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TORSION THEORIES AND RINGS OF QUOTIENTS OF MORITA EQUIVALENT RINGS

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A ring of left quotients $Q_{\mathscr{T}}$ of a ring R can be constructed relative to any hereditary torsion class $\mathscr{T}$ of left R-modules. For Morita equivalent rings R and S we construct a one-to-one correspondence between the hereditary torsion classes (strongly complete Serre classes) of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ and describe the resulting correspondence between the strongly complete filters of left ideals of R and S. We show that the proper rings of left quotients of R and S relative to corresponding hereditary torsion classes are Morita equivalent. Applications are made to the maximal and the classical rings of left quotients and the corresponding torsion theories.

A *torsion theory* for the category ${}_R\mathfrak{M}$ of unitary left modules over an associative ring R with identity has been defined by Dickson [3] to be a pair $(\mathscr{T}, \mathscr{F})$ of classes of left R-modules such that

(a) $\mathscr{T} \cap \mathscr{F} = \{0\}$

(b) $\mathscr{T}$ is closed under homomorphic images

(c) $\mathscr{F}$ is closed under submodules

(d) for every left R-module M there exists a submodule $T(M)$ of M with $T(M) \in T$ and $M/T(M) \in \mathscr{F}$.

A class $\mathscr{T}(\mathscr{F})$ of left modules is called a *torsion* (*torsion-free*) class if there is a (necessarily unique) class $\mathscr{F}(\mathscr{T})$ such that $(\mathscr{T}, \mathscr{F})$ is a torsion theory. A torsion class closed under submodules is said to be *hereditary*. By [3, Theorem 2.3] a class $\mathscr{T}$ is a hereditary torsion class if and only if it is closed under submodules, homomorphic images, extensions, and arbitrary direct sums. Walker and Walker [13] call such a class a *strongly complete Serre class*. Gabriel [4] has shown that for a ring R there is a one-to-one correspondence between the strongly complete Serre classes of ${}_R\mathfrak{M}$ and the strongly complete filters F of left ideals of R given by the mapping

$$\mathscr{T} \longrightarrow F(\mathscr{T}) = \{I \leqq R \mid R/I \in \mathscr{T}\}$$

where $I \leqq R$ denotes that I is a left ideal of R. The inverse correspondence is given by

$$F \longrightarrow \mathscr{T}(F) = \{M \in {}_R\mathfrak{M} \mid (0\colon m) \in F \text{ for all } m \in M\}$$

where $(0\colon m) = \{r \in R \mid rm = 0\}$. We say a strongly complete filter F of left ideals of R is *faithful* if $(0\colon r) \in F$ implies $r = 0$ for each $r \in R$. A strongly complete Serre class $\mathscr{T}$ is called a *faithful Serre* *class* if $F(\mathscr{T})$ is faithful. Viewing $\mathscr{T}$ as a hereditary torsion class this is equivalent to the requirement that ${}_RR$ is torsion-free.

1. **Rings of quotients.** Throughout this section $\mathscr{T}$ will denote a faithful Serre class of ${}_R\mathfrak{M}$ with associated filter F. Then $(\mathscr{T}, \mathscr{F})$ is a torsion theory for ${}_R\mathfrak{M}$ and ${}_RR \in \mathscr{F}$ where

$$\mathscr{F} = \{M \in {}_R\mathfrak{M} \mid \mathrm{Hom}_R(T, M) = 0 \text{ for all } T \in \mathscr{T}\} .$$

Let $\mathscr{A}$ denote the quotient category of ${}_R\mathfrak{M}$ relative to $\mathscr{T}$ as defined in [4] and let

$$R_{\mathscr{T}} = \mathrm{Hom}_{\mathscr{A}}(R, R) = \varinjlim_{I \in F} \mathrm{Hom}_R(I, R)$$

the endomorphism ring of R as an object of $\mathscr{A}$. The opposite ring of $R_{\mathscr{T}}$ is denoted by $Q_{\mathscr{T}}$ and is called the *ring of left quotients of R relative to $\mathscr{T}$*. The natural ring anti-isomorphism of R and $\mathrm{Hom}_R(R, R)$ induces a one-to-one ring homomorphism $\varphi: R \rightarrow Q_{\mathscr{T}}$. We usually identify R as a unital subring of $Q_{\mathscr{T}}$. More generally, for each left R-module M let

$$M_{\mathscr{T}} = \mathrm{Hom}_{\mathscr{A}}(R, M) = \varinjlim_{R/I,\, M' \in \mathscr{T}} \mathrm{Hom}_R(I, M/M') .$$

