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TOPOLOGIES FOR QUOTIENT FIELDS OF COMMUTATIVE INTEGRAL DOMAINS

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In this paper topologies for the quotient field K of a commutative integral domain A are investigated. The topologies for K are defined so that convergence in K is stronger than convergence in A whenever A is a topological ring.

In particular, the Mikusinski field of operators is the quotient field of many commutative integral domains which are also topological rings. Each of these rings leads to a topological convergence notion in the Mikusinski field, which is stronger than the convergence notion introduced originally by Mikusinski. (The latter has recently been shown to be nontopological.)

In general, the algebraic and topological structures considered are not necessarily compatible; however, the question of compatibility is investigated. Necessary and sufficient conditions are given for the topology on A to be the restriction to A of the topology defined on K. In a theorem of S. Warner, necessary and sufficient conditions have been given for the neighborhood filter of zero in A to be a fundamental system of neighborhoods of zero for a topology on K. Moreover K, with this topology, is a topological field with A topologically embedded in K as an open set. For rings satisfying the conditions of this theorem, the topology for K which is defined in this paper is shown to reduce to that specified by Warner.

Let C_R^∞ denote the set of all infinitely differentiable, complex valued functions of a real variable with the support of each function contained in some right half-line. Endowed with the operations of addition and convolution, C_R^∞ becomes a commutative ring which has no divisors of zero. The quotient field of the ring C_R^∞ will be denoted by the symbol M. It is isomorphic to the field of Mikusiński operators [8]. If C_R^∞ is assigned the topology $\mathscr{T}^*$, in which a sequence $(\alpha_n \mid n \in Z^+)$ converges if the supports of the elements α_n are uniformly bounded on the left and the derivative sequences $(\alpha_n^{(k)} \mid n \in Z^+)$ converge uniformly on compact sets for all $k \in Z^+ \cup \{0\}$, then $(C_R^\infty, \mathscr{T}^*)$ is a topological ring.

Mikusiński has introduced a convergence concept for M which is equivalent to the following definition. If $(a_n \mid n \in Z^+)$ is a sequence in M, then $(a_n \mid n \in Z^+)$ converges if there exists a nonzero $p \in C_R^\infty$ such that $(pa_n \mid n \in Z^+)$ is a sequence in C_R^∞ which converges in the space $(C_R^\infty, \mathscr{T}^*)$ [6, pg. 144]. T. K. Boehme has shown that this convergence is nontopological in the sense that there is no topology for M in which sequential convergence is given by Mikusiński's definition [2]. E. F. Wagner has defined an analogous convergence concept for nets and filters and has shown that this leads to a limit space structure on M which is also nontopological [9].

It seems natural to ask how Mikusiński convergence can be modified so that it becomes topological. R. A. Struble has introduced such a modification [7], which has the property that the restriction of the resulting topology to the right-sided Schwartz distributions, which are embedded algebraically in M, is the topology which is ordinarily associated with them. The topology introduced by Struble is also defined by a convergence concept for sequences and appears to be unwieldy.

S. Warner has given necessary and sufficient conditions for a topological ring which has no zero divisors to be openly embeddable in a topological division ring [10, Theorem 5]. It is easy to see that the ring $(C_R^\infty, \mathscr{T}^*)$ does not satisfy these conditions. Consequently, there is no topology on M which makes M a topological field with C_R^∞ topologically embedded as an open set. Using some recent results of Boehme, we will prove an even stronger result concerning M; namely, there is no topology on M such that C_R^∞ is topologically embedded in M and multiplication in M is continuous. Essentially this means that M cannot be topologized in a "nice" way and efforts to "extend" the topology of C_R^∞ to M must be channelled in other directions.

In this paper we present a method for topologizing the quotient field of any commutative ring which has no zero divisors, using any topology which may be assigned to the ring. If the ring satisfies the conditions given by Warner in [10], then the topology which we will define has the property that the quotient field with this topology is a topological field with the ring topologically embedded as an open set. In general, however, the field topology will reflect only part of the algebraic and the topological structure of the ring and will not necessarily be compatible with the field structure. Although the ensuing development is applicable to very general algebraic and topological settings, it is strongly motivated by the unsatisfactory situation afforded by the Mikusiński operators. The field M will frequently be used as an example.

Throughout this paper, the symbol A will denote a commutative ring which has no zero divisors and K will denote the quotient field of A. We will use the symbol A^* to represent the set of nonzero elements of A. A topology on a set will be a collection of open sets and a neighborhood will be a set containing an open set. We will always assume that there is a topology associated with the set of elements of A and this topology will be denoted by $\mathscr{T}$. The topology $\mathscr{T}$ is not necessarily compatible with the algebraic structure of A.

Whenever we consider a ring of functions in which ring multiplication is the convolution of functions, multiplication will be represented by the symbol, *. For terminology concerning nets and filters, the reader should refer to [5] and [3].

The following development will be divided into two sections. In the first section we will deal with the definition and characterizations of a topology for the set of elements of the quotient field K. The second section will examine some specific properties of this topology relative to the algebraic and topological structures of A.

1. The definition and characterizations of the topology. Before defining a topology for the quotient field of an arbitrary commutative integral domain, let us examine the specific problem of extending the topology $\mathscr{T}^*$ of C_R^∞ to M.

LEMMA 1. *Let $\mathscr{T}'$ be any topology on M with the following properties.*

(i) $\mathscr{T}'|C_R^\infty > \mathscr{T}^*$. *(The restriction of $\mathscr{T}'$ to C_R^∞ is finer than $\mathscr{T}^*$.)*

(ii) *For each nonzero $p \in C_R^\infty$, the mapping $\xi_p\colon x \mapsto px$ of M into M is continuous.*

Then sequential convergence in $(M, \mathscr{T}')$ is stronger than Mikusiński convergence.

Proof. Let $(a_n | n \in Z^+)$ be a sequence in M and let $a \in M$ such that $(a_n | n \in Z^+) \xrightarrow{\mathscr{T}'} a$. (The net $(a_n | n \in Z^+)$ converges to a in the topology $\mathscr{T}'$.) A theorem of T. K. Boehme implies that any countable collection of elements in C_R^∞ has a common multiple in C_R^∞ [1]. This implies that there exists a nonzero element p in C_R^∞ such that $pa \in C_R^\infty$ and, for every $n \in Z^+$, $pa_n \in C_R^\infty$. Since multiplication by an element of C_R^∞ is continuous in $(M, \mathscr{T}')$, $(pa_n | n \in Z^+) \xrightarrow{\mathscr{T}'} pa$. But $\mathscr{T}'|C_R^\infty > \mathscr{T}^*$ and therefore $(pa_n | n \in Z^+) \xrightarrow{\mathscr{T}^*} pa$. This implies that $(a_n | n \in Z^+)$ Mikusiński-converges to a.

