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ON UNCONDITIONALLY CONVERGING SERIES AND BIORTHOGONAL SYSTEMS IN A BANACH SPACE

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Our main result is as follows: Let B be a Banach space containing no subspace isomorphic (linearly homeomorphic) to l_∞, and let $\{(b_n,\beta_n)\}$ be a biorthogonal sequence in B such that (β_n) is total. If $x \in B$ then $\sum_{n=1}^{\infty} \beta_n(x)b_n$ converges unconditionally to x if and only if for every sequence (a_n) of 0's and 1's there exists $y \in B$ with $\beta_n(y) = a_n\beta_n(x)$ for all n. This theorem improves previous results of Kadec and Pelczynski.

Similar results are obtained in the context of biorthogonal decompositions of a Banach space into separable subspaces.

1. Preliminaries. We follow the notation of [2] for the most part, and we also refer the reader to [2] for various results concerning unconditional convergence. We recall that a sequence of pairs $\{(b_n, \beta_n)\}$ is called a *biorthogonal sequence* in the Banach space B if for all m and n, $b_m \in B$, $\beta_n \in B^*$, and $\beta_m(b_n) = \delta_{mn}$; (β_n) is said to be *total* (in B) if given $x \in B$ with $\beta_n(x) = 0$ for all n, then $x = 0$. Finally, we denote the space of all bounded scalar-valued sequences by l_∞.

2. The Main Result. We first need the following lemma, due to Seever [8]:

LEMMA 1. *Let X be a Banach space and $T: X \to l_\infty$ be a bounded linear map such that for every $a \in l_\infty$ with $a_n = 0$ or 1 for all n, there exists $x \in X$ with $Tx = a$. Then $T(X) = l_\infty$.*

Proof. Our hypotheses imply that T has dense range; thus it is enough to show that T has closed range. If not, then T^* does not have closed range, so there exists a sequence (γ_n) in l_∞^* with $\|\gamma_n\| \to \infty$ and $\|T^*(\gamma_n)\| = 1$ for all n. But if $a \in l_\infty$ and $a_n = 0$ or 1 for all n, then choosing $x \in X$ with $Tx = a$, we have that

$$\sup_n |\gamma_n(a)| = \sup_n |T^*\gamma_n(x)| \leqq \|x\| < \infty .$$

Thus identifying l_∞ with $C(\beta N)$ (the space of continuous scalar-valued functions on the Stone-Čech compactification of N) and each γ_n with a complex regular Borel measure on βN, we have by a theorem of Dieudonne [3] (c.f. also the *Correction*, pp. 311–313 of [7]) that

$\sup_n \|\gamma_n\| < \infty$, a contradiction.

THEOREM 1. *Let B be a Banach space containing no subspace isomorphic to l_∞, and let $\{(b_n, \beta_n)\}$ be a biorthogonal sequence in B such that (β_n) is total. Let $x \in B$. Then*

(1) $\sum_{n=1}^{\infty} \beta_n(x) b_n$ *converges unconditionally to x*

if and only if

(2) *Given $a \in l_\infty$ with $a_n = 0$ or 1 for all n, there exists $y \in B$ such that $\beta_n(y) = a_n \beta_n(x)$ for all n.*

Proof. Let $x \in B$. If $\sum \beta_n(x) b_n$ converges unconditionally, then it is subseries convergent; thus "(1) $\Rightarrow$ (2)" is immediate. Now suppose that (2) holds. We shall prove that $\Sigma \beta_n(x) b_n$ converges unconditionally. Since (β_n) is total in B it then follows that the limit is x.

Let M be the set of all $a \in l_\infty$ such that there exists $y \in B$ with $\beta_n(y) = a_n \beta_n(x)$ for all n. Given such an a, there is a unique y satisfying the above. We then define $\|a\| = \|a\|_\infty + \|y\|$. It is easily verified that M is a Banach space under this norm. Thus the inclusion map $T: M \to l_\infty$ is continuous and satisfies the hypotheses of Lemma 1. Hence $M = l_\infty$, so T^{-1} is continuous. Thus the mapping U given by $\beta_n(U(a)) = a_n \beta_n(x)$ for all n, is a continuous linear mapping of l_∞ into the Banach space B, which by hypothesis contains no subspace isomorphic to l_∞. Hence by [7, Cor. 1.4], U is weakly compact.

Given a subseries $\sum_k \beta_{n_k}(x) b_{n_k}$, let a be the characteristic function of (n_k). If a subsequence of the partial sums of this subseries, (S_k), converges weakly to $z \in B$, then $\beta_n(z) = \lim_{k \to \infty} \beta_n(S_k) = a_n \beta_n(x)$ for all n; thus $U(a) = z$. Since the partial sums of this subseries are contained in a weakly sequentially compact set (the image under U of the unit ball of l_∞), it follows that the subseries itself converges weakly to $U(a)$. Hence $\Sigma \beta_n(x) b_n$ is weakly subseries convergent, so by the Orlicz-Pettis Theorem it is unconditionally convergent.