Using the composition of morphisms in $\mathscr{A}$ each $M_{\mathscr{T}}$ is a right $R_{\mathscr{T}}$-module and thus a left $Q_{\mathscr{T}}$-module. The ring homomorphism φ induces a left R-module structure on $M_{\mathscr{T}}$ and there is a natural left R-homomorphism $\varphi_M: M \rightarrow M_{\mathscr{T}}$ given by $\varphi_M(m) = [\rho_m]$, the equivalence class of ρ_m in $M_{\mathscr{T}}$, where for each $m \in M$, $\rho_m: R \rightarrow M$ by $\rho_m(r) = rm$. As shown in [13] for each left R-module M, $\ker \varphi_M = T(M) = \{m \in M \mid (0: m) \in F\}$.

A left R-module M is said to be *$\mathscr{T}$-injective* if for every exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow T \longrightarrow 0$$

of left R-modules with $T \in \mathscr{T}$, the associated sequence

$$0 \longrightarrow \mathrm{Hom}_R(T, M) \longrightarrow \mathrm{Hom}_R(L, M) \longrightarrow \mathrm{Hom}_R(K, M) \longrightarrow 0$$

is exact. By [13, Proposition 4.2] for each left R-module M

$$E_{\mathscr{T}}(M) = \{x \in E(M) \mid (M: x) \in F\}$$

is $\mathscr{T}$-injective and is (up to isomorphism) the unique minimal $\mathscr{T}$-injective module containing M where $E(M)$ is an injective envelope of M. We call $E_{\mathscr{T}}(M)$ a *$\mathscr{T}$-injective envelope* of M. The following lemmas are consequences of [4, Proposition 4, page 413] but the proof is included for the sake of completeness.

LEMMA 1.1. *For each* $M \in \mathscr{F}$, $E_{\mathscr{T}}(M) \cong M_{\mathscr{T}}$ *as left R-modules.*

Proof. For each $x \in E_{\mathscr{T}}(M)$, $(M: x) \in F$. Define $\lambda: E_{\mathscr{T}}(M) \rightarrow M_{\mathscr{T}}$ by $\lambda(x) = [\rho_x]$ for each $x \in E_{\mathscr{T}}(M)$ where $\rho_x(r) = rx$ for each $r \in (M: x)$. It is easily checked that λ is additive.

By [3, Theorem 2.9] $\mathscr{F}$ is closed under injective envelopes. Thus $E(M)$ and hence $E_{\mathscr{T}}(M) \in \mathscr{F}$. If $x \in E_{\mathscr{T}}(M)$ and $\lambda(x) = 0$, then $Ix = 0$ for some $I \in F$. Since $E_{\mathscr{T}}(M) \in \mathscr{F}$ this implies $x = 0$. Thus λ is one-to-one.

Let $[f] \in M_{\mathscr{T}}$ be represented by $f: I \rightarrow M$ with $I \in F$. Since $E_{\mathscr{T}}(M)$ is $\mathscr{T}$-injective and contains M, f extends to an R-homomorphism $\bar{f}: R \longrightarrow E_{\mathscr{T}}(M)$. Let $x = \bar{f}(1) \in E_{\mathscr{T}}(M)$. Then $\lambda(x) = [f]$ so λ is onto.

Finally, for $x \in E_{\mathscr{T}}(M)$ and $r \in R$ one checks that $\lambda(rx) = r\lambda(x)$.

In the special case that $M = {}_RR$ we have the following.

LEMMA 1.2. *As left R-modules,* $Q_{\mathscr{T}} \cong E_{\mathscr{T}}(R)$.

From this we get the following proposition which will be used later in studying Morita equivalence of quotient rings.