LEMMA 2. *Let $\mathscr{T}'$ be any topology on M with the following properties.*

(i) $\mathscr{T}^* > \mathscr{T}'|C_R^\infty$

(ii) *For each $a \in M$, the mapping $\xi_a\colon x \mapsto ax$ of M into M is continuous.*

Then Mikusiński convergence of sequences is stronger than sequential convergence in $(M, \mathscr{T}')$.

Proof. Let $(a_n | n \in Z^+)$ be a sequence in M and let $a \in M$ such that $(a_n \mid n \in Z^+)$ Mikusiński-converges to a. Then there exists a non-zero element p in C_R^∞ such that $(pa_n \mid n \in Z^+)$ is a sequence in C_R^∞, $pa \in C_R^\infty$ and $(pa_n \mid n \in Z^+) \xrightarrow{\mathscr{T}^*} pa$. Since $\mathscr{T}^* > \mathscr{T}' \mid C_R^\infty$, $(pa_n \mid n \in Z^+) \xrightarrow{\mathscr{T}'} pa$. But $\xi_{1/p}$ is continuous in $(M, \mathscr{T}')$ and therefore $(a_n \mid n \in Z^+) \xrightarrow{\mathscr{T}'} a$.

In [2] Boehme has shown that there is no topology on M which has the collection of Mikusiński-convergent sequences for its sequential convergence class. Combining this result with Lemma 1 and Lemma 2, we obtain the following theorem.

THEOREM 1. *There is no topology on M in which multiplication is continuous and for which $(C_R^\infty, \mathscr{T}^*)$ is topologically embedded in M.*

We will now examine the more general situation of an arbitrary commutative ring A with no zero divisors, and its associated quotient field K. For each $p \in A^*$, define a mapping ϕ_p from A into K by $\phi_p(\alpha) = \alpha/p$, $\alpha \in A$. Denote the image of A under the mapping ϕ_p by the symbol A_p. Let $\mathscr{T}_p$ be the finest topology on A_p which renders the mapping ϕ_p continuous. That is, $\mathscr{T}_p = \{0_p \subset A_p \mid 0_p = 0/p, 0 \in \mathscr{T}\}$. Since A has no zero divisors, $\alpha_1/p = \alpha_2/p$, $\alpha_1, \alpha_2 \in A$, if and only if $\alpha_1 = \alpha_2$. Consequently, ϕ_p is a bijection. Therefore $(A_p, \mathscr{T}_p)$ is homeomorphic to $(A, \mathscr{T})$. For each $\alpha \in A$, let $\mathscr{N}(\alpha)$ be the $\mathscr{T}$-neighborhood filter of α and if $a \in A_p$, let $\mathscr{N}_p(a)$ be the $\mathscr{T}_p$-neighborhood filter of a. We note that $K = \bigcup_{p \in A^*} A_p$. If $(a_\mu \mid \mu \in \mathrm{M})$ is a net in K, let $\mathrm{M}_p = \{\mu \in \mathrm{M} \mid a_\mu \in A_p\}$. Clearly if the net $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in A_p, then $(a_\mu \mid \mu \in \mathrm{M}_p)$ is a subnet which is in A_p.

DEFINITION 1. Let $(a_\mu \mid \mu \in \mathrm{M})$ be a net in K and let $a \in K$. Then $(a_\mu \mid \mu \in \mathrm{M})$ is K-convergent to a, written $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} a$, if the following condition is satisfied. Whenever $a \in A_p$, the net $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in the space A_p and $(a_\mu \mid \mu \in \mathrm{M}_p) \xrightarrow{\mathscr{T}_p} a$.

The obvious generalization of Mikusiński convergence is the following. Let $(a_\mu \mid \mu \in \mathrm{M})$ be a net in K and let $a \in K$. Then $(a_\mu \mid \mu \in \mathrm{M})$ Mikusiński-converges to a if and only if for some $p \in A^*$, $a \in A_p$, $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in A_p and $(a_\mu \mid \mu \in \mathrm{M}_p) \xrightarrow{\mathscr{T}_p} a$. Clearly K-convergence is stronger than Mikusiński convergence. We will now show that K-convergence is topological. This could be done directly by proving that the collection of K-convergent nets is the convergence class of a topology on K; however, it is slightly more interesting to give an analogous definition of K-convergence of filters, show that it is topological and then prove that it is equivalent to K-convergence of nets.

DEFINITION 2. If $\mathscr{F}$ is a filter on K and $a \in K$, then $\mathscr{F}$ is K-convergent to a, written $\mathscr{F}\tau_K a$, if and only if whenever $a \in A_p$, $\mathscr{F}$ is finer than the $\mathscr{T}_p$-neighborhood filter of a.

Clearly if $\mathscr{F}\tau_K a$ and $\mathscr{G} > \mathscr{F}$, then $\mathscr{G}\tau_K a$. Moreover, if $\mathscr{N}(a) = \bigcap_{\mathscr{F}\tau_K a} \mathscr{F}$, then $\mathscr{N}(a)\tau_K a$. Now for each $a \in K$, the collection of filters which K-converge to a is the collection of filters which are finer than $\mathscr{N}(a)$. Obviously $\mathscr{N}(a)$ is a candidate for the neighborhood filter of a in some topology. In [3, pg. 19, Proposition 2], sufficient conditions are given for a collection of filters on a set to uniquely determine a topology in which the specified filters are the neighborhood filters. The fact that these conditions are satisfied by the collection $\{\mathscr{N}(a) \mid a \in K\}$ constitutes the proof of Theorem 2; however, first we will prove the following lemma.

LEMMA 3. *For each $a \in K$, $\mathscr{B}(a) = \{N_p(a) \mid a \in A_p$ and $N_p(a) \in \mathscr{N}_p(a)$ for some $p \in A^*\}$ is a subbase for the filter $\mathscr{N}(a)$.*

Proof. Since every element of $\mathscr{B}(a)$ contains the point a, $\mathscr{B}(a)$ is a subbase for a filter on K. Let $\mathscr{B}'(a)$ be the collection of all finite intersections of elements of $\mathscr{B}(a)$ and let $\mathscr{B}''(a)$ be the filter generated by $\mathscr{B}'(a)$. Then $\mathscr{B}''(a)$ is the coarsest filter containing $\mathscr{B}(a)$. Now if $\mathscr{F}\tau_K a$, then $\mathscr{F}$ contains $\mathscr{B}(a)$ and so $\mathscr{B}''(a) < \mathscr{F}$. Therefore $\mathscr{B}''(a) < \mathscr{N}(a)$. On the other hand, if $a \in A_p$, then clearly $\mathscr{B}''(a) > \mathscr{N}_p(a)$ which implies that $\mathscr{B}''(a)\tau_K a$. Consequently $\mathscr{B}''(a) > \mathscr{N}(a)$.