REMARKS. (I) If B is separable, then B contains no subspace isomorphic to the (nonseparable) space l_∞, so Theorem 1 holds. In this case one can apply a theorem of Grothendieck [5, p. 168] in the proof, rather than the generalization given by [7, Cor. 1.4].

(II) Suppose that B is separable. Kadec and Pelczynski proved the equivalence of (1) and (2) under the above hypotheses together with the added assumption that the norm $\|x\| = \sup\{|x^*(x)|\}$ (the supremum taken over x^* in the linear span of (β_n) with $\|x^*\| \leqq 1$), is equivalent to the original norm of B. They also proved that $\Sigma \beta_n(x) b_n$ converges unconditionally to x if for *all* $a \in l_\infty$ there exists $y \in B$ such that $\beta_n(y) = a_n \beta_n(x)$ for all n, [6, Thms. 4 and 5, resp.]

(III) An earlier version for Theorem 1 contained the unnecessary hypothesis that (b_n) be fundamental in B. The authors are indebted to Professor Ivan Singer for pointing this out.

(IV) It is crucial that B contain no subspace isomorphic to l_∞, since if B equals l_∞ itself, then the obvious biorthogonal system satisfies (2) for all $x \in B$. The assumption that the biorthogonal set of pairs be denumerable, however, is irrelevant; see Remark (I) at the end of the paper. It is also crucial that (β_n) be total, for consider the following biorthogonal sequence $\{(b_n, \beta_n)\}$ in a separable Hilbert space H:

Let (e_n) be a complete orthonormal sequence in H; let (y_n) be a sequence such that for each n there are infinitely many indices m such that $y_m = y_n$, such that $y_2 = y_{2j}$ for all j, and such that $\{y_n: n = 1, 2, \cdots\} = \{e_{2n-1} : n = 1, 2, \cdots\}$; put $b_n = e_{2n} + y_n$ and $\beta_n = e^*_{2n}$ for all n (where $e^*_{2n}(x) = \langle x, e_{2n}\rangle$, $x \in H$). Now let $x = \sum_{n=1}^{\infty} (1/n)\, e_{4n}$. Then the span of (b_n) is dense in H, yet

(i) for every $a \in l_\infty$ there exists $y \in H$ with $\beta_n(y) = a_n\beta_n(x)$ for all n, and

(ii) $\lim_{n\to\infty} \|\sum_{j=1}^{n} \beta_j(x) b_j\| = \infty$.

(V) If B satisfies the hypotheses of Theorem 1 and (2) holds for *all* $x \in B$, then by Theorem 1 (b_n) is an unconditional basis for B, and in particular B is separable. This result, for B separable, has been announced by William J. Davis, David W. Dean, and Ivan Singer [A.M.S. Notices 17 (1970), 437].

(VI) The argument of the second paragraph of Theorem 1, in the context of Harmonic Analysis, is due to Figá-Talamanca (see [4], p. 347).

3. **Biorthogonal Decompositions.** We wish now to state a similar result concerning biorthogonal decompositions; first some preliminaries:

Given a Banach space B and a collection $\{M_\alpha, P_\alpha\}_{\alpha \in A}$ we say that $\{M_\alpha, P_\alpha\}$ is a *biorthogonal decomposition* in B if for each $\alpha \in A$, M_α is a closed linear subspace of B and P_α is a bounded linear projection of B onto M_α with $P_\alpha(x) = 0$ whenever $x \in M_\beta$ and $\beta \neq \alpha$. We say that $\{M_\alpha, P_\alpha\}$ is *complete* if the linear span of $\{M_\alpha\}$ is dense in B and if $P_\alpha(x) = 0$ for all α implies $x = 0$.

Let now the Banach space B and $\{M_\alpha\}_{\alpha \in A}$, a collection of closed linear subspaces of B, be given. For $A_1 \subseteqq A$, let $S(A_1)$ denote the closed linear span of $\{M_\alpha\}_{\alpha \in A_1}$. We have:

PROPOSITION. *Assume $S(A) = B$. There is a complete biorthogonal decomposition $\{M_\alpha, P_\alpha\}_{\alpha \in A}$ of B, corresponding to $\{M_\alpha\}_{\alpha \in A}$ if and only if both of the following conditions hold:*

(1) *$S(A_1) \cap S(A \sim A_1) = (0)$ for all $A_1 \subseteqq A$.*
(2) *$S(\{\alpha\}) + S(A \sim \{\alpha\}) = B$ for all $\alpha \in A$.*

Proof. The "only if" part is trivial. Suppose now that (1) and (2) hold. Then fixing $\alpha \in A$, (1) and (2) imply that

$$B = S(\{\alpha\}) \oplus S(A \sim \{\alpha\}) .$$

Thus letting P_α be the projection onto $S(\{\alpha\})$ with kernel $S(A \sim \{\alpha\})$, P_α is bounded by the Closed Graph Theorem, whence $\{M_\alpha, P_\alpha\}_{\alpha \in A}$ is a biorthogonal decomposition of B.