PROPOSITION 1.3. *If* $\mathscr{T}$ *is any faithful Serre class of* ${}_R\mathfrak{M}$, *then* $Q_{\mathscr{T}} \cong \mathrm{End}_R(E_{\mathscr{T}}(R))^0$ *as rings.*

Proof. Let $f \in \mathrm{End}_R(Q_{\mathscr{T}})$ and let $q, x \in Q_{\mathscr{T}}$. Then for each $r \in (R: q)$, $r(qf(x) - f(qx)) = 0$. But $(R: q) \in F$ and $Q_{\mathscr{T}} \in \mathscr{F}$. Thus $qf(x) = f(qx)$. It follows that $\mathrm{End}_R(Q_{\mathscr{T}}) = \mathrm{End}_{Q_{\mathscr{T}}}(Q_{\mathscr{T}})$. Using the natural ring anti-isomorphism and (1.2) we have

$$Q_{\mathscr{T}} \cong \mathrm{End}_{Q_{\mathscr{T}}}(Q_{\mathscr{T}})^0 = \mathrm{End}_R(Q_{\mathscr{T}})^0 \cong \mathrm{End}_R(E_{\mathscr{T}}(R))^0 .$$

We now investigate more closely the relationship between the ring of left quotients $Q_{\mathscr{T}}$ and the torsion theory $(\mathscr{T}, \mathscr{F})$. As previously noted $\ker \varphi_M = T(M)$ for each left R-module M where φ_M is the natural R-homomorphism from M to $M_{\mathscr{T}}$. For each left R-module M, $\varphi_M = \theta_M \eta_M$ where

$$\eta_M: M \longrightarrow Q_{\mathscr{T}} \otimes_R M \quad \text{by} \quad \eta_M(m) = 1 \otimes m$$

and

$$\theta_M: Q_{\mathscr{T}} \otimes_R M \longrightarrow M_{\mathscr{T}} \quad \text{by} \quad \theta_M(x \otimes m) = x\varphi_M(m)$$

for each $m \in M$ and each $x \in Q_{\mathscr{T}}$. Thus in general we have $\ker \eta_M \subseteq T(M)$.

THEOREM 1.4. *Let* $\mathscr{T}$ *be a strongly complete Serre class of* ${}_R\mathfrak{M}$.

Then $T(M) = \ker \eta_M$ *for every left R-module M if and only if* $Q_{\mathscr{T}}\varphi(I) = Q_{\mathscr{T}}$ *for all* $I \in F = F(\mathscr{T})$. *Moreover* $Q_{\mathscr{T}}$ *is flat as a right R-module whenever* $T(M) = \ker \eta_M$ *for all M.*

Proof. If $Q_{\mathscr{T}}\varphi(I) = Q_{\mathscr{T}}$ for all $I \in F$, then θ_M is an isomorphism for each left R-module M by [13, Theorem 3.2]. Hence $\ker \varphi_M = \ker \eta_M = T(M)$ for every M.

Conversely if $\ker \eta_M = T(M)$ for every left R-module M, then $R/I = \ker \eta_{R/I}$ for each $I \in F$. Thus $Q_{\mathscr{T}} \otimes_R R/I = 0$ for every $I \in F$. Hence for each $I \in F$ the mapping $Q_{\mathscr{T}} \otimes_R I \longrightarrow Q_{\mathscr{T}} \otimes_R R$ is an isomorphism. Thus $Q_{\mathscr{T}} = Q_{\mathscr{T}}\varphi(I)$ for each $I \in F$. The last remark follows by [13, Corollary 3.3].

We conclude this section indicating two important special cases of this result.

A left ideal I of R is said to be *dense* if $(I: a)b \neq 0$ for all a, b in R with $b \neq 0$. The strongly complete faithful filter D of dense left ideals of R is maximal among all the strongly complete faithful filters of left ideals of R. The corresponding faithful Serre class

$$\mathscr{T}' = \{M \in {}_R\mathfrak{M} \mid (0: m) \in D \text{ for all } m \in M\}$$

is thus maximal among all the faithful Serre classes of ${}_R\mathfrak{M}$ and coincides with the $E(R)$-torsion class considered by Jans [6]. The ring of left quotients of R relative to $\mathscr{T}'$ is called the *maximal ring of left quotients* of R and is denoted by $Q({}_RR)$.