THEOREM 2. *There is a unique topology on K with the property that a filter converges to a point if and only if it K-converges to that same point.*

Proof. Let a be a given element of K. Since $\mathscr{N}(a)$ is a filter, every subset of K which contains a set of $\mathscr{N}(a)$ is an element of $\mathscr{N}(a)$ and, moreover, $\mathscr{N}(a)$ has the finite intersection property. By Lemma 3 if $N(a) \in \mathscr{N}(a)$, then $a \in N(a)$ since every element of $\mathscr{B}(a)$ contains a. Also as a result of Lemma 3, there exists a finite intersection, $\bigcap_i N_{p_i}(a)$, of elements of $\mathscr{B}(a)$, which is contained in $N(a)$. Hence there exist open sets $O_{p_i} \in \mathscr{T}_{p_i}$ such that $O_{p_i}(a) \in \mathscr{N}_{p_i}(a)$ and $O_{p_i}(a) \subset N_{p_i}(a)$. Therefore $\bigcap_i O_{p_i}(a) \subset \bigcap_i N_{p_i}(a) \subset N(a)$. Moreover, $\bigcap_i O_{p_i}(a) \in \mathscr{N}(a)$. Let y be an arbitrary element of $\bigcap_i O_{p_i}(a)$. Since the sets $O_{p_i}(a)$ are open, they are elements of $\mathscr{B}(y)$. Consequently, $\bigcap_i O_{pi}(a) \in \mathscr{N}(y)$ and because $\bigcap_i O_{p_i}(a) \subset N(a)$, $N(a) \in \mathscr{N}(y)$.

It remains to be shown that K-convergence of nets and K-convergence of filters are equivalent. For a given net, its associated net filter is the collection of all sets which the net is "eventually in". In [5, pg. 83], it is shown that every filter is the net filter of some net. Therefore it is sufficient to prove the following theorem.

THEOREM 3. *Let $(a_\mu \mid \mu \in \mathrm{M})$ be a net in K and let $a \in K$. Then $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} a$ if and only if its associated net filter, $\mathscr{F}(a_\mu \mid \mu \in \mathrm{M})$, K-converges to a.*

Proof. Suppose $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} a$. Then if $a \in A_p$, $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in every $\mathscr{T}_p$-neighborhood of a which implies that

$$\mathscr{F}(a_\mu \mid \mu \in \mathrm{M}) > \mathscr{N}_p(a) .$$

Therefore $\mathscr{F}(a_\mu \mid \mu \in \mathrm{M})\tau_K a$. Conversely, suppose $\mathscr{F}(a_\mu \mid \mu \in \mathrm{M})\tau_K a$. Then if $a \in A_p$, $\mathscr{F}(a_\mu \mid \mu \in \mathrm{M}) > \mathscr{N}_p(a)$ which implies that $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in every $\mathscr{T}_p$-neighborhood of a. Therefore $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in A_p, $(a_\mu \mid \mu \in \mathrm{M}_p) \xrightarrow{\mathscr{T}_p} a$, and hence $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} a$.

Since K-convergence is topological, the topology determined by K-convergence will be denoted by $\mathscr{T}_K$. Moreover, since K-convergence of nets and K-convergence of filters are equivalent, we will use whichever notion of K-convergence is most appropriate to a particular situation.

The following theorem gives a simple characterization of the topology $\mathscr{T}_K$.

THEOREM 4. *$\mathscr{T}_K$ is the coarsest topology on K for which the collection $\mathscr{S} = \{O_p \mid O_p \in \mathscr{T}_p$ for some $p \in A^*\}$ is a collection of open sets.*

Proof. Let $O_p \in \mathscr{S}$ and suppose $(a_\mu \mid \mu \in \mathrm{M})$ is a net in K which K-converges to $a \in O_p$. Then $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}_p} a$ which implies that $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in O_p. Therefore $O_p \in \mathscr{T}_K$. Let $\mathscr{T}'$ be any topology on K with the property that $\mathscr{S} \subset \mathscr{T}'$. If $(a_\mu \mid \mu \in \mathrm{M})$ is a net in K which $\mathscr{T}'$-converges to a, then $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in every $\mathscr{T}_p$-open-neighborhood of a. This implies that $(a_\mu \mid \mu \in \mathrm{M})$ is eventually in A_p, $(a_\mu \mid \mu \in \mathrm{M}_p) \xrightarrow{\mathscr{T}_p} a$, and hence $(a_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} a$. Therefore $\mathscr{T}_K$ is coarser than $\mathscr{T}'$.

Now we can make the following two observations. First, in view of Theorem 4, the topology $\mathscr{T}_K$ could have been defined as that topology on K which has the collection $\mathscr{S}$ as a subbase. From this point of view, Definition 1 and Definition 2 characterize convergence in this topology. Second, the algebraic characteristics of the ring A

and the field K are not essential in defining $\mathscr{T}_K$. In general, if $\{(\mathscr{B}_\ell, \mathscr{T}_\ell) \mid \ell \in \mathscr{L}\}$ is an arbitrary, indexed collection of topological spaces, then we may define a topology on $\bigcup_{\ell \in \mathscr{L}} B_\ell$ by taking the collection $\mathscr{S}_{\mathscr{L}} = \{O_\ell \mid O_\ell \in \mathscr{T}_\ell$ for some $\ell \in \mathscr{L}\}$ as a subbase. This topology is the coarsest one in which all the injection maps $i_\ell: B_\ell \to \bigcup_{\ell \in \mathscr{L}} B_\ell$ are open mappings. Convergence in this topology is characterized by definitions which are analogous to Definition 1 and Definition 2.

2. Properties of the topology. A few facts concerning the relationship between $(A, \mathscr{T})$ and $(K, \mathscr{T}_K)$ are immediate consequences of the characterizations of $\mathscr{T}_K$ which have already been given. For instance, it follows from Lemma 3 that if $(A, \mathscr{T})$ is a Hausdorff space, then $(K, \mathscr{T}_K)$ is a Hausdorff space, since distinct points of K are always elements of a common A_p-space and have disjoint neighborhoods in that space. There are several observations that can be made as a result of Theorem 4. Clearly if $(A, \mathscr{T})$ is a discrete space, then $(K, \mathscr{T}_K)$ is a discrete space. Also, it is obvious that for each $p \in A^*$, $\mathscr{T}_K \mid A_p$ is finer than $\mathscr{T}_p$. Another result of Theorem 4 is that if $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ are comparable topologies on A with $\mathscr{T}^{(1)}$ finer than $\mathscr{T}^{(2)}$, then the corresponding topologies of K-convergence, $\mathscr{T}_K^{(1)}$ and $\mathscr{T}_K^{(2)}$, have the same relationship. It is easy to construct examples to show that if $\mathscr{T}^{(1)}$ is strictly finer than $\mathscr{T}^{(2)}$, then $\mathscr{T}_K^{(1)}$ may be strictly finer than $\mathscr{T}_K^{(2)}$. Two major questions which remain to be answered are; "Under what conditions is $(A, \mathscr{T})$ topologically embedded in $(K, \mathscr{T}_K)$?", and "What is the relationship between the topology $\mathscr{T}_K$ and the algebraic structure of K?" It is the purpose of this section to examine these two questions.

