s} \lambda^j c_j$, $g(\lambda) = \sum_{j=-r}^{-1} \lambda^j c_j$, and $h(\lambda) = \sum_{j=1}^{r} \lambda^j \bar{c}_{-j}$. Since $\lambda^{-j} = (\bar{\lambda})^j$ for $\lambda \in C$ it follows that Re $g(\lambda) =$ Re $h(\lambda)$, and therefore Re $[f(\lambda) + g(\lambda)] =$ Re $[f(\lambda) + h(\lambda)]$. Now $f(\lambda) + h(\lambda)$ is a polynomial in λ; by Lemma 1 there exists $\lambda \in C$ such that $f(\lambda) + h(\lambda) \geq 0$. Thus $f(\lambda) + h(\lambda) + c_0 \geq c_0$, and

$$\left| \sum_{j=-r}^{s} \lambda^j c_j \right| \geq \mathrm{Re} \left(\sum_{j=-r}^{s} \lambda^j c_j \right) = \mathrm{Re}\, [f(\lambda) + h(\lambda) + c_0] \geq c_0 .$$

LEMMA 3. *Given any vectors* $x_{-r}, \cdots, x_s$ *in a Banach space* X *with* $x_0 \neq 0$ *there exists* $\lambda \in C$ *such that*

$$\left\| \sum_{j=-r}^{s} \lambda^j x_j \right\| \geqq \| x_0 \| .$$

Proof. By the Hahn-Banach Theorem choose $x^* \in X^*$ with $\| x^* \| = 1$ and $x^*(x_0) = \| x_0 \|$. It suffices to find $\lambda \in C$ with

$$\left| x^*\left(\sum_{j=-r}^{s} \lambda^j x_j \right) \right| \geqq | x^*(x_0) | .$$

Set $c_j = x^*(x_j)$ and apply Lemma 2: there exists $\lambda \in C$ such that

$$\left| x^*\left(\sum_{j=-r}^{s} \lambda^j x_j \right) \right| = \left| \sum_{j=-r}^{s} \lambda^j c_j \right| \geqq | c_0 | = | x^*(x_0) | .$$

Proof of Theorem 1. Necessity is proved in [2, Theorem 1]. To prove sufficiency, suppose that T is not uniformly expansive. Then for each positive integer n there exists $x_n \in X$ with $\| x_n \| = 1$ and max $\{\| T^n x_n \|, \| T^{-n} x_n \|\} < 2$. For infinitely many n we produce a vector $y_n \in X$ and a number $\lambda_n \in C$ such that $\| (T - \lambda_n^{-1})\, y_n \| / \| y_n \| \to 0$. This will suffice. In fact, if $\mu \in C$ is a limit point of $\{\lambda_n^{-1}\}$ choose a subsequence $\{\lambda_m^{-1}\}$ of $\{\lambda_n^{-1}\}$ with $\lambda_m^{-1} \to \mu$. Then

$$\| (T - \mu)\, y_m \| / \| y_m \| \leqq \| (T - \lambda_m^{-1})\, y_m \| / \| y_m \| + | \lambda_m^{-1} - \mu | .$$

The right-hand side approaches 0 as $m \to \infty$, so that $\mu \in \Pi\,(T)$.

To construct y_n we must consider two cases. Define

$$\phi(n) = \max_{k=-n,0} \sup_{\lambda \in C} \left\| \sum_{j=k}^{k+n-1} \lambda^j T^j x_n \right\| .$$

Case 1. $\phi(n)$ is unbounded. Fix n, choose k where the maximum in the definition of ϕ is attained, and let λ_n be the $\lambda \in C$ where the supremum is attained. Define

$$y_n = \sum_{j=k}^{k+n-1} \lambda_n^j T^j x_n$$

so that $\| y_n \| = \phi(n)$. Now

$$(T - \lambda_n^{-1})\, y_n = \lambda_n^{n-1}\; T^n x_n - \lambda_n^{-1}\; x_n \text{ if } k = 0 \;,$$

and

$$(T - \lambda_n^{-1})\; y_n = \lambda_n^{-1} x_n - \lambda_n^{-n-1} T^{-n} x_n \text{ if } k = -n \;.$$

In either event,

$$\| (T - \lambda_n^{-1})\; y_n \| \leqq 3. \quad \text{Thus } \| (T - \lambda_n^{-1})\; y_n \| / \| y_n \| \leqq 3/\phi(n) \;.$$

Since $\phi(n)$ is unbounded, $3/\phi(n_j) \to 0$ for some subsequence $n_j \to \infty$.