For each left R-module ${}_RM$ we let $Z({}_RM)$ denote the set of all elements of ${}_RM$ whose annihilator is an essential left ideal of R. Then $Z({}_RM)$ is a submodule of ${}_RM$ called the *singular submodule* of ${}_RM$. For a ring R with $Z({}_RR) = 0$, a left ideal is dense if and only if it is essential. For such rings $Q({}_RR)$ is von Neumann regular. (See [7]) Moreover for a ring R with $Z({}_RR) = 0$, $Q({}_RR)$ is semisimple (with minimum condition) if and only if $Q({}_RR)I = Q({}_RR)$ for all essential left ideals of R by [11, Theorem 1.6] or [13, Theorem 4.19]. Combining these facts with (1.4) we get the following results of Sandomierski [11].

PROPOSITION 1.5. *Let R be a ring with* $Z({}_RR) = 0$. *Then* $Z(M) = \ker \eta_M$ *where* $\eta_M: M \longrightarrow Q({}_RR) \otimes_R M$ *via* $\eta_M(m) = 1 \otimes m$ *for every left R-module M if and only if* $Q({}_RR)$ *is semisimple. Moreover, if* $Q({}_RR)$ *is semisimple it is flat as a right R-module.*

Let U denote the set of two-sided nonzero divisors of R, let $F_C = \{I \leqq R \mid I \cap U \neq \varnothing\}$ and let

$$\mathscr{T}_C = \{M \in {}_R\mathfrak{M} \mid (0: m) \in F_C \quad \text{for all} \quad m \in M\}\ .$$

A ring R is said to be *left Ore* if for all $a \in R$ and $d \in U$ there exist

$a' \in R$ and $d' \in U$ such that $d'a = a'd$. One checks that F_C is a strongly complete faithful filter of left ideals of R and $\mathscr{T}_C$ is a faithful Serre class of ${}_R\mathfrak{M}$ if and only if R is left Ore. For any left Ore ring R, the ring of left quotients of R relative to $\mathscr{T}_C$ is denoted by $Q_C(R)$ and is called the *classical ring of left quotients* of R. For a left Ore ring R, $Q_C(R)$ has the following properties:

(a) $d \in U$ implies d^{-1} exists in $Q_C(R)$

(b) for each $q \in Q_C(R)$, there exists $a \in R$ and $d \in U$ with $q = d^{-1}a$.

For a left Ore ring R, every $I \in F_C$ contains an invertible element of $Q_C(R)$. Hence $Q_C(R)I = Q_C(R)$ for every $I \in F_C$. Applying (1.4) we have the following results of Levy [8].

PROPOSITION 1.6. *Let R be a left Ore ring. Then for each left R-module M, the kernel of the mapping η_M: $M \longrightarrow Q_C(R) \otimes_R M$ defined by $\eta_M(m) = 1 \otimes m$ is $T_C(M) = \{m \in M \mid (0: m) \in F_C\}$. Moreover $Q_C(R)$ is flat as a right R-module.*

2. Morita equivalence of quotient rings. Morita has shown that two rings R and S have equivalent categories of unitary left modules if and only if $S \cong \mathrm{End}_R(P_R)$ for some right R-progenerator P_R where a right R-module P_R is called a *progenerator* if it is finitely generated projective and if the right regular module R_R is isomorphic to a direct summand of a direct sum of copies of P_R. (See [1] or [10]) Two such rings are said to be *Morita equivalent*. Throughout this paper we assume $S = \mathrm{End}_R(P_R)$ with P_R a progenerator. Then the functors

$$G = P \otimes_R (\quad): {}_R\mathfrak{M} \longrightarrow {}_S\mathfrak{M}$$

and

$$H = P^* \otimes_S (\quad): {}_S\mathfrak{M} \longrightarrow {}_R\mathfrak{M}$$

are inverse category equivalences where $P^* = \mathrm{Hom}_R(P, R)$ is a left R-progenerator.