For each $p \in A^*$, let ξ_p be the mapping of A into A defined by $\xi_p(x) = px$, $x \in A$.

LEMMA 4. *If for each $p \in A^*$ the mapping ξ_p is continuous, then $\mathscr{T}$ is finer than $\mathscr{T}_K \mid A$.*

Proof. Since the mappings ξ_p, $p \in A^*$, are continuous, if

$$(\alpha_\mu \mid \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha ,$$

then for every $p \in A^*$, $(\alpha_\mu p \mid \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. From the way in which A is algebraically embedded in K, it follows that the elements of A are in every A_p-space. Specifically, if $p \in A^*$, then for each $\mu \in \mathrm{M}$, α_μ is identified with $\alpha_\mu p/p$ and α is identified with $\alpha p/p$. Therefore $(\alpha_\mu p \mid \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$ implies that $(\alpha_\mu \mid \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}_p} \alpha$. But this is true for every $p \in A^*$ and so we have $(\alpha_\mu \mid \mu \in \mathrm{M}) \xrightarrow{K} \alpha$.

In view of Lemma 4, we can make the following observation. If $(A, \mathscr{T})$ is a topological ring (recall that A is always a commutative ring which has no zero divisors), then K-convergence is a generalization of ring convergence in the sense that every $\mathscr{T}$-convergent net is K-convergent. There may, however, be nets in A which do not converge in $(A, \mathscr{T})$ but which are K-convergent. In fact, the following example shows that this is the case when $(A, \mathscr{T}) = (C_R^\infty, \mathscr{T}^*)$.

EXAMPLE 1. $\mathscr{T}$ may be strictly finer than $\mathscr{T}_K | A$.

If $(A, \mathscr{T}) = (C_R^\infty, \mathscr{T}^*)$ and $K = M$, we will denote the topology of K-convergence on M by $\mathscr{T}_K^*$. Then $\mathscr{T}^*$ is finer than $\mathscr{T}_K^* | C_R^\infty$ because $(C_R^\infty, \mathscr{T}^*)$ is a topological ring. Let $(\alpha_n | n \in Z^+)$ be a sequence in C_R^∞ with the following properties.

(i) For each $n \in Z^+$, the support of α_n is contained in $[0, 1/n]$.

(ii) For each $n \in Z^+$, $\max_t |\alpha_n(t)| = 1$.

Now if p is a nonzero element of C_R^∞, then $(\alpha_n{}^* p | n \in Z^+) \xrightarrow{\mathscr{T}^*} 0$ which implies that $(\alpha_n | n \in Z^+) \xrightarrow{\mathscr{T}_p} 0$. Therefore $(\alpha_n | n \in Z^+) \xrightarrow{\mathscr{T}^*_K} 0$, but $(\alpha_n | n \in Z^+)$ does not converge in $\mathscr{T}^*$.

THEOREM 5. *$\mathscr{T} = \mathscr{T}_K | A$ if and only if $\mathscr{T}$ has the following property: If $(\alpha_\mu | \mu \in \mathrm{M})$ is a net in A and $\alpha \in A$, then $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$ for every $p \in A^*$ if and only if $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$.*

Proof. Suppose $\mathscr{T} = \mathscr{T}_K | A$. Let $(\alpha_\mu | \mu \in \mathrm{M})$ be a net in A with $\alpha \in A$ such that for each $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. Now for each $p \in A^*$, $\alpha_\mu = (\alpha_\mu p/p)$, $\mu \in \mathrm{M}$, and $\alpha = (\alpha p/p)$. Therefore $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$ implies that $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}_p} \alpha$. Consequently $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{K} \alpha$, and since $\mathscr{T} = \mathscr{T}_K | A$, $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$. On the other hand, if $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$ and $\mathscr{T} = \mathscr{T}_K | A$, then for every $p \in A^*$, $((\alpha_\mu p/p) | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}_p} (\alpha p/p)$ which implies that

$$(\alpha_\mu p | \mu \in \mu) \xrightarrow{\mathscr{T}} \alpha p \ .$$

Conversely, suppose $\mathscr{T}$ has the specified property. Then for each $p \in A^*$, the mapping ξ_p is continuous. By Lemma 4, $\mathscr{T} > \mathscr{T}_K | A$. Let $(\alpha_\mu | \mu \in \mathrm{M})$ be a net in A and let $\alpha \in A$ such that $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{K} \alpha$. Then for every $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$ and consequently

$$(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha. \quad \text{Therefore } \mathscr{T}_k | A > \mathscr{T} \ .$$

COROLLARY a. *If A has an identity e and if for every $p \in A^*$ the mapping ξ_p is continuous, then $\mathscr{T} = \mathscr{T}_K | A$.*

Proof. Clearly the existence of an identity implies that the condition given in Theorem 5 is satisfied.

COROLLARY b. *Suppose that $(A, \mathscr{T})$ is a topological ring. If there exists $p' \in A^*$ such that $p'\mathscr{N}_{\mathscr{T}}(0) < \mathscr{N}_{\mathscr{T}}(0)$, then $\mathscr{T} = \mathscr{T}_K | A$.*

Proof. Since multiplication is continuous in A, we have $\mathscr{N}_{\mathscr{T}}(0) < p'\mathscr{N}_{\mathscr{T}}(0)$. Therefore the given condition is equivalent to the requirement that $p'\mathscr{N}_{\mathscr{T}}(0)$ be a base for $\mathscr{N}_{\mathscr{T}}(0)$. Because $(A, \mathscr{T})$ is a topological ring, if $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$, then for every $p \in A^*$, $(\alpha_\mu p | (\mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. Let $(\alpha_\mu | \mu \in \mathrm{M})$ be a net in A and let $\alpha \in A$ such that for every $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. Since $p'\mathscr{N}_{\mathscr{T}}(0) < \mathscr{N}_{\mathscr{T}}(0)$, if $N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$, it follows that $p'N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$. Therefore $(\alpha_\mu p' - \alpha p' | \mu \in \mathrm{M})$ is eventually in $p'N_{\mathscr{T}}(0)$ which implies that $(\alpha_\mu - \alpha | \mu \in \mathrm{M})$ is eventually in $N_{\mathscr{T}}(0)$. Consequently $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$.

