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ON EXTENSIONS OF HOMEOMORPHISMS TO HOMEOMORPHISMS

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Let $h; P \to Q$ be a homeomorphism between two compact subsets of the topological spaces X and Y respectively.

Conditions on the decompositions of $X \backslash P$ and $Y \backslash Q$ are found such that there exists a homeomorphism H of X onto Y which is an extension of h.

It is shown that if P and Q are compact subsets of the one dimensional space R_ω consisting of all rational points of the Hilbert space l_2 then any homeomorphism between P and Q can be extended to a homeomorphism of R_ω onto itself. Thus an example of a one dimensional space having a very high degree of homogenity is obtained.

A generalization of a theorem of B. Knaster and M. Reichbach (Reichaw) is also given.

Let $h: P \to Q$ be a homeomorphism between two compact subsets $P \subset X$, $Q \subset Y$ of the topological spaces X and Y. The problem of finding conditions under which there exists a homeomorphism $H: X \to Y$ which is an extension of h has been considered by a number of authors (see [7], [9]). It was shown ([7]) that one can extend homeomorphisms given between two compact subsets of the Cantor set to a self homeomorphisms of the Cantor set under certain conditions. There are other examples where one can extend homeomorphisms in totally disconnected spaces.

In this paper theorems on extensions of homeomorphisms between subsets of two topological spaces to a homeomorphism of the whole spaces are proved. Some results concerning the degree of homogenity of spaces are obtained. The theorems obtained here apply mostly to totally disconnected spaces.

In § 1 a generalizations of a theorem of B. Knaster and M. Reichbach (see [7]) from metric separable spaces to regular spaces is given. It is applied to extend homeomorphisms in a non separable lacunar subset of some Banach space.

In § 2 a theorem an extension of homeomorphisms in metric spaces is proved. It is applied to the subspace R_ω of the Hilbert space l_2 consisting of all points $x = \{x_n\}_1^\infty$ such that x_n is rational for each n. As was shown by P. Erdös (see [4] or [5] p. 13) R_ω has dimension 1. We show that *every* homeomorphism between two compact subsets of the space R_ω can be extended to a self homeomorphism of R_ω. Thus an example of a *finite dimensional*, but *not zero dimensional* space having a very high degree of homogenity is obtained. This result is related to a problem posed by B. Knaster in [12]. At the end some problems concerning extensions of homeomorphisms in the Knaster-Kuratowski biconnected set ([8], or [5]) are posed.

NOTATION. In the sequel we use the logical connectives $\vee$ (or) $\wedge$ (and) $\Rightarrow$ (implies). N denotes the set of natural numbers Z the set of integers and R the set of real numbers. card (A) or $\overline{\overline{A}}$ denotes the cardinality of A, nbd. stands for "neighborhood" and $S(p, \varepsilon)$ denotes the ball of radius ε and centre p in a metric space. Finally all homeomorphisms, are "onto".

1. In this section two theorems on extensions of homeomorphisms to homeomorphisms are proved. The first theorem generalizes Theorem 1 of [7] from separable metric spaces to regular spaces. The second theorem follows from the first one and is applied to extend homeomorphisms in lacunar[1] subspace of some Banach space.

DEFINITION 1.1. A directed set[2] A will be called sequentially directed if $A = \bigcup_{i=1}^{\infty} A_i$ where A_i are disjoint and the ordering in A is defined by: Two elements of the same A_i are incomparable and if $\alpha' \in A_i$ $\alpha'' \in A_j$ and $i < j$ then $\alpha' < \alpha''$.

We note that in a sequentially directed set every non cofinal subset has an upper bound.

DEFINITION 1.2. Let $A = \{\alpha | \alpha \in A\}$, $B = \{\beta | \beta \in B\}$ be directed sets. A map $f: A \to B$ will be called cofinality preserving if:

(i) $\alpha_1 \neq \alpha_2 \Rightarrow f(\alpha_1) \neq f(\alpha_2)$

(ii) for every cofinal subset $C \subset A, f(C)$ is a cofinal subset of B.

(iii) for every cofinal subset D, $D \subset f(A)$, $f^{-1}(D)$ is a cofinal subset of A.

A map f satisfying conditions (i) and (ii) will be called semi cofinality preserving.

LEMMA 1.1. *Let A and B be directed sets. Let f and g be cofinality preserving maps $f: A \to B$, $g: B \to A$. Then there exists a bijection $k: A \to B$ such that k is also cofinality preserving and for every $\alpha \in A$ either $k(\alpha) = f(\alpha)$ or $k(\alpha) = g^{-1}(\alpha)$.*

Proof. The proof is similar to the proof of the Cantor Bernstein theorem. The following lemma is trivial.

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[1] A lacunar space is a space in which every compact set is nowhere dense. (See [**10**]).

[2] A directed set A is a partially ordered set such that for every $\alpha', \alpha'' \in A$ there exists $\alpha''' \in A$ with $\alpha''' > \alpha' \wedge \alpha''' > \alpha''$ ([**6**]).

LEMMA 1.2. *Let A and B be two directed sets in which every non cofinal subset has an upper bound. If $f: A \to B$ is monotone and semi cofinality preserving then f is cofinality preserving.*

DEFINITION 1.3. Let $P \subset X$ be a compact subset of X. A decomposition of $X \backslash P$ is a family $\{X_\alpha | \alpha \in A\}$ such that $X \backslash P = \bigcup_{\alpha \in A} X_\alpha$, where X_α are open, closed and disjoint subsets of X and A is a directed set of indices.