Case 2. $\phi(n)$ is bounded. Assume that $\phi(n) \leqq A$ for all n and define

$$y_n = \sum_{j=-n}^{-1} (n + 1 + j)\, \lambda_n^j T^j x_n + \sum_{j=0}^{n-1} (n - j)\; \lambda_n^j T^j x_n \;,$$

where we choose $\lambda_n \in C$ by Lemma 3 to insure that $\| y_n \| \geqq n$, the norm of the term with index 0.

$$\begin{aligned} \| (T - \lambda_n^{-1})\, y_n \| &= \left\| - \sum_{j=-n}^{-1} \lambda_n^{j-1}\; T^j x_n + \sum_{j=1}^{n} \lambda_n^{j-1}\; T^j x_n \right\| \\ &\leqq \left\| \sum_{j=-n}^{-1} \lambda_n^j\; T^j x_n \right\| + \left\| T\Bigl(\sum_{j=0}^{n-1} \lambda_n^j T^j x_n \Bigr) \right\| \\ &\leqq A(1 + \| T \|) \;. \end{aligned}$$

Hence

$$\| (T - \lambda_n^{-1})\; y_n \| / \| y_n \| \leqq A(1 + \| T \|)/n \to 0 \;.$$

Note that the hypothesis that T is not uniformly expansive is not used in Case 2. But it is easy to see directly (by Lemma 3) that T is not uniformly expansive if $\phi(n)$ is bounded. Note also that it follows immediately from Theorem 1 that a hyperbolic automorphism is uniformly expansive.

2. Density. Denote the class of all hyperbolic automorphisms of a fixed Banach space X by $\mathscr{H}$, of uniformly expansive by $\mathscr{U}\mathscr{E}$, of expansive by $\mathscr{E}$, of all automorphisms by $\mathscr{I}$, and of all bounded linear operators by $\mathscr{B}$. If dim $X < \infty$ then $\mathscr{H} = \mathscr{U}\mathscr{E} = \mathscr{E}$ and is precisely the class of all automorphisms whose spectrum is disjoint from C. In general the situation is much different.

THEOREM 2. *Let X be separable infinite dimensional Hilbert space. Then:*

(1) $\mathscr{H} \subset \mathscr{U}\mathscr{E} \subset \mathscr{E} \subset \mathscr{I} \subset \mathscr{B}$;

(2) *$\mathscr{H}$ and $\mathscr{U}\mathscr{E}$ are open (in $\mathscr{B}$, in the uniform operator topology) but $\mathscr{E}$ is not;*

(3) *no class is dense in the next larger.*

The tools necessary for the proof are two results on semicontinuity of pieces of the spectrum due to Halmos and Lumer.

THEOREM A. [4, Theorem 2] *$\Pi(T)$ and $\Lambda(T)$ are upper semicontinuous: to every $T \in \mathscr{B}$ and every open set G containing $\Pi(T)$ [respectively, $\Lambda(T)$] there corresponds a positive number ε such that $\Pi(S) \subset G$ [$\Lambda(S) \subset G$] whenever $\|S-T\| < \varepsilon$.*

THEOREM B. [4, Theorem 3] *$\Lambda(T)\backslash\Pi(T)$ is lower semicontinuous: to every $T \in \mathscr{B}$ and every compact set K contained in $\Lambda(T)\backslash\Pi(T)$ there corresponds a positive number ε such that $K \subset \Lambda(S)\backslash\Pi(S)$ whenever $\| S - T\| < \varepsilon$.*

Proof of Theorem 2. (2) If $T \in \mathscr{H}$ then $\Lambda(T) \cap C = \varnothing$. By semicontinuity, $\Lambda(S) \cap C = \varnothing$ for S sufficiently near T. Since $\mathscr{I}$ is open, $S \in \mathscr{H}$. The proof for $\mathscr{U}\mathscr{E}$ is identical. To see that $\mathscr{E}$ is not open fix an orthonormal base $\{e_n\}_1^\infty$ and let T be the diagonal operator $Te_n = n/(n+1)\, e_n$. T is expansive [2, Example 2]. Given $\varepsilon > 0$ let $Se_n = Te_n$ for $|1 - n/(n+1)| \geqq \varepsilon$ and $Se_n = e_n$ otherwise. Then $\| S - T\| < \varepsilon$ but S is not expansive since $1 \in \Pi_0(S)$.