COROLLARY c. *If $(A, \mathscr{T})$ is a compact, Hausdorff, topological ring, then $\mathscr{T} = \mathscr{T}_K | A$.*

Proof. Since $(A, \mathscr{T})$ is a topological ring, if $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$, then for every $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. Let $(\alpha_\mu | \mu \in \mathrm{M})$ be a net in A and let $\alpha \in A$ such that for every $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. Let $(\beta_\lambda | \lambda \in \Lambda)$ be an arbitrary subnet of $(\alpha_\mu | \mu \in \mathrm{M})$. Since $(A, \mathscr{T})$ is compact, there exists a subnet $(\delta_\gamma | \gamma \in \Gamma)$ of $(\beta_\lambda | \lambda \in \Lambda)$ and $\delta \in A$ such that $(\delta_\gamma | \gamma \in \Gamma) \xrightarrow{\mathscr{T}} \delta$. If $p \in A^*$, then $(\delta_\gamma p | \gamma \in \Gamma) \xrightarrow{\mathscr{T}} \delta p$. But $(\delta_\gamma p | \gamma \in \Gamma)$ is a subnet of $(\alpha_\mu p | \mu \in \mathrm{M})$ which, by assumption, converges to αp. Therefore $(\delta_\gamma p | \gamma \in \Gamma) \xrightarrow{\mathscr{T}} \alpha p$ and since $(A, \mathscr{T})$ is Hausdorff, $\delta p = \alpha p$ which implies that $\delta = \alpha$. Now every subnet of $(\alpha_\mu | \mu \in \mathrm{M})$ has a subnet which converges to α. Consequently $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$.

There are several important subsets of K which warrant special consideration, among which are A itself and the A_p-spaces. Another important subset of K is the intersection of all of the A_p-spaces. It is well known that the elements of K may be identified algebraically as either quotients (equivalence classes of ordered pairs of elements of A), or as partial homomorphisms of ideals of A into A whose domains are maximal in the sense that the partial homomorphisms cannot be extended to properly larger ideals [4]. In the latter situation, $\bigcap_{p \in A^*} A_p$

is identifiable as the set of those partial homomorphisms defined on all of A. This collection of mappings is denoted $\mathrm{Hom}_A(A, A)$. If $a \in A$, then a may be identified with the homomorphism $\xi_a: A \to A$ where $\xi_a(x) = ax$, $x \in A$. Hence $A \subset \mathrm{Hom}_A(A, A)$. Now $\mathrm{Hom}_A(A, A)$ may be considered as a collection of functions which map a common domain into a topological space. One way of topologizing such a function space is to use the so-called "weak" topology, the topology of pointwise convergence. Let $\mathscr{P}$ denote this topology. It is not difficult to see that $\mathscr{T}_K | \mathrm{Hom}_A(A, A) = \mathscr{P}$. An immediate corollary to Theorem 5 is that $(A, \mathscr{T})$ is topologically embedded in $(K, \mathscr{T}_K)$ if and only if it is topologically embedded in $(\mathrm{Hom}_A(A, A), \mathscr{P})$.

If $A = C_R^\infty$ and $K = M$, Struble has shown that $\mathrm{Hom}_A(A, A)$ is isomorphic to the collection of all right-sided Schwartz distributions [7]. The usual topology assigned to distribution is a "weak" topology. In this case it can be shown that these right-sided distributions are embedded both algebraically and topologically in the Mikusiński operator field.

In general, A and $\mathrm{Hom}_A(A, A)$ do not need to be either open or closed subsets of $(K, \mathscr{T}_K)$. In his paper on compact rings [10], Warner considers the problem of openly embedding a topological ring, which has no divisors of zero, in a division ring. The following theorem shows that if $(A, \mathscr{T})$ is a topological ring, then a weakened version of a condition used by Warner in [10, Theorem 5] is sufficient to guarantee that both $(A, \mathscr{T})$ and $(\mathrm{Hom}_A(A, A), \mathscr{P})$ are openly embedded in $(K, \mathscr{T}_K)$.

THEOREM 6. *Suppose that $(A, \mathscr{T})$ is a topological ring with the additional property that for each $N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$, there exists $p \in A^*$ such that $pN_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$. Then*

(i) $\mathscr{T}_K | A = \mathscr{T}$

(ii) $A \in \mathscr{T}_K$

(iii) $\mathrm{Hom}_A(A, A) \in \mathscr{T}_K$.

Proof.

(i) Let $(\alpha_\mu | \mu \in \mathrm{M})$ be a net in A with $\alpha \in A$ such that

$$(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$$

for every $p \in A^*$. Then $(\alpha_\mu p - \alpha p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} 0$ for every $p \in A^*$. Let $N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ and choose $p' \in A^*$ such that $p'N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$. Then $(\alpha_\mu p' - \alpha p' | \mu \in \mathrm{M})$ is eventually in $p'N_{\mathscr{T}}(0)$ which implies that $(\alpha_\mu - \alpha | \mu \in \mathrm{M})$ is eventually in $N_{\mathscr{T}}(0)$. Therefore $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$. Since $(A, \mathscr{T})$ is a topological ring, if $(\alpha_\mu | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha$, then for every $p \in A^*$, $(\alpha_\mu p | \mu \in \mathrm{M}) \xrightarrow{\mathscr{T}} \alpha p$. By Theorem 5, $\mathscr{T} = \mathscr{T}_K | A$.

(ii) Choose $p' \in A^*$ such that $p'A \in \mathscr{N}_{\mathscr{T}}(0)$. If $x \in p'A$, then there exists $y \in A$ such that $x = p'y$ and hence $x + p'A = p'y + p'A = p'A$. Therefore $p'A \in \mathscr{N}_{\mathscr{T}}(x)$. Now since $p'A$ is in the neighborhood filter of each of its points, it is an open subset of $(A, \mathscr{T})$. Consequently if $\alpha \in A$, then $p'A \in \mathscr{N}_{\mathscr{T}}(p'\alpha)$. But α has the representation $p'\alpha/p'$ in K, and since $p'A \in \mathscr{N}_{\mathscr{T}}(p'\alpha)$, by Lemma 3, $p'A/p' = A \in \mathscr{N}(\alpha)$. Thus A is in the $\mathscr{T}_K$-neighborhood filter of each of its points and hence is an open subset of $(K, \mathscr{T}_K)$.

(iii) We have shown in (ii) that it is possible to choose $p' \in A^*$ such that $p'A$ is an open subset of $(A, \mathscr{T})$. If $a \in \mathrm{Hom}_A(A, A)$, then there exists $\alpha \in A$ such that $a = \alpha/p'$. Now $\alpha + p'A \in \mathscr{N}_{\mathscr{T}}(\alpha)$ and by Lemma 3, $(\alpha + p'A)/p' \in \mathscr{N}(a)$. However, $(\alpha + p'A)/p' = \alpha/p' + p'A/p' = a + A$ which is a subset of $\mathrm{Hom}_A(A, A)$. Consequently $\mathrm{Hom}_A(A, A) \in \mathscr{N}(a)$. Now $\mathrm{Hom}_A(A, A)$ is in the $\mathscr{T}_K$-neighborhood filter of each of its points and hence is an open subset of $(K, \mathscr{T}_K)$.

The remainder of this paper will be devoted to an examination of the compatibility of the topology $\mathscr{T}_K$ with the algebraic structure of K.

DEFINITION 3. For each $a \in K$, let $D(a) = \{p \in A^* \mid a \in A_p\}$.

Note that $D(a) \cup \{0\}$ is an ideal in A. It is, in fact, the domain of a when a is identified as a partial homomorphism.