A decomposition $\{X_\alpha | \alpha \in A\}$ of $X \backslash P$ is called regular if the following conditions hold:
(1) for every $p \in P$ and every nbd. U_p of p there exists an α_0 and a nbd. $\tilde{U}$ of p, $\tilde{U} \subset U_p$, such that for $\alpha > \alpha_0$ $X_\alpha \cap \tilde{U} \neq \emptyset \Rightarrow X_\alpha \subset U_p$.
(2) for every $p \in P$, every nbd. U_p of p and every α_0 the set $U_p \backslash \cup \{X_\alpha | \alpha \not> \alpha_0\}$ is a nbd. of p.
(3) for every cofinal subset C of A there exists a point $p \in P$ and a cofinal subset $C_p \subset C$ such that for every nbd. U_p of p there exists α_0 such that $U_p \cap X_\alpha \neq \emptyset$ for $\alpha \in C_p \wedge \alpha > \alpha_0$. ($\alpha_0$ depends on U_p).

DEFINITION 1.4. Let $\{X_\alpha | \alpha \in A\}$ and $\{Y_\beta | \beta \in B\}$ be decompositions of $X \backslash P$ and $Y \backslash Q$ respectively. Let $h: P \to Q$ be a homeomorphism. We say (similarly to [7]) that $\{X_\alpha\}$ and $\{Y_\beta\}$ approach P and Q according to h if the following properties hold:
(4) There exists a cofinality preserving map $f: A \to B$ such that X_α is homeomorphic with $Y_{f(\alpha)}$.
(4a) There exists a cofinality preserving map $g: B \to A$ such that Y_β is homeomorphic with $X_{g(\beta)}$.
(5) for every pair of points (p, q) with $p \in P, q = h(p) \in Q$ and for every nbd. V of q there exists a nbd. U of p and α_0 such that for $\alpha > \alpha_0$

$$X_\alpha \cap U \neq \emptyset \Rightarrow Y_{f(\alpha)} \cap V \neq \emptyset$$

(5a) for every pair of points (q, p) with $q \in Q$, $p = h^{-1}(q) \in P$ and for every nbd. U of p there exists a nbd. V of q and β_0 such that for $\beta > \beta_0$.

$$Y_\beta \cap V \neq \emptyset \Rightarrow X_{g(\beta)} \cap U \neq \emptyset$$

THEOREM 1.1. *Let X and Y be regular spaces and let $h: P \to Q$ be a homeomorphism between compact subsets $P \subset X, Q \subset Y$. Let $\{X_\alpha | \alpha \in A\}$ and $\{Y_\beta | \beta \in B\}$ be regular decompositions of $X \backslash P$ and $Y \backslash Q$ respectively and let $\{X_\alpha\}$ and $\{Y_\beta\}$ approach P and Q according to h. Then there exists an extension of h to a homeomorphism $H: X \to Y$.*

Proof. Let $\theta_\alpha: X_\alpha \to Y_{f(\alpha)}$ $\psi_\beta: Y_\beta \to X_{g(\beta)}$ be the homeomorphisms given by (4) and (4a). Let k be the cofinality preserving map of A

onto B given by Lemma 1. Denote:

$$A_f = \{\alpha | \alpha \in A \wedge k(\alpha) = f(\alpha)\}$$

and

$$A_g = \{\alpha | \alpha \in A \wedge k(\alpha) = g^{-1}(\alpha)\} .$$

Define H by:

$$H(x) = \begin{cases} h(x) & x \in P \\ \theta_\alpha(x) & x \in X_\alpha \wedge \alpha \in A_f \\ \psi_\alpha^{-1}(x) & x \in X_\alpha \wedge \alpha \in A_g \backslash A_f . \end{cases}$$

Clearly H is a one-to-one mapping of X onto Y. By the symmetry of our assumptions it suffices to prove that H is continuous. Continuity of H is obvious at every point $x \in X \backslash P$. We shall show that for every point $q \in Q$ and an arbitrary nbd. V of q there exists a nbd. U of $p = h^{-1}(q)$ such that $H(U) \subset V$.

Denote by $\hat{H}$ the map H restricted to $X \backslash P$. It suffices to show that there exists a nbd. $\hat{U}$ of p such that $\hat{H}(\hat{U}) \subset V$.

Let (1a) (2a) and (3a) denote properties obtained from (1), (2) and (3) by replacing X, p, P, U, α, C by Y, q, Q, V, β, D respectively. Let $\tilde{V} \subset V$ be the nbd. of q given by (1a). By (5) there exists a nbd. $\hat{U}_1$ of p contained in U such that

$$(6) \qquad \alpha > \alpha_0 \wedge X_\alpha \cap \hat{U}_1 \neq \emptyset \Rightarrow \theta_\alpha(X_\alpha) \cap \tilde{V} \neq \emptyset .$$

By (2) there exists a nbd. U_1 of p such that

$$(7) \qquad X_\alpha \cap U_1 \neq \emptyset \Rightarrow \theta_\alpha(X_\alpha) \subset V .$$

If there are no sets X_α contained in U_1 for which H is defined by ψ_α^{-1} then obviously H is continuous at p. Thus it remains to consider the case that there exist sets X_α satisfying:

$$(8) \qquad X_\alpha \subset U_1$$

(9) for X_α, H is defined by ψ_α^{-1}

$$(10) \qquad H(X_\alpha) \not\subset V .$$

We denote these sets by $\hat{X}_\alpha$ and the set of their indices by $A' = A'(U_1)$. We prove first the following proposition (*).

(*) The nbd. U_1 can be chosen so that $A'(U_1)$ is not cofinal.