If $\mathscr{T}(R)$ is any strongly complete Serre class of ${}_R\mathfrak{M}$, then

$$\mathscr{T}(S) = \{M \in {}_S\mathfrak{M} \mid H(M) \in \mathscr{T}(R)\}$$

is a strongly complete Serre class of ${}_S\mathfrak{M}$ since H preserves exactness and direct sums. The mapping pairing each $\mathscr{T}(R)$ with $\mathscr{T}(S)$ as defined above gives a one-to-one correspondence between the strongly complete Serre classes of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$. Henceforth $\mathscr{T}(R)$ and $\mathscr{T}(S)$ will denote corresponding strongly complete Serre classes of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ respectively. By our introductory remarks there are (unique) classes $\mathscr{F}(R)$ and $\mathscr{F}(S)$ such that $(\mathscr{T}(R), \mathscr{F}(R))$ and $(\mathscr{T}(S), \mathscr{F}(S))$ are hereditary torsion theories for ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ respectively. Moreover,

$$\mathscr{F}(S) = \{M \in {}_S\mathfrak{M} \mid H(M) \in \mathscr{F}(R)\} .$$

PROPOSITION 2.1. *$\mathscr{T}(R)$ is faithful if and only if $\mathscr{T}(S)$ is faithful.*

Proof. If $\mathscr{T}(R)$ is faithful, then ${}_RR \in \mathscr{F}(R)$. Hence by [3, Theorem 2.3] every finitely generated projective left R-module is in $\mathscr{F}(R)$. But $H({}_SS) \cong {}_RP^*$ is a finitely generated projective left R-module, so $H({}_SS) \in \mathscr{F}(R)$. Thus ${}_SS \in \mathscr{F}(S)$, so $\mathscr{T}(S)$ is faithful. The converse follows by a dual argument.

Throughout the remainder of this paper unless otherwise noted we restrict our attention to the case where $\mathscr{T}(R)$ and $\mathscr{T}(S)$ and faithful.

We let $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ denote the rings of left quotients of R and S relative to $\mathscr{T}(R)$ and $\mathscr{T}(S)$ respectively as defined in § 1. Before examining the Morita equivalence of $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ we need a few observations on $\mathscr{T}$-injectivity. Using routine arguments with the category equivalences G and H one gets the following.

LEMMA 2.2. *Let M be a left R-module. Then M is $\mathscr{T}(R)$- injective if and only if $G(M)$ is $\mathscr{T}(S)$-injective.*

PROPOSITION 2.3. *Let M be a left R-module with $\mathscr{T}(R)$-injective envelope $E_{\mathscr{T}(R)}(M)$. Then $G(E_{\mathscr{T}(R)}(M)$ is a $\mathscr{T}(S)$-injective envelope of $G(M)$.*

Proof. By the lemma, $G(E_{\mathscr{T}(R)}(M))$ is a $\mathscr{T}(S)$-injective extension of $G(M)$. Using the fact that G induces an isomorphism between the lattices of submodules of $E_{\mathscr{T}(R)}(M)$ and $G(E_{\mathscr{T}(R)}(M))$ one checks that $G(E_{\mathscr{T}(R)}(M))$ is a minimal $\mathscr{T}(S)$-injective extension of $G(M)$.

Two left R-modules M and N are said to be *similar* if each is isomorphic to a direct summand of a finite direct sum of copies of the other. Observing that finite direct sums of $\mathscr{T}(R)$-injective modules are $\mathscr{T}(R)$-injective one checks that similar left R-modules have similar $\mathscr{T}(R)$-injective envelopes. Since the left R-module ${}_RP^*$ is a progenerator and is thus similar to ${}_RR$ we have $E_{\mathscr{T}(R)}({}_RP^*)$ is similar to $E_{\mathscr{T}(R)}({}_RR)$.

To simplify our notation we let $E_{\mathscr{T}}(R) = E_{\mathscr{T}(R)}({}_RR)$, $E_{\mathscr{T}}(P^*) = E_{\mathscr{T}(R)}({}_RP^*)$ and $E_{\mathscr{T}}(S) = E_{\mathscr{T}(S)}({}_SS)$. Then using (2.3) and the fact that $G(P^*) \cong {}_SS$, we have

$$\begin{aligned} \operatorname{End}_R(E_{\mathscr{T}}(P^*)) &\cong \operatorname{End}_S(G(E_{\mathscr{T}}(P^*))) \\ &\cong \operatorname{End}_S(E_{\mathscr{T}}(G(P^*))) \\ &\cong \operatorname{End}_S(E_{\mathscr{T}}(S)) . \end{aligned}$$