THEOREM 7. *Let a and b be elements of K such that $D(a + b) = D(a) \cap D(b)$. Then, if addition is continuous in $(A, \mathscr{T})$, the mapping $f: K \times K \to K$ defined by $f(x, y) = x + y$, $(x, y) \in K \times K$, is continuous at the point (a, b).*

Proof. Let $\mathscr{N}(a + b)$ be the $\mathscr{T}_K$-neighborhood filter of $a + b$ and let $N(a + b)$ be an arbitrary element of $\mathscr{N}(a + b)$. By Lemma 3, there exists a finite intersection, $\bigcap_i N_{p_i}(a + b)$, of $\mathscr{T}_{p_i}$-neighborhoods of $a + b$ contained in $N(a + b)$. Since $D(a + b) = D(a) \cap D(b)$, both a and b are elements of each A_{p_i}-space. Moreover, addition is continuous in each A_{p_i}-space and hence, for each i, there exists $N_{p_i}(a) \in \mathscr{N}_{p_i}(a)$ and $N_{p_i}(b) \in \mathscr{N}_{p_i}(b)$ such that $f(N_{p_i}(a) \times N_{p_i}(b)) \subset N_{p_i}(a + b)$. Therefore we have $f(\bigcap_i N_{p_i}(a) \times \bigcap_i N_{p_i}(b)) \subset \bigcap_i N_{p_i}(a + b) \subset N(a + b)$. If $\mathscr{N}(a, b)$ is the neighborhood filter of (a, b) in $K \times K$, then a base for $\mathscr{N}(a, b)$ is the collection of all sets of the form $N(a) \times N(b)$ where $N(a) \in \mathscr{N}(a)$ and $N(b) \in \mathscr{N}(b)$. Now by Lemma 3 we have $\bigcap_i N_{p_i}(a) \times \bigcap_i N_{p_i}(b) \in \mathscr{N}(a, b)$. Therefore f is continuous at (a, b).

COROLLARY. *If addition is continuous in $(A, \mathscr{T})$, then addition is continuous in* $(\mathrm{Hom}_A(A, A), \mathscr{P})$.

Proof. If a and b are elements of $\mathrm{Hom}_A(A, A)$, then $D(a) = D(b) = D(a + b) = A^*$.

THEOREM 8. *Let a and b be elements of K such that $D(ab) \subset D(a) \cdot D(b)$. ($D(a) \cdot D(b) = \{p \in A \mid p = qr,\ q \in D(a)$ and $r \in D(b)\}$). Then, if multiplication is continuous in $(A, \mathscr{T})$, the mapping $f\colon K \times K \to K$ defined by $f(x, y) = xy$, $(x, y) \in K \times K$, is continuous at the point (a, b).*

Proof. Let $\mathscr{N}(ab)$ be the $\mathscr{T}_K$-neighborhood filter of ab and let $N(ab)$ be an arbitrary element of $\mathscr{N}(ab)$. By Lemma 3, there exists a finite intersection, $\bigcap_i N_{p_i}(ab)$, of $\mathscr{T}_{p_i}$-neighborhoods of ab contained in $N(ab)$. Now, for each i, we have the following. Since $D(ab) \subset D(a) \cdot D(b)$, there exist $q_i \in A^*$ and $r_i \in A^*$ such that $a \in A_{q_i}$, $b \in A_{r_i}$, and $p_i = q_i r_i$. Therefore there exist ring elements α_i and β_i such that $a = \alpha_i/q_i$, $b = \beta_i/r_i$, and $ab = \alpha_i\beta_i/q_i r_i = \alpha_i\beta_i/p_i$. Moreover, there exists $N_{\mathscr{T}}(\alpha_i\beta_i) \in \mathscr{N}_{\mathscr{T}}(\alpha_i\beta_i)$ such that $N_{p_i}(ab) = N_{\mathscr{T}}(\alpha_i\beta_i)/p_i$. But multiplication is continuous in $(A, \mathscr{T})$, and therefore there exist $N_{\mathscr{T}}(\alpha_i) \in \mathscr{N}_{\mathscr{T}}(\alpha_i)$ and $N_{\mathscr{T}}(\beta_i) \in \mathscr{N}_{\mathscr{T}}(\beta_i)$ such that $N_{\mathscr{T}}(\alpha_i) \cdot N_{\mathscr{T}}(\beta_i) \subset N_{\mathscr{T}}(\alpha_i\beta_i)$. Let $N_{q_i}(a) = N_{\mathscr{T}}(\alpha_i)/q_i$ and $N_{r_i}(b) = N_{\mathscr{T}}(\beta_i)/r_i$. Then $N_{q_i}(a) \in \mathscr{N}_{q_i}(a)$ and $N_{r_i}(b) \in \mathscr{N}_{r_i}(b)$. Now we have

$$N_{q_i}(a) \cdot N_{r_i}(b) = \frac{N_{\mathscr{T}}(\alpha_i)}{q_i} \cdot \frac{N_{\mathscr{T}}(\beta_i)}{r_i} = \frac{N_{\mathscr{T}}(\alpha_i) \cdot N_{\mathscr{T}}(\beta_i)}{q_i r_i}$$
$$= \frac{N_{\mathscr{T}}(\alpha_i) \cdot N_{\mathscr{T}}(\beta_i)}{p_i} \subset \frac{N_{\mathscr{T}}(\alpha_i\beta_i)}{p_i} = N_{p_i}(ab) .$$

That is, for each i, $f(N_{q_i}(a) \times N_{r_i}(b)) \subset N_{p_i}(ab)$. Therefore we have $f(\bigcap_i N_{q_i}(a) \times \bigcap_i N_{r_i}(b)) \subset \bigcap_i N_{p_i}(ab) \subset N(ab)$ and since $\bigcap_i N_{q_i}(a) \times \bigcap_i N_{r_i}(b) \in \mathscr{N}(a, b)$, f is continuous at the point (a, b).

COROLLARY. *If multiplication is continuous in $(A, \mathscr{T})$ and $A = A^2$, then multiplication is continuous in* $(\mathrm{Hom}_A(A, A), \mathscr{P})$.

Proof. If a and b are elements of $\mathrm{Hom}_A(A, A)$, then $D(a) = D(b) = D(ab) = A^*$. Since $A = A^2$ and A has no divisors of zero, $A^* = (A^*)^2$. Therefore $D(a) \cdot D(b) = (A^*)^2 = A^* = D(ab)$.

Theorems 7 and 8 give algebraic conditions which are sufficient for addition and multiplication to be locally continuous operations in $(K, \mathscr{T}_K)$. Since for every $a \in K$, $D(-a) = D(a)$, it is clear that if additive inversion is continuous in $(A, \mathscr{T})$, then it is also continuous in $(K, \mathscr{T}_K)$. Combining this fact with the corollaries to Theorems 7 and 8 yields the interesting result that if $(A, \mathscr{T})$ is a topological ring and $A = A^2$, then $(\mathrm{Hom}_A(A, A), \mathscr{P})$ is a topological ring.