Indeed, suppose that for some U_1, A' is cofinal. By definition of $\tilde{V}$ there exists for this U_1 a cofinal subset of indices α' such that

$$(11) \qquad \hat{X}_{\alpha'} \subset U_1 \text{ and } \psi_{\alpha'}^{-1}(\hat{X}_{\alpha'}) \cap \tilde{V} = \emptyset$$

hence there exists a cofinal subset $A'' = A''(U_1)$ of indices α'' and a point q_1 such that (3a) is satisfied. Clearly $q_1 \notin \tilde{V}$. By (3) there exists a cofinal subset $A''' = A'''(U_1)$ of indices $\alpha'''(A''' \subset A'')$ and a point p_1 such that every nbd. U of p_1 intersects all sets $\hat{X}_{\alpha'''}$ with $\alpha''' > \alpha_0'''$ (α_0''' depends on U). By regularity of X and by $\hat{X}_{\alpha'} \subset U_1$

(12) $$p_1 \in \bar{U}_1 \text{ (the closure of } U_1) .$$

By (5) and (5a) we have $h(p_1) = q_1$.

Indeed assume $h^{-1}(q_1) = p_2 \neq p_1$. Let U_1 and U_2 be disjoint nbd's of p_1 and p_2 and let $\tilde{U}_1 \subset U_1$, $\tilde{U}_2 \subset U_2$ be the nbd's of p_1 and p_2 given by (1). There exists an index α_0''' such that for $\alpha''' > \alpha_0'''$ we have $\hat{X}_\alpha''' \cap \tilde{U}_1 \neq \emptyset$.

According to (5a) there exists for $\tilde{U}_2$ a nbd. V of q_1 and a β_0 such that for $\beta > \beta_0$: $Y_\beta \cap V \neq \emptyset \Rightarrow \hat{X}_{\alpha'''} \cap \tilde{U}_2 \neq \emptyset$ but this is impossible since $\hat{X}_{\alpha'''} \subset U_1$ for $\alpha > \alpha_0'''$.

Suppose now to the contrary that (*) does not hold. Then for every U_1 there exists a point p_{u_1} and a point q_{u_1} such that

$$p_{u_1} \in \bar{U}_1,\ q_{u_1} \in \tilde{V} \text{ and } h(p_{u_1}) = q_{u_1} .$$

But then the generalized sequence $\{p_{u_1}\}$ converges to p which contradicts the continuity of h at p. Thus (*) holds. Consider now the set $\hat{U} = U_1 \setminus \cup \{\hat{X}_\alpha \mid \alpha \in A'\}$. By (*) A' is not a cofinal subset of A. Thus by (2) $\hat{U}$ is a nbd. of p and $\hat{H}(\hat{U}) \subset V$. Theorem 1 is proved.

From now on X and Y will denote metric spaces and the decompositions of $X \setminus P$ and $Y \setminus Q$ will be assumed to have sequentially directed sets of indices A and B, $A = \bigcup_{i=1}^\infty A_i$, $B = \bigcup_{i=1}^\infty B_i$.

THEOREM 1.2. *Let $h: P \to Q$ be a homeomorphism. The following conditions are sufficient for the existence of a homeomorphism $H: X \to Y$ which is an extension of h:*

(13) *for every i, $\bar{\bar{A}}_i = \bar{\bar{B}}_i = M$ where M is some fixed infinite cardinal.*

(14) *for every $\alpha \in A_i$*

$$\delta(X_\alpha) < \frac{1}{2^i}$$

(14a) *for every $\beta \in B_i$*

$$\delta(Y_\beta) < \frac{1}{2^i}$$

(15) *for every $\alpha \in A_i$*

$$d(i) < \rho(X_\alpha, P) < \frac{1}{2^{i-1}}$$

(15a) *for every $\beta \in B_i$*

$$d(i) < \rho(Y_\beta, Q) < \frac{1}{2^{i-1}}$$

where $d(i) > 0$.

(16) *for every* α, β *there exists a homeomorphism*

$$h_{\alpha\beta}: X_\alpha \to Y_\beta .$$

(17) *for every* $p \in P$ *and* $\varepsilon > 0$ *there exists an* i_0 *such that*

$$\operatorname{card}\{\alpha \mid \alpha \in A_{i_0} \wedge X_\alpha \cap S(P, \varepsilon) \neq \emptyset\} = M$$

(17a) *for every* $q \in Q$ *and* $\varepsilon > 0$ *there exists a* j_0 *such that*

$$\operatorname{card}\{\beta \mid \beta \in B_{j_0} \wedge Y_\beta \cap S(q, \varepsilon) \neq \emptyset\} = M .$$

Proof. It suffices to show that all assumptions of Theorem 1 are satisfied. Clearly $\{X_\alpha \mid \alpha \in A\}$ and $\{Y_\beta \mid \beta \in B\}$ are regular decompositions of $X \backslash P$ and $Y \backslash Q$. To show that $\{X_\alpha\}$ and $\{Y_\beta\}$ approach P and Q according to h it suffices to construct (by the symmetry of our assumptions) a monotone semi-cofinality preserving map $f: A \to B$ such that (4) and (5) hold. Let us well order A_i and B_i into type $\omega(M)$ where $\omega(M)$ denotes the first ordinal of cardinality M.