Now suppose that $x \in B$ and $P_\alpha(x) = 0$ for all α. There exist finite subsets $A_n \subseteqq A$ and elements $x_n \in S(A_n)$ such that $x_n \to x$. Since $\lim_{n\to\infty} P_\alpha(x_n) = P_\alpha(x) = 0$ for all $\alpha \in A$, we claim that one can choose a subsequence (n_k), subsets $B_k \subseteqq A_{n_k}$ and elements $y_k \in S(B_k)$ such that $B_k \cap B_j = \emptyset$ for k even and j odd, and such that $y_k \to x$. To see this, assume (as we may) that $A_n \subseteqq A_{n+1}$ for all n. Put $n_0 = 1$; having chosen n_k, let $m = \# A_{n_k}$ and choose $n_{k+1} > n_k$ such that $n \geqq n_{k+1}$ and $\alpha \in A_{n_k}$ implies $\|P_\alpha(x_n)\| < (m(k+1))^{-1}$. This defines (n_k); now put $B_k = A_{n_k} \sim A_{n_{k-1}}$ and $y_k = x_{n_k} - \sum_{\alpha \in A_{n_{k-1}}} P_\alpha(x_{n_k})$ for $k = 1, 2, \cdots$.

Let $A_1 = \bigcup_{k=1}^\infty B_{2k}$. Then $y_{2k} \to x$, $y_{2k+1} \to x$, so

$$x \in S(A_1) \cap S(A \sim A_1) = (0) .$$

REMARK: If each M_α is finite-dimensional and $S(A) = B$ then (2) is automatically satisfied. Thus a sequence $\{b_n\}_{n \in N}$ in B corresponds to a complete biorthogonal sequence $\{(b_n, \beta_n)\}$ in B if and only if $S(N) = B$ and $S(N_1) \cap S(N \sim N_1) = (0)$ for all $N_1 \subseteqq N$.

THEOREM 2. *Let B be a Banach space and let $\{M_\alpha\}_{\alpha \in A}$ be a collection of closed separable subspaces with dense span such that*
(1) *$S(A_1) \cap S(A \sim A_1) = (0)$ for all $A_1 \subseteqq A$.*
(2) *$S(\{\alpha\}) + S(A \sim \{\alpha\}) = B$ for all $\alpha \in A$.*
Then $\{x \mid \exists x_\alpha \in M_\alpha$ such that Σx_α converges unconditionally to $x\} = \cap \{S(A_1) + S(A \sim A_1) \mid A_1 \subseteqq A\}$.

Proof. By the preceding proposition, $\{M_\alpha, P_\alpha\}_{\alpha \in A}$ is a complete biorthogonal decomposition of B, where P_α is the projection onto $S(\{\alpha\})$ with kernel $S(A \sim \{\alpha\})$. Since $S(A) = B$, if $x \in B$ we have that $P_\alpha(x) = 0$ for all but a countable number of α's, say $\{\alpha_n\}$. If $x \in \cap \{S(A_1) + S(A \sim A_1) \mid A_1 \subseteqq A\}$, then given $a \in l_\infty$, $a_n = 0$ or 1, by letting $A_1 = \{\alpha_n \mid a_n = 1\}$ we have that there exists $y \in B$ such that $P_{\alpha_n}(y) = a_n P_{\alpha_n}(x)$ for all n and $P_\alpha(y) = 0$, $\alpha \notin \{\alpha_n\}$. All such y's are contained in the separable Banach space $S(\{\alpha_n\})$. With this observation, the proof is similar to the proof of Theorem 1, with (β_n) replaced by $\{P_\alpha\}$.

We conclude with several remarks:

(1) Assume that B contains no subspace isomorphic to l_∞. Then Theorem 2 admits the following generalization: *Let* $\{M_\alpha, P_\alpha\}_{\alpha \in A}$ *be a biorthogonal decomposition of* B *such that* $x \in B$ *and* $P_\alpha(x) = 0$ *for all* α *implies* $x = 0$, *and let* $x \in B$ *be such that for every function* $a: A \to \{0, 1\}$ *with* $a^{-1}\{1\}$ *countable, there exists* $y \in B$ *with* $P_\alpha(y) = a(\alpha)P_\alpha(x)$ *for all* $\alpha \in A$. *Then* $P_\alpha(x) = 0$ *for all but countably many* α's, *and* $\Sigma P_\alpha(x)$ *converges unconditionally to* x. The proof proceeds as in the proof of Theorem 1; one deduces that for each countable subset A_0 of A, $\sum_{\alpha \in A_0} P_\alpha(x)$ converges unconditionally in norm, from which the conclusion easily follows.

(2) For the special case in which $S(A_1) + S(A \sim A_1) = B$ for all $A_1 \subseteqq A$, Theorem 2 was proven in [1].

(3) Theorem 2 applies to the Banach space $L_p(G)$, $1 \leqq p < \infty$, where G is a compact topological group and each M_α is the finite-dimensional subspace generated by the character of an irreducible unitary representation of G. If G is abelian, a direct proof is available, using the existence of approximate identities for L_p which are bounded in the L_1-norm.

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