The following examples demonstrate that multiplication and multiplicative inversion are not necessarily continuous operations in $(K, \mathscr{T}_K)$.

Example 2. Multiplication is not necessarily continuous in $(K, \mathscr{T}_K)$.

Let $(A, \mathscr{T}) = (C_R^\infty, \mathscr{T}^*)$ and $K = M$. Choose a nonzero element ϕ of C_R^∞ such that ϕ has compact support. For each $n \in Z^+$, let $m_n = \sup_t |\phi^{(n)}(t)|$. Consider the sequence $(f_n = s^n/nm_n | n \in Z^+)$, where s^n is the operator (homomorphism mapping C_R^∞ into itself) which maps a function in C_R^∞ to its nth derivative. If α is a nonzero element of C_R^∞, then for each $n \in Z^+, f_n$ has the representation $(\alpha^{(n)}/nm_n)/\alpha$. Choose a real number $\lambda > 1$ and let $\xi(t) = \phi(\lambda t)$. Then

$$\left(\frac{\xi^{(n)}/nm_n}{\xi} \middle| n \in Z^+\right) = \left(\frac{\lambda^n\phi^{(n)}(\lambda t)/nm_n}{\xi} \middle| n \in Z^+\right) \overset{\mathscr{T}_\xi}{\not\to} 0$$

$$\text{since}\left(\frac{\lambda^n\phi^{(n)}(\lambda t)}{nm_n} \middle| n \in Z^+\right) \overset{\mathscr{T}^*}{\not\to} 0 \; .$$

Hence $(f_n | n \in Z^+) \overset{K}{\not\to} 0$. If, however, ψ is any nonzero element of C_R^∞, then

$$\left(\frac{(\phi * \psi)^{(n)}}{nm_n} \middle| n \in Z^+\right) = \left(\frac{\phi^{(n)} * \psi}{nm_n} \middle| n \in Z^+\right) \xrightarrow{\mathscr{T}^*} 0 \; ,$$

and since $(C_R^\infty, \mathscr{T}^*)$ is a topological ring, by Lemma 4, it follows that

$$\left(\frac{(\phi * \psi)^n}{nm_n} \middle| n \in Z^+\right) \xrightarrow{K} 0 \; .$$

For each $n \in Z^+$, let $a_n = (\phi * \psi)^{(n)}/nm_n$ and let $b_n = (\phi * \psi)^{-1}$. Let $a = 0$ and $b = (\phi * \psi)^{-1}$. Now $(a_n | n \in Z^+) \xrightarrow{K} a$ and $(b_n | n \in Z^+) \xrightarrow{K} b$; however, $(a_nb_n | n \in Z^+) = (f_n | n \in Z^+) \overset{K}{\not\to} 0 = ab$. Therefore multiplication is not continuous on M.

Example 3. Multiplicative inversion is not necessarily continuous in $(K, \mathscr{T}_K)$.

Let $(A, \mathscr{T}) = (C_R^\infty, \mathscr{T}^*)$ and $K = M$. Consider the sequence $(1 - s/n | n \in Z^+)$. This is a sequence in M which clearly K-converges to the multiplicative identity; however, Mikusiński has shown that $((1 - s/n)^{-1} | n \in Z^+)$ does not converge according to his definition [6, pg. 147]. Therefore $((1 - s/n)^{-1} | n \in Z^+)$ does not K-converge. Consequently, multiplicative inversion is not continuous on M.

If addition is to be continuous in $(K, \mathscr{T}_K)$, then for each $a \in K$, the $\mathscr{T}_K$-neighborhood filter of a must be the translate to a of the

$\mathscr{T}_K$-neighborhood filter of zero. We will now discuss sufficient conditions on $(A, \mathscr{T})$ for $(K, \mathscr{T}_K)$ to have this property.

Suppose that $(A, \mathscr{T})$ is a topological ring. Then for each $p \in A^*$, the mapping $x \mapsto px$ is a continuous mapping of A into itself. Consequently, $p\mathscr{N}_{\mathscr{T}}(0)$ is a filter base for a filter which is finer than $\mathscr{N}_{\mathscr{T}}(0)$. In general, if p and q are distinct elements of A^*, then $p\mathscr{N}_{\mathscr{T}}(0)$ and $q\mathscr{N}_{\mathscr{T}}(0)$ are not equivalent filter bases; however, if for every pair (p, q) of elements of A^*, $p\mathscr{N}_{\mathscr{T}}(0)$ and $q\mathscr{N}_{\mathscr{T}}(0)$ are equivalent filter bases, then for each $p \in A^*$, $p\mathscr{N}_{\mathscr{T}}(0)$ and $p^2\mathscr{N}_{\mathscr{T}}(0)$ are equivalent filter bases. In this case, given $N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$, there exists $N'_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ such that $pN'_{\mathscr{T}}(0) \subset p^2N_{\mathscr{T}}(0)$ which implies that $N'_{\mathscr{T}}(0) \subset pN_{\mathscr{T}}(0)$. Therefore $pN_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ and consequently, $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$. Conversely, if for each $p \in A^*$, $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$, then for every pair (p, q) of elements of A^*, $p\mathscr{N}_{\mathscr{T}}(0)$ and $q\mathscr{N}_{\mathscr{T}}(0)$ are equivalent filter bases.

LEMMA 5. *Let (R, T) be any topological ring. The following conditions on (R, T) are equivalent.*

(1) *Given an open neighborhood O of zero and a nonzero element p of R, then pO is an open set.*

(2) *Given a nonzero element p of R, then $p\mathscr{N}_T(0)$ is a base for $\mathscr{N}_T(0)$. ($\mathscr{N}_T(0)$ is the T-neighborhood filter of zero.)*

Proof.

(1) implies (2): Let p be a nonzero element of R and let $N_T(0) \in \mathscr{N}_T(0)$. Since (R, T) is a topological ring, the mapping $x \mapsto px$ is a continuous mapping of R into itself. Consequently, there exists an open neighborhood O of zero such that] $pO \subset N_T(0)$. By hypothesis, pO is an open neighborhood of zero. Therefore $p\mathscr{N}_T(0)$ is a base for $\mathscr{N}_T(0)$.

(2) implies (1): Let O be an open neighborhood of zero and let p be a nonzero element of R. Let $p\alpha$ be an arbitrary element of pO. Then O is a neighborhood of α. Consequently, there exists $O' \in \mathscr{N}_T(0)$ such that $O = \alpha + O'$. By hypothesis, $p\mathscr{N}_T(0)$ is a base for $\mathscr{N}_T(0)$. Therefore pO' is a neighborhood of zero. Now $pO = p\alpha + pO'$ and hence pO is an element of $\mathscr{N}_T(p\alpha)$. Therefore pO is in the neighborhood filter of each of its points which implies that pO is an open set.