Let $j: N \to N$ satisfy:

$$\frac{1}{2^{j(i)-1}} < \frac{d(i)}{2} \quad \text{for all} \quad i \in N \tag{18}$$

$$j(1) < j(2) \cdots < j(i-1) < j(i) \cdots \tag{19}$$

$$j(i) > j_0 \quad \text{where} \quad j_0 \quad \text{satisfies (17a) with } \varepsilon = \frac{d(i)}{2} \tag{20}$$

For every X_α (where $\alpha \in A_i$) there exists a point $p_\alpha \in P$ such that $\rho(X_\alpha, p_\alpha) = \rho(X_\alpha, P)$. Take the point $q_\alpha = h(p_\alpha) \in Q$. By (17a) and (20) there exists an injection $f_i: A_i \to B_{j(i)}$ such that,

$$Y_{\beta(\alpha)} \cap S\left(q_\alpha, \frac{d(i)}{2}\right) \neq \emptyset .$$

The union $f = \bigcup_{i=1}^{\infty} f_i$ is a semi cofinality preserving monotone map $f: A \to B$ satisfying:

$$\begin{cases} \alpha \in A_i \Rightarrow f(\alpha) \in B_{j(i)} \\ \rho(Y_{f(\alpha)}, h(p_\alpha)) < \rho(X_\alpha, p_\alpha) . \end{cases} \tag{21}$$

Thus (4) holds. By (18), (19), (21) also (5) holds.

Let $\mathscr{M}_\Sigma$ denote the Banach space of all bounded functions from an infinite set Σ to the reals,

$$\mathscr{M}_\Sigma = \{f \mid f\colon \Sigma \to R, \sup_{\sigma \in \Sigma} |f(\sigma)| < \infty\}$$

with the norm $\|f\| = \sup_{\sigma \in \Sigma} |f(\sigma)|$.

Let W denote the subset of $\mathscr{M}_\Sigma$ consisting of rational valued functions. Clearly W is a lacunar space. In the following theorem an application of Theorem 2 to extension of homeomorphisms in the space W is given

THEOREM 1.3. *Each homeomorphism $h\colon P \to Q$ between compact subsets of W has an extension to a self homeomorphism of W.*

Proof. We shall write $W \backslash P (W \backslash Q)$ as union

$$\cup \{X_\alpha \mid \alpha \in A\} (\cup \{Y_\beta \mid \beta \in B\})$$

where $X_\alpha (Y_\beta)$ are "cubes" such that all assumptions of Theorem 2 will be satisfied.

For every function $\alpha \in Z^\Sigma$ from Σ to the set of integers Z denote by X^1_α the cube:

$$X^1_\alpha = \{f \mid f \in W \wedge \sqrt{2} + \alpha(\sigma) < f(\sigma) < \sqrt{2} + \alpha(\sigma) + 1 \text{ for all } \sigma\} . \tag{22}$$

All such cubes are homeomorphic, mutually disjoint, closed and open subsets of W. Let $\mathscr{F}_1 = \{X^1_\alpha \mid \alpha \in Z^\Sigma\}$. Clearly $\overline{\overline{\mathscr{F}_1}} = 2^{\overline{\overline{\Sigma}}}$. Define

$$A_1 = \{\alpha \mid \alpha \in Z^\Sigma \wedge \rho(X^1_\alpha, P) > 1\} . \tag{23}$$

Since P is bounded (as a compact set) we have also: $\overline{\overline{A}}_1 = 2^{\overline{\overline{\Sigma}}}$. For every function $\alpha \in Z^\Sigma$ denote by X^2_α the cube:

$$X^2_\alpha = \left\{f \mid f \in W \wedge \sqrt{2} + \frac{\alpha(\sigma)}{6} < f(\sigma) < \sqrt{2} + \frac{\alpha(\sigma)+1}{6} \text{ for all } \sigma\right\} . \tag{24}$$

Let $\mathscr{F}_2 = \{X^2_\alpha \mid \alpha \in Z^\Sigma \wedge X^2_\alpha \subset W \backslash \bigcup_{\alpha \in A_1} X^1_\alpha\}$ and define

$$A_2 = \{\alpha \mid \alpha \in Z^\Sigma \wedge \rho(X^2_\alpha, P) > \tfrac{1}{6} \wedge X^2_\alpha \in \mathscr{F}_2\} . \tag{25}$$

There exists for every $p \in P$ at least one cube $X^1_{\alpha_0}$ in $\mathscr{F}_1$ such that $X^1_{\alpha_0} \cap P = \emptyset$ and $X^1_{\alpha_0} \cap S(p, 1) \neq \oslash$, (hence $\alpha_0 \notin A_1$). The set $\{X^2_\alpha \mid X^2_\alpha \subset X^1_{\alpha_0} \wedge \alpha \in A_2\}$ has cardinality $2^{\overline{\overline{\Sigma}}}$ and therefore also $\overline{\overline{A}}_2 = 2^{\overline{\overline{\Sigma}}}$. Thus (17) holds for $\varepsilon = 1$ and $i_0 = 2$.

By induction we define sets A_i for $i = 3, 4 \cdots$ and sets of cubes $\{X_\alpha \mid \alpha \in A_i\}$ satisfying:

$$A_i = 2^{\overline{\overline{\Sigma}}} \qquad (i = 1, 2 \cdots) \tag{26}$$

(27) $$\delta(X_\alpha) < 1/6^{i-1} \quad \text{for} \quad \alpha \in A_i$$

(28) $$\frac{1}{6^{i-1}} < \rho(X_\alpha, P) < \frac{2}{6^{i-2}} \quad \text{for} \quad \alpha \in A_i \,.$$

Obviously (26) (27) (28) imply (13) (14) (15). Taking $i_0 = i + 1$ for $\varepsilon > 1/6^{i-1}$ $(i = 1, 2, \cdots)$ one obtains that (17) holds with $M = 2^{\overline{\overline{\Sigma}}}$. Also $W \backslash P = \cup \{X_\alpha \mid \alpha \in A\}$ where $A \bigcup_{i=1}^{\infty} A_i$.