Thus by (1.3)

$$Q_{\mathscr{T}(R)} \cong \mathrm{End}_R(E_{\mathscr{T}}(R))^0$$

and

$$Q_{\mathscr{T}(S)} \cong \mathrm{End}_S(E_{\mathscr{T}}(S))^0 \cong \mathrm{End}_R(E_{\mathscr{T}}(P^*))^0.$$

Hirata [5, Theorem 1.5] has shown that for similar left R-modules M and N, the rings $E = \mathrm{End}_R(M)^0$ and $E' = \mathrm{End}_R(N)^0$ are Morita equivalent. (The opposite rings arise from our convention of regarding mappings as operating on the left.) Moreover $\mathrm{Hom}_R(M, N)$ is a progenerator both as a left E-module and as a right E'-module. Similarly $\mathrm{Hom}_R(N, M)$ is a progenerator both as a left E'-module and as a right E-module.

Letting $M = E_{\mathscr{T}}(R)$ and $N = E_{\mathscr{T}}(P^*)$ we conclude that the rings $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ are Morita equivalent and that $\mathrm{Hom}_R(E_{\mathscr{T}}(P^*), E_{\mathscr{T}}(R))$ is a progenerator both as a left $Q_{\mathscr{T}(S)}$-module and as a right $Q_{\mathscr{T}(R)}$-module.

Since $P \otimes_R E_{\mathscr{T}}(R)$ is $\mathscr{T}(S)$-injective and

$$0 \longrightarrow S \longrightarrow E_{\mathscr{T}}(S) \longrightarrow E_{\mathscr{T}}(S)/S \longrightarrow 0$$

is an exact sequence of left S-modules with $E_{\mathscr{T}}(S)/S \in \mathscr{T}(S)$,

$$0 \longrightarrow \mathrm{Hom}_S(E_{\mathscr{T}}(S)/S, P \otimes_R E_{\mathscr{T}}(R)) \longrightarrow \mathrm{Hom}_S(E_{\mathscr{T}}(S), P \otimes_R E_{\mathscr{T}}(R))$$
$$\longrightarrow \mathrm{Hom}_S(S, P \otimes_R E_{\mathscr{T}}(R)) \longrightarrow 0$$

is an exact sequence of right $Q_{\mathscr{T}(R)}$-modules. But $\mathrm{Hom}_S(E_{\mathscr{T}}(S)/S, P \otimes_R E_{\mathscr{T}}(R)) = 0$ since $E_{\mathscr{T}}(S)/S \in \mathscr{T}(S)$ and $P \otimes_R E_{\mathscr{T}}(R) \in \mathscr{F}(S)$. Hence as a right $Q_{\mathscr{T}(R)}$-module

$$\begin{aligned} \mathrm{Hom}_R(E_{\mathscr{T}}(P^*), E_{\mathscr{T}}(R)) &\cong \mathrm{Hom}_S(E_{\mathscr{T}}(S), P \otimes_R E_{\mathscr{T}}(R)) \\ &\cong \mathrm{Hom}_S(S, P \otimes_R E_{\mathscr{T}}(R)) \\ &\cong P \otimes_R E_{\mathscr{T}}(R) \cong P \otimes_R Q_{\mathscr{T}(R)} . \end{aligned}$$

Summarizing, we have the following theorem.

THEOREM 2.4. *Let $\mathscr{T}(R)$ be a faithful Serre class of ${}_R\mathfrak{M}$ and let $\mathscr{T}(S)$ be the corresponding faithful Serre class of ${}_S\mathfrak{M}$. Then the rings of left quotients $Q_{\mathscr{T}(R)}$ and $Q_{\mathscr{T}(S)}$ are Morita equivalent. Moreover $P \otimes_R Q_{\mathscr{T}(R)}$ is a right $Q_{\mathscr{T}(R)}$-progenerator with*

$$Q_{\mathscr{T}(S)} \cong \mathrm{End}_{Q_{\mathscr{T}(R)}}(P \otimes_R Q_{\mathscr{T}(R)}) .$$

Let F_R be a free right R-module of rank n. Then $\mathrm{End}_R(F_R) \cong R_n$ and $\mathrm{End}_{Q_{\mathscr{T}(R)}}(F \otimes_R Q_{\mathscr{T}(R)}) \cong (Q_{\mathscr{T}(R)})_n$.