THEOREM 9. *Suppose that $(A, \mathscr{T})$ is a topological ring. If for every $p \in A^*$, $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$, then $\mathscr{K}(0) = \mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}(0)$ and $\mathscr{N}(a) = a + \mathscr{N}(0)$ for every $a \in K$.*

Proof. By Lemma 3, for every $a \in K$, $\mathscr{B}(a) = \{N_p(a) | a \in A_p$ and

$N_p(a) \in \mathscr{N}_p(a)$ for some $p \in A^*\}$ is a subbase for $\mathscr{N}(a)$. If $N_p(0) \in \mathscr{B}(0)$, then there exists $N_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ such that $N_p(0) = N_{\mathscr{T}}(0)/p$. Since $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$, there exists $N'_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ such that $pN'_{\mathscr{T}}(0) \subset N_{\mathscr{T}}(0)$. Therefore $N_p(0) = N_{\mathscr{T}}(0)/p \supset pN'_{\mathscr{T}}(0)/p = N'_{\mathscr{T}}(0)$. This implies that $\mathscr{B}(0) < \mathscr{K}(0)$. On the other hand, if $N_{\mathscr{T}}(0) \in \mathscr{K}(0)$ and $p \in A^*$, then $pN_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$. Now $N_{\mathscr{T}}(0) = pN_{\mathscr{T}}(0)/p$ which is an element of $\mathscr{B}(0)$. This implies that $\mathscr{K}(0) < \mathscr{B}(0)$. Therefore $\mathscr{K}(0)$ and $\mathscr{B}(0)$ are equivalent subbases. However, since $\mathscr{K}(0)$ is a filter on A, it is a filter base on K. Consequently, $\mathscr{K}(0)$ and $\mathscr{B}(0)$ are bases for the filter $\mathscr{N}(0)$. For each $a \in K$, let $\mathscr{K}(a) = a + \mathscr{K}(0)$. Clearly $\mathscr{K}(a)$ is a base for the filter $a + \mathscr{N}(0)$. If $a \in A_p$ and $N_p(a) \in \mathscr{N}_p(a)$, then there exists $\alpha \in A$ and $N_{\mathscr{T}}(\alpha) \in \mathscr{N}_{\mathscr{T}}(\alpha)$ such that $a = \alpha/p$ and $N_p(a) = N_{\mathscr{T}}(\alpha)/p$. Since $(A, \mathscr{T})$ is a topological ring, there exists $N_{\mathscr{T}}(0)$ such that $N_{\mathscr{T}}(\alpha) = \alpha + N_{\mathscr{T}}(0)$. Moreover, $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$. Therefore there exists $N'_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$ such that $pN'_{\mathscr{T}}(0) \subset N_{\mathscr{T}}(0)$. Now we have

$$N_p(a) = \frac{N_{\mathscr{T}}(\alpha)}{p} = \frac{\alpha + N_{\mathscr{T}}(0)}{p} \subset \frac{\alpha + pN'_{\mathscr{T}}(0)}{p} = a + N'_{\mathscr{T}}(0) .$$

This implies that $\mathscr{B}(a) < \mathscr{K}(a)$. Conversely, if $a + N_{\mathscr{T}}(0) \in \mathscr{K}(a)$, choose $p \in A^*$ such that $a \in A_p$. Now let $\alpha \in A$ such that $a = \alpha/p$. Since $p\mathscr{N}_{\mathscr{T}}(0)$ is a base for $\mathscr{N}_{\mathscr{T}}(0)$, we have $pN_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(0)$. This implies that $\alpha + pN_{\mathscr{T}}(0) \in \mathscr{N}_{\mathscr{T}}(\alpha)$. Consider

$$a + N_{\mathscr{T}}(0) = \frac{\alpha}{p} + \frac{pN_{\mathscr{T}}(0)}{p} = \frac{\alpha + pN_{\mathscr{T}}(0)}{p} .$$

This is an element of $\mathscr{B}(a)$ and consequently $\mathscr{K}(a) < \mathscr{B}(a)$. Therefore $\mathscr{B}(a)$ is a filter base which is equivalent to $\mathscr{K}(a)$. Since $\mathscr{B}(a)$ is a base for $\mathscr{N}(a)$ and $\mathscr{K}(a)$ is a base for $a + \mathscr{N}(0)$, we have $\mathscr{N}(a) = a + \mathscr{N}(0)$.

What we have now demonstrated is that if $(A, \mathscr{T})$ is a topological ring which satisfies either of the conditions of Lemma 5, then $(K, \mathscr{T}_K)$ is homogeneous in the sense that the $\mathscr{T}_K$-neighborhood filter of any point is the translate to that point of the $\mathscr{T}_K$-neighborhood filter of zero. Moreover, the neighborhood filter of zero in $(A, \mathscr{T})$ is a base for the neighborhood filter of zero in $(K, \mathscr{T}_K)$. Also, since $(A, \mathscr{T})$ satisfies one of the conditions of Lemma 5, by Theorem 6 it follows that A is topologically embedded in $(K, \mathscr{T}_K)$ as an open set.

In [10, Theorem 5], Warner places the following conditions on a topological ring which has no divisors of zero.

(1) Given an open neighborhood O of zero and a nonzero ring element p, then pO and Op are open sets.

(2) The collection of ring elements which have an inverse relative

to the circle composition ($x \circ y = x + y - xy$) is an open set, and the mapping which sends an element of this open set to this inverse is continuous.

He concludes that these conditions are both necessary and sufficient for the ring to be algebraically embeddable in a division ring, where the neighborhood filter of zero in the original ring is a fundamental system of neighborhoods of zero for a topology on the division ring. Moreover, the specified topology on the division ring is compatible with the division ring structure and the original ring is topologically embedded as an open set. Therefore, by Lemma 5 and Theorem 9, we conclude that these conditions on $(A, \mathscr{T})$ are necessary and sufficient for $(K, \mathscr{T}_K)$ to be a topological field with A topologically embedded as an open set. In the process of proving this theorem of Warner's, condition (2) is used only to establish the continuity of multiplicative inversion in the division ring. Hence we conclude that $(K, \mathscr{T}_K)$ is a topological ring with A topologically embedded as an open set if and only if $(A, \mathscr{T})$ satisfies one of the conditions of Lemma 5.

Several questions concerning the topology $\mathscr{T}_K$ are suggested by this paper. For instance, what hypotheses are required for $(K, \mathscr{T}_K)$ to be a topological field without A necessarily being an open set? By Theorem 5, Corollary c, if $(A, \mathscr{T})$ is compact and Hausdorff, then it is topologically embedded in $(K, \mathscr{T}_K)$. What further hypotheses, if any, are needed to insure that $(K, \mathscr{T}_K)$ is at least a topological ring? There is also, of course, the observation that the concept of K-convergence provides a method for topologizing the Mikusiński field. In fact, the various algebraic models which generate the Mikusiński field lead to several topologies of K-convergence on it. What properties do they possess and how are they related?

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