(3) $\mathscr{I}$ is not dense in $\mathscr{B}$: [3, Problem 109].

$\mathscr{E}$ is not dense in $\mathscr{I}$: let $\{e_n\}_{-\infty}^{\infty}$ be an orthonormal base and let T be the backward bilaterial weighted shift defined by $Te_n = 2e_{n-1}$ for $n \geqq 1$, $Te_n = 1/2\, e_{n-1}$ for $n \leqq 0$. Then [2, Example 4]

$$\Pi_0(T) = \{1/2 < |\lambda| < 2\}$$

so that $T \notin \mathscr{E}$. Now $\Lambda(T^*)\backslash\Pi(T^*) = \{1/2 < |\lambda| < 2\}$; by Theorem B if $\| S^* - T^* \|$ is small then $C \subset \Lambda(S^*)\backslash\Pi(S^*) \subset \Gamma(S^*)$. Hence $C \subset \Pi_0(S)$ and $S \notin \mathscr{E}$.

$\mathscr{H}$ is not dense in $\mathscr{U}\mathscr{E}$: in fact $\mathscr{U}\mathscr{E}\backslash\mathscr{H}$ is open. If $T \in \mathscr{U}\mathscr{E}\backslash\mathscr{H}$ then $\Pi(T) \cap C = \varnothing$ but $\Lambda(T) \cap C \neq \varnothing$. So there exists a compact set $K \subset C \cap [\Lambda(T)\backslash\Pi(T)]$. By Theorem B, $K \subset \Lambda(S)$ for $\| S - T\|$ small, so that $S \notin \mathscr{H}$.

$\mathscr{U}\mathscr{E}$ is not dense in $\mathscr{E}$: let X be represented as H^2 (of the unit circle) and let T be the multiplication operator $Tf(e^{it}) = (e^{it} + 3/2) f(e^{it})$. Let $A_r = \{|\lambda - 3/2| \leqq r\}$. Either direct calculation or appeal to the spectral properties of Toeplitz operators ([1], for instance) shows that $\Lambda(T) = A_1$, $\Pi_0(T) = \varnothing$, $\Pi(T) = \operatorname{bdy} A_1$, and $\Gamma(T) = \operatorname{int} A_1$. By Theorems A and B there exists $\varepsilon > 0$ such that if $\| S - T \| < \varepsilon$ then $A_{3/4} \subset \Gamma(S)$ and $\Lambda(S) \subset A_{3/2}$. Now the arc $\alpha(t) = e^{it}$, $0 \leqq t \leqq \pi/2$, on the unit circle has $\alpha(0) \in A_{3/4}$ and $\alpha(\pi/2) \notin A_{3/2}$. Thus $\alpha(t) \in \operatorname{bdy} \Lambda(S)$ for some t; hence $\Pi(S) \cap C \neq \varnothing$ and $S \notin \mathscr{U}\mathscr{E}$. To verify that T is expansive let $a \in [0, \pi]$ with $|e^{ia} + 3/2| = 1$. Fix $f \in H^2$ with $\|f\|_2 = 1$. Then either

$$1/2\pi \int_{-a}^{a} |f(e^{it})|^2 \, dt \geqq 1/2 \text{ or } 1/2\pi \int_{a}^{2\pi - a} |f(e^{it})|^2 \, dt \geqq 1/2 \, .$$

If the former holds choose $-a < b < c < a$ with

$$1/2\pi \int_{b}^{c} |f(e^{it})|^2 \, dt \geqq 1/4 \, ,$$

let $K = \min \{|e^{ib} + 3/2|, |e^{ic} + 3/2|\} > 1$, and choose an integer n with $K^n \geqq 4$. If $m \geqq n$

$$\begin{aligned} \| T^m f \|_2^2 &= 1/2\pi \int_0^{2\pi} |e^{it} + 3/2|^{2m} |f(e^{it})|^2 \, dt \\ &\geqq 1/2\pi \int_b^c |e^{it} + 3/2|^{2m} |f(e^{it})|^2 \, dt \\ &\geqq K^{2m} 1/2\pi \int_b^c |f(e^{it})|^2 \, dt \\ &\geqq 4 \, . \end{aligned}$$

If the other alternative holds then $\| T^{-m} f \|_2 \geqq 2$ for large m. Hence T is expansive.

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Received August 3, 1970.

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