COROLLARY 2.5. *Let $\mathscr{T}(R)$ be a faithful Serre class of ${}_R\mathfrak{M}$ and* *let $\mathscr{T}(R_n)$ be the corresponding faithful Serre class of ${}_{R_n}\mathfrak{M}$. Then* $Q_{\mathscr{T}(R_n)} \cong (Q_{\mathscr{T}(R)})_n$.

Previously in this section we described a one-to-one correspondence between the strongly complete Serre classes of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$. We conclude this section by describing the resulting correspondence between the strongly complete filters of left ideals of R and S.

By hypothesis $S = \mathrm{End}_R(P_R)$ with P_R a progenerator. Since P_R is finitely generated and projective, by the Dual Basis Lemma [2, Proposition VII, 3.1] there exist $x_1, \cdots, x_n \in P$ and $f_1, \cdots, f_n \in P^*$ such that

$$x = \sum x_i f_i(x) \quad \text{and} \quad f = \sum f(x_i) f_i$$

for all $x \in P$ and all $f \in P^*$.

For each left ideal I of R, let

$$\bar{I} = \{s \in S \mid s(x_i) \in PI \quad \text{for all} \quad i = 1, \cdots, n\} = \cap (0 \colon {}_S\bar{x}_i)$$

where $\bar{x}_i$ is the canonical image in P/PI of x_i. Similarly, for each left ideal J of S, let

$$\bar{J} = \{r \in R \mid rf_i \in P^*J \quad \text{for all} \quad i = 1, \cdots, n\} = \cap (0 \colon {}_R\bar{f}_i)$$

where $\bar{f}_i$ is the canonical image in P^*/P^*J of f_i.

If $I \in F(R)$, the strongly complete filter of left ideals corresponding to $\mathscr{T}(R)$, then $G(R/I) = P \otimes_R R/I \cong P/PI \in \mathscr{T}(S)$. Thus $(0 \colon {}_S\bar{x}_i) \in F(S)$, the strongly complete filter of left ideals corresponding to $\mathscr{T}(S)$, for all $i = 1, \cdots, n$. It follows that $\bar{I} \in F(S)$.

Similarly, if $J \in F(S)$, then $H(S/J) = P^* \otimes_S S/J \cong P^*/P^*J \in \mathscr{T}(R)$. Thus $(0 \colon {}_R\bar{f}_i) \in F(R)$ for all $i = 1, \cdots, n$. Thus $\bar{J} \in F(R)$.

Finally, if $J \in F(S)$ and $I = \bar{J}$ one checks that $\bar{I} \leqq J$. Thus we have shown the following.

PROPOSITION 2.6. *Let $\mathscr{T}(R)$ and $\mathscr{T}(S)$ be corresponding strongly complete Serre classes of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ with associated filters of left ideals $F(R)$ and $F(S)$ and let J be a left ideal of S. Then $J \in F(S)$ if and only if there exists an $I \in F(R)$ with $\bar{I} \leqq J$.*

3. Applications.

In this section the results of the preceding section and applied to the maximal and the classical rings of left quotients.

Let $\mathscr{T}'(R)$ and $\mathscr{T}'(S)$ denote the maximal faithful Serre classes of ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$. By virtue of their maximality $\mathscr{T}'(R)$ and $\mathscr{T}'(S)$ correspond as in § 2. Hence as a special case of (2.4) we have the following.

THEOREM 3.1. *The maximal rings of left quotients of Morita* *equivalent rings are Morita equivalent.*

COROLLARY 3.2. *Let R and S be Morita equivalent rings. Then $Q({}_RR)$ is von Neumann regular if and only if $Q({}_SS)$ is von Neumann regular. Consequently, $Z({}_RR) = 0$ if and only if $Z({}_SS) = 0$.*

In the following let R be a left Ore ring and let $\mathscr{T}_C(R)$ and $F_C(R)$ be as defined in § 1. As usual let $S = \mathrm{End}_R(P_R)$ with P_R a right R-progenerator. It is unknown whether S is necessarily left Ore. Indeed, we do not know whether the ring of $n \times n$ matrices over a left Ore ring is left Ore for $n > 1$ unless additional requirements are placed on $Q_C(R)$. (See Small [12, Theorem 2.28]) As a partial result we shall show that S is left Ore if R is commutative.