Similarly $W \backslash Q$ can be decomposed into sets Y_β, $\beta \in B = \bigcup_{i=1}^{\infty} B_i$. Finally assumption (16) of Theorem 2 is satisfied since all the cubes X_α and Y_β are homeomorphic.

2. In this section a theorem on extension of homeomorphisms to homeomorphisms in metric spaces is proved. It is applied to extend homeomorphisms in the one-dimensional space R_ω of all points with rational coordinates in the Hilbert space l_2. We show that each homeomorphism between two compact subsets of R_ω can be extended to a self homeomorphism of R_ω. Thus an example of a *finite* dimensional but not zero dimensional space having a very high degree of homogenity is obtained.

DEFINITION 2.1. Let $\{X_\alpha \mid \alpha \in A\}$ be a decomposition of $X \backslash P$. For every $\alpha \in A$ let $p_\alpha \in P$ be any point such that $\rho(X_\alpha, P) = \rho(X_\alpha, p_\alpha)$. The sets X_α will be called thin with respect to P if the following conditions hold:

(29) $$\frac{1}{2^i} < \rho(X_\alpha, P) < \frac{1}{2^{i-1}} \quad \text{for} \quad \alpha \in A_i \qquad (\text{for} \quad i = 2, 3 \cdots)$$
$$\frac{1}{2} < \rho(X_\alpha, P) \qquad \text{for} \quad \alpha \in A$$

(30) for every $p \in P$ and $\varepsilon > 0$ there exists an $i_0 = i_0(p, \varepsilon)$ and $\delta = \delta(p, \varepsilon) > 0$ such that

$$i > i_0 \wedge \alpha \in A_i \wedge S(p, \delta) \cap X_\alpha \neq \varnothing \Rightarrow p_\alpha \in S(p, \varepsilon) \,.$$

If moreover

(31) for every $i = 1, 2, \cdots \overline{\overline{A}}_i = M$ where M is an infinite cardinal and for every $p \in P$ and every $d > 1/2^i$ there exist M indices α' satisfying $\alpha' \in A_i$, $\rho(p_{\alpha'}, p) < 1/4^i$ and $\rho(X_{\alpha'}, P) < Kd$ where $K > 1$ is a fixed number then the sets X_α will be called M dense with respect to P.

LEMMA 2.1. Let $\{X_\alpha \mid \alpha \in A\}$ and $\{Y_\beta \mid \beta \in B\}$ be decompositions of $X \backslash P$ and $Y \backslash Q$. Let $h: P \rightarrow Q$ be a homeomorphism. The following assumptions suffice for the existence of an extension of h to a homeomorphism $H: X \rightarrow Y$.

(32) The sets X_α are thin with respect to P.

(32a) The sets Y_β are thin with respect to Q.

(33) There exists an injection $\phi\colon A \to B$ such that

$$\alpha \in A_i \Rightarrow \phi(\alpha) \in B_i$$

(33a) There exists an injection $\psi\colon B \to A$ such that

$$\beta \in B_i \Rightarrow \psi(\beta) \in A_i$$

(34) There exist homeomorphisms $f_\alpha\colon X_\alpha \to Y_{\phi(\alpha)}$ satisfying

$$\frac{1}{K}\rho(x, p_\alpha) < \rho(f_\alpha(x), h(p_\alpha) < K\rho(x, p_\alpha)$$

for every $x \in X$ where $K > 1$ is a fixed number

(34a) There exist homeomorphisms $g_\beta\colon Y_\beta \to X_{\psi(\beta)}$ satisfying

$$\frac{1}{K}\rho(y, q_\beta) < \rho(g_\beta(y), h^{-1}(q_\beta)) < K\rho(y, q_\beta)$$

for every $y \in Y_\beta$ where $K > 1$ is a fixed number, and $q_\beta \in Q$ is any point of Q for which $\rho(Y_\beta, Q) = \rho(Y_\beta, q_\beta)$.

(35) for every cofinal (in A) sequence $\{\alpha_s\}$ of indices $\rho(h(p_{\alpha_s}), q_{\phi(\alpha_s)}) \to 0$ for $s \to \infty$

(35a) for every cofinal (in B) sequence $\{\beta_s\}$ of indices $\rho(h^{-1}(q_{\beta s}), p_{\psi(\beta_s)}) \to 0$ for $s \to \infty$.

Proof. By (33) and (33a) there exists a one-to-one mapping θ of A onto B. Denoting $A_\phi = \{\alpha \mid \alpha \in A, \theta(\alpha) = \phi(\alpha)\}$ and $A_\psi = \{\alpha \mid \alpha \in A, \theta(\alpha) = \psi^{-1}(\alpha)\}$ one can assume (by Lemma 1.1) that θ is defined so that $A = A_\phi \cup A_\psi$. We define H by:

$$H(x) = \begin{cases} h(x) & x \in P \\ f_\alpha(x) & x \in X_\alpha \wedge \alpha \in A_\phi \\ g_\alpha^{-1}(x) & x \in X \wedge \alpha \in A_\psi \setminus A_\phi \,. \end{cases}$$

As in Theorem 1.1 it suffices to show that H is continuous at an arbitrary point $p \in P$. Let $V = S(q, \varepsilon)$ be a given nbd. of $q = h(p)$. By the continuity of h we have:

(a) There exists a nbd. U_1 of p such that

$$x \in U_1 \cap P \Rightarrow H(x) = h(x) \in V \,.$$

Let U_2 be a nbd. of p such that $h(U_2 \cap P) \subset S(q, \varepsilon/8)$. By (32) and (30) there exists a $\delta > 0$ such that $X_\alpha \cap S(p, \delta) \neq \varnothing \Rightarrow p_\alpha \in U_2$. Let $\delta_1 = \min(\varepsilon/3K, \delta)$ and let $U_3 = U_2 \cap S(p, \delta_1)$. Then

$$x \in X_\alpha \cap U_3 \Rightarrow \Big[\rho(f_\alpha(x), q) < \rho(f_\alpha(x), h(p_\alpha) + \rho(h(p_\alpha), q) < K\rho(x, p_\alpha) + \frac{\varepsilon}{8} < \varepsilon\Big] \,.$$

Hence
(b) There exists a nbd. U_3 of p such that

$$x \in X_\alpha \cap U_3 \Rightarrow f_\alpha(x) \in V .$$

We now show that
(c) There exists a nbd. U_4 of p such that

$$x \in X_{\psi(\beta)} \cap U_4 \Rightarrow g_\beta^{-1}(x) \in V .$$

Indeed, otherwise there exists a cofinal sequence of indices $\{\alpha_s\}$ such that $x_s \in X_{\alpha_s}$, $x_s \to p$ and $\rho(g_{\beta_s}^{-1}(x_s), q) > \varepsilon$ where $\beta_s = \psi^{-1}(\alpha_s)$. But then $p_{\alpha_s} \to p$ and by (35a) also $h^{-1}(q_{\beta_s}) \to p$. Thus $q_{\beta_s} \to q$. Now by (34a)

$$\rho(y_s q_{\beta_s}) < K\rho(x_s, h^{-1}(q_{\beta_s})) \quad \text{where} \quad y_s = g_{\beta_s}^{-1}(x_s) .$$

This is however impossible because $\rho(y_s, q_{\beta_s}) > \varepsilon/2$ and $\rho(x_s, h^{-1}(q_{\beta_s})) < \rho(x_s, p) + \rho(p, h^{-1}(q_{\beta_s})) \to 0$. It follows by (a) (b) and (c) that $H(U_1 \cap U_3 \cap U_4) \subset V$.

THEOREM 2.1. *Let* $\{X_\alpha | \alpha \in A\}$ *and* $\{Y_\beta | \beta \in B\}$ *be decompositions of* $X\backslash P$ *and* $Y\backslash Q$ *and let* $h: P \to Q$ *be a homeomorphism.*

Denote for every i and every $\alpha \in A_i$ by $x_\alpha \in X_\alpha$ a point satisfying $\rho(x_\alpha, P) - \rho(X_\alpha, P) < 1/4^i$.

Similarly denote for every i and every $\beta \in B_i$ by $y_\beta \in Y_\beta$ a point satisfying $\rho(y_\beta, Q) - \rho(Y_\beta, Q) < 1/4^i$.

The following conditions are sufficient for the existence of a homeomorphism $H: X \to Y$ *which is an extension of* h.
(36) X_α are thin and $\aleph_0$ dense with respect to P.
(36a) Y_β are thin and $\aleph_0$ dense with respect to Q.
(37) for every $\alpha \in A_i$ and $\beta \in B_i$ there exists a homeomorphism $f_{\alpha\beta}$: $X_\alpha \to Y_\beta$ such that

$$f_{\alpha\beta}(x_\alpha) = y_\beta$$

and such that for every $x \in X_\alpha$

$$\rho(f_{\alpha\beta}(x), y_\beta) \begin{cases} < \operatorname{Max}\left\{\dfrac{1}{4^i}, 2\rho(x_\alpha, x)\right\} \\ > \rho(x_\alpha, x) - \dfrac{1}{4^i} \end{cases}$$

(37a) for every $\beta \in B_i$ and $\alpha \in A_i$ there exists a homeomorphism $g_{\beta\alpha}$: $Y_\beta \to X_\alpha$ such that

$$g_{\beta\alpha}(y_\beta) = x_\alpha$$

and such that for every $y \in Y_\beta$

$$\rho(g_{\beta\alpha}(y), x_\alpha)\begin{cases} < \mathrm{Max}\left\{\dfrac{1}{4^i}, 2\rho(y_\beta, y)\right\} \\ > \rho(y, y_\beta) - \dfrac{1}{4^i} \end{cases}$$

(38) for every $x \in X_\alpha$

$$\rho(x, p_\alpha) > \frac{1}{2}\rho(x, x_\alpha) + \frac{1}{2}\rho(x_\alpha, p_\alpha)$$

(38a) for every $y \in Y_\beta$

$$\rho(y, q_\beta) > \frac{1}{2}\rho(y, y_\beta) + \frac{1}{2}\rho(y_\beta, q_\beta)\ .$$

Proof. It suffices to define mappings $\phi: A \to B$ and $\psi: B \to A$ and homeomorphisms $f_\alpha: X_\alpha \to Y_{\phi(\alpha)}$ and $g_\beta: y_\beta \to X_{\psi(\beta)}$ so that all assumptions of Lemma 2.1 will be satisfied. We note first that by (36) and (36a) assumptions (32) and (32a) hold. Now denote for a fixed i by $\{\alpha_n\}_{n=1}^\infty$ the sequence of elements of A_i and define $\phi: A_i \to B_i$ by induction. Suppose that ϕ has already been defined for $\alpha_1 \cdots \alpha_n$. Define $\phi(\alpha_{n+1})$ as follows: By (31) there exists a set Y_β satisfying:

$$(39)\qquad \begin{cases} \text{(a)} & \beta \in B_i \\ \text{(b)} & \beta \notin \{\phi(\alpha_1) \cdots \phi(\alpha_n)\} \\ \text{(c)} & \rho(q_\beta, h(p_\alpha)) < \dfrac{1}{4^i} \\ \text{(d)} & \rho(Y_\beta, Q) < K\rho(X_\alpha, P) \quad \text{here} \quad \alpha = \alpha_{n+1}\ . \end{cases}$$

Put $\phi(\alpha_{n+1}) = \beta$ and define $f_\alpha = f_{\alpha,\phi(\alpha)}: X_\alpha \to Y_{\phi(\alpha)}$ as the homeomorphism given by (37). Thus $\phi: A_i \to B_i$ is defined for every i and so $\phi: A \to B$ is defined.