As indicated in § 2,

$$\mathscr{T}(S) = \{M \in {}_S\mathfrak{M} \mid H(M) \in \mathscr{T}_C(R)\}$$

is a faithful Serre class of ${}_S\mathfrak{M}$ with associated filter $F(S)$ given by

$$F(S) = \{J \leqq S \mid \bar{I} \leqq J \quad \text{for some} \quad I \in F_C(R)\}\ .$$

Let

$$F_C(S) = \{J \leqq S \mid J \cap U(S) \neq \varnothing\}$$

where $U(S)$ denotes the set of nonzero divisors of S and let

$$\mathscr{T}_C(S) = \{M \in {}_S\mathfrak{M} \mid (0\colon m) \in F_C(S) \quad \text{for all} \quad m \in M\}\ .$$

If $\mathscr{T}_C(S) = \mathscr{T}(S)$ or equivalently if $F_C(S) = F(S)$, then S is left Ore and $Q_C(R)$ and $Q_C(S)$ are Morita equivalent.

THEOREM 3.3. *If R is commutative, then S is left Ore and $Q_C(R)$ and $Q_C(S)$ are Morita equivalent.*

Proof. We show $F_C(S) = F(S)$. Let $J \in F(S)$. Then there exists $I \in F_C(R)$ with $\bar{I} \leqq J$. Let $d \in I \cap U(R)$ and define $\rho_d \in S$ by $\rho_d(x) = xd$ for each $x \in P$. Then $\rho_d \in \bar{I}$ since $\rho_d(x) \in PI$ for all $x \in P$. For all $s \in S$ and all $x \in P$, $\rho_d s(x) = s\rho_d s(x) = s(x)d$. *If* $\rho_d s = 0$ then $f_i(s(x))d = 0$ for $i = 1, \cdots, n$. Since $d \in U(R)$ and $f_i(s(x)) \in R$ this implies that $f_i(s(x)) = 0$ for $i = 1, \cdots, n$. Therefore $s(x) = \sum x_i f_i(s(x)) = 0$ for all $x \in P$. Hence $s = 0$ so $\rho_d \in U(S)$. Thus $\rho_d \in J \cap U(S)$ so $J \in F_C(S)$. Therefore $F(S) \subseteqq F_C(S)$.

Conversely, let $J \in F_C(S)$ and let $s \in J \cap (S)$. Let F_R be a free right R-module of rank n with $F_R = P_R \oplus P_R'$ for some P_R' and let $\Lambda\colon \mathrm{End}_R(F_R) \to R_n$ be a unital ring isomorphism. Using the fact that P_R is a progenerator one checks that $\bar{s} \in \mathrm{End}_R(F_R)$ defined by $\bar{s}(p, p') = (s(p), p')$ is a nonzero divisor of $\mathrm{End}_R(F_R)$. Since $\Lambda(\bar{s})$ is a nonzero divisor of R_n and R is commutative, det $\Lambda(\bar{s}) \in U(R)$. (See McCoy [9]). Thus letting $I = Rd$, we have $I \in F_C(R)$. Let s' denote the restriction of Λ^{-1} (adj $\Lambda(\bar{s})$) to P_R. Then $s's = \rho_d$ where $\rho_d(x) = xd$ for each $x \in P$ and since $s \in J$, $\rho_d \in J$. Let $t \in \bar{I}$. Define $t' \in S$ by

$$t'(x) = \sum_{i,j=1}^{n} x_j r_{ij} f_i(x) \quad \text{for each} \quad x \in P \quad \text{where}$$

$$t(x_i) = \sum_{j=1}^{n} x_j r_{ij} d \in PI \quad \text{for} \quad i = 1, \cdots, n .$$

Then one checks that $t = t'\rho_d$ and since $\rho_d \in J$, $t \in J$. Hence $\bar{I} \leqq J$ so $J \in F(S)$ by (2.6). Therefore $F_C(S) \subseteqq F(S)$. Thus we have shown that $F_C(S) = F(S)$ and by our previous remarks the theorem follows.

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