Similarly we define $\psi: B \to A$ and $g_\beta: Y_\beta \to X_{\psi(\beta)}$ (again using (31) and (37a)).

Obviously (33) and (33a) are satisfied. By (39c) also (35) and (35a) hold. By (29), (37), (38) and (39), (denoting $\phi(\alpha)$ by β) we have:

$$\begin{aligned}\rho(f_\alpha(x), h(p_\alpha)) &< \rho(f_{\alpha\beta}(x), f_{\alpha\beta}(x_\alpha)) + \rho(f_{\alpha\beta}(x_\alpha), q_\beta) + \rho(q_\beta, h(p_\alpha)) \\ &< \frac{1}{4^i} + 2\rho(x_\alpha, x) + K\rho(x_\alpha, p_\alpha) + \frac{3}{4^i} \\ &< (K+6)\rho(x, p_\alpha)\end{aligned}$$

and

$$\begin{aligned}\rho(f_\alpha(x), h(p_\alpha)) &> \rho(f_\alpha(x), q_\beta) - \rho(q_\beta, h(p_\alpha)) \\ &> \frac{1}{2}\rho(f_{\alpha\beta}(x), f_{\alpha\beta}(x_\alpha)) + \frac{1}{2}\rho(y_\beta, q_\beta) - \rho(q_\beta, h(p_\alpha)) \\ &> \frac{1}{(K+6)}\rho(x, p_\alpha)\ .\end{aligned}$$

Thus also property (34) of Lemma 2.1 holds with K replaced by $K+6$.

REMARK. One could define the notion of "thin and M dense" using sequences of numbers $\{t_k\}$, $\{r_k\}$ satisfying $t_k \to 0$, $r_k/_{t_k} \to 0$ instead of the sequences $\{2^{-k}\}$, $\{4^{-k}\}$ used.

Extension of homeomorphisms in R_ω. We shall show now that in R_ω every homeomorphism between two compact subsets can be extended to a self homeomorphism of R_ω. Before proving this we introduce some definitions and notations.

The n-dimensional cube $C = \{(x_1 \cdots x_n) \mid a_i \leqq x_i \leqq a_i + l,\ i = 1, \cdots n\}$ will be denoted by $[a_1 a_2 \cdots a_n; l]$.

Every cube $[a_1 \cdots a_n; l]$ can be divided into 2^n cubes $C_{i_1, i_2 \cdots i_n}$ of the form:

$$C_{i_1, i_2, \cdots i_n} = \left\{(x_1 \cdots x_n) \mid a_j + i_j \frac{l}{2} \leqq x_j \leqq a_j + \frac{l}{2}(i_j + 1) \right. \\ \left. \text{for every} \quad j = 1 \cdots n \right\}$$

where i_j equals 0 or 1. Let $\hat{C}_1, \hat{C}_2 \cdots \hat{C}_{2n}$, be the sequence of these cubes ordered lexicographically. By induction we define (as above) cubes $\hat{C}_{ij}$ which divide the cube $\hat{C}_i$ into 2^n cubes and more generally $\hat{C}_{ij \cdots k}$.

For a given cube C, let $Q(C)$ denote the set of cubes:

$$\{\hat{C}_j \mid j = 2 \cdots 2^n\} \cup \{\hat{C}_{1j} \mid j = 2 \cdots 2^n\} \cup \{\hat{C}_{11j} \mid j = 2 \cdots 2^n\} \cup \cdots .$$

Let $\langle C \rangle = \langle a_1 \cdots a_n, l \rangle$ denote the cylinder (in l_2) over the cube C i.e.

$$\langle C \rangle = \{\{x_i\}_{i=1}^{\infty} \mid \{x_i\} \in l_2 \quad \text{and} \quad \forall_{i \leqq n} (a_i \leqq x_i \leqq a_i + l)\} .$$

We call the n-dimensional cube $C = [a_1 \cdots a_n, l]$ the base of the cylinder $\langle C \rangle$, and define $Q\langle C \rangle$ as the set of $\aleph_0$ cylinders in l_2 whose base is one of the cubes in the set $Q(C)$.

π_n denotes the projection of l_2 on the subspace of all points of the form $(x_1 \cdots x_n 0 \cdots 0 \cdots)$.

Finally for a compact subset $P \subset l_2$ and for a set X_α disjoint with P we denote by p_α any point of P for which $\rho(X_\alpha, P) = \rho(X_\alpha, p_\alpha)$ and by x_α any point of X_α satisfying $\rho(x_\alpha, P) - \rho(X_\alpha, P) < \varepsilon_\alpha$ where ε_α is given. The following two lemmas are trivial:

LEMMA 2.2. *Let S be a compact subset of R_ω and let $\varepsilon > 0$. There exists n_0 such that for every point $s \in S$ and every $n > n_0$ $\rho(s, \pi_n(s)) < \varepsilon$.*

Proof. Indeed it suffices to take any finite $\varepsilon/3$ net $\{s_1 s_2 \cdots s_k\}$ in S and choose n_0 such that $\rho(s_i, \pi_{n_0}(s_i)) < \varepsilon/3$ for every $i = 1, 2, \cdots k$.

LEMMA 2.3. *The cylinders of the form*

$$\langle r_1 + \sqrt{2}, r_2 + \sqrt{2}, \cdots r_n + \sqrt{2}; l\rangle$$

are for every $n \in N$ and every sequence $l, r_1 \cdots r_n$ of rational numbers closed and open subsets of R_ω.

THEOREM 2.2. *Any homeomorphism $h: P \to Q$ between two compact subsets P and Q of R_ω can be extended to a self homeomorphism of R_ω.*

Proof. It suffices to decompose the sets $R_\omega \backslash P$ and $R_\omega \backslash Q$ so that all assumptions of Theorem 2.1 hold. Let $\varepsilon_1 > 0$ and let n_1 be a natural number such that Lemma 2.2 holds with $S = P$ $n_0 = n_1$ and $\varepsilon = \varepsilon_1$. Consider the collection F_1 of all cylinders of the form

$$\left\langle \frac{k_1}{4^{n_1}} + \sqrt{2}, \frac{k_2}{4^{n_1}} + \sqrt{2}, \cdots \frac{k_{n_1}}{4^{n_1}} + \sqrt{2}; \frac{1}{4^{n_1}} \right\rangle$$

where $k_1 \cdots k_{n_1}$ are integers. F_1 is a set of mutually disjoint cylinders. Choose from F_1 the set of all cylinders $\langle C\rangle$ satisfying $\rho(\langle C\rangle, P) > \frac{1}{2}$ and denote it by G_1. Take the (countable) set of cylinders

$$\bigcup_{\langle C\rangle \in G_1} Q\langle C\rangle$$

and denote it by $\{X_\alpha \mid \alpha \in A_1\}$ where A_1 is countable. (By Theorem 2.1 one has to decompose $R_\omega \backslash P$ into sets X_α where $\alpha \in \bigcup_{i=1}^\infty A_i$ and the sets A_i have to be disjoint sets of indices. Therefore we do not assume that A_1 is the set of integers).

Let ε_2 satisfy $0 < \varepsilon_2 < \varepsilon_1$ and let n_2 be any natural number such that Lemma 2.2 holds with $S = P$ $n_0 = n_2$ and $\varepsilon = \varepsilon_2$.

Decompose the set $R_\omega \backslash \cup \{\langle C\rangle \mid \langle C\rangle \in G_1\}$ into cylinders of the form:

$$\left\langle \frac{k_1}{4^{n_2}} + \sqrt{2}, \frac{k_2}{4^{n_2}} + \sqrt{2}, \cdots \frac{k_{n_2}}{4^{n_2}} + \sqrt{2}; \frac{1}{4^{n_2}} \right\rangle$$

where $k_1 \cdots k_{n_2}$ are integers, and denote the obtained set of cylinders by F_2. F_2 is a set of mutually disjoint cylinders. Let

$$G_2 = \left\{\langle C\rangle \mid \langle C\rangle \in F_2 \wedge \rho(\langle C\rangle, P) > \frac{1}{2^2}\right\}.$$

Take the (countable) set of cylinders $\bigcup_{\langle C\rangle \in G_2} Q\langle C\rangle$ and denote it by $\{X_\alpha \mid \alpha \in A_2\}$ where A_2 is countable. By induction one can define for a given sequence $\varepsilon_k \to 0$ $(0 < \varepsilon_k < \varepsilon_{k-1})$ and a sequence of natural numbers $n_k (n_k > n_{k-1})$ countable sets of cylinders $\{X_\alpha | \alpha \in A_k\}$ for every $k = 1, 2 \cdots$.

Clearly $R_\omega \backslash P = \cup \{X_\alpha | \alpha \in A\}$ where $A = \bigcup_{i=1}^\infty A_i$.

Similarly $R_\omega \backslash Q = \cup \{Y_\beta | \beta \in B\}$ where $B = \bigcup_{i=1}^\infty B_i$. Also we can choose the same sequences $\{\varepsilon_k\}$ and $\{n_k\}$ for both decompositions. It is easy to show that for sufficiently fast decreasing sequence of numbers ε_k (for example $\varepsilon_k < 1/8^k$ the sets $X_\alpha(Y_\beta)$ are thin and $\aleph_0$ dense with respect to $P(Q)$).

Obviously every cylinder X_α, $\alpha \in A_k$ is homeomorphic to every cylinder Y_β, $\beta \in B_k$. Also for every pair of points $x_\alpha \in X_\alpha (\alpha \in A_k) y_\beta \in Y_\beta (\beta \in B_k)$ there exists a homeomorphism $f_{\alpha\beta}: X_\alpha \to Y_\beta$ so that $f_{\alpha\beta}(x_\alpha) = y_\beta$ and such that (37) is satisfied. Finally (38), (38a) follow from simple geometric properties of the Hilbert space l_2.

Theorem 2.2 is proved.

We conclude with two problems. Let X denote the biconnected set defined by Knaster and Kuratowski ([8] or [5] p. 22) and let $p \in X$ be the point such that $X \backslash \{p\}$ is totally disconnected.

PROBLEM 1. Can each homeomorphism between two compact subsets of $X \backslash \{p\}$ be extended to a self homeomorphism of X?

In connection with the result obtained in Theorem 2.2 one can ask:

PROBLEM 2. Does there exist for $n = 2, 3 \cdots$ n-dimensional space X where every homeomorphism between two compact subsets can be extended to a self homeomorphism of X?

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