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MAXIMAL SUBGROUPS AND CHIEF FACTORS OF CERTAIN GENERALIZED SOLUBLE GROUPS

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It is shown by means of a generalization of a result of R. Baer and D. H. McLain that if G is a locally polycyclic group and if the chief factors of every finitely generated subgroup of G have finite rank at most equal to r, then every maximal subgroup of G has index dividing the r th power of some prime. This answers a question about locally supersoluble groups raised by the first author. In addition, examples are furnished to show that neither of the properties "all chief factors are finite" and "all maximal subgroups have finite index" implies the other.

1. Both R. Baer [1, p. 419] and D. H. McLain [9] have given proofs of the fact that maximal subgroups of locally nilpotent groups are normal. An easy generalization of this, which seems hitherto to have escaped notice, is

THEOREM A. *Let K be a normal locally nilpotent subgroup of a locally Noetherian group G. If H is a maximal subgroup of G, then $H \cap K$ is normal in G.*

The proof is similar to McLain's proof of the special case when $K = G$. Suppose $H \cap K$ is not normal in G. Then there exists x in G such that $(H \cap K)^x \not\leq H \cap K$. Since K is normal in G, it follows that $(H \cap K)^x$ is not contained in H. Since H is maximal, $G = \langle H, (H \cap K)^x \rangle$. Thus, there exist finitely generated subgroups P of H and Q of $H \cap K$ such that $x \in R = \langle P, Q^x \rangle$. Letting $Q_1 = Q^R$, we deduce that $R = P(Q^x)^R = PQ^R = PQ_1$. Therefore

$$Q_1 = (Q^P)^{Q_1} .$$

Since R is finitely generated, it is Noetherian. Thus its subgroup Q_1 is finitely generated. Since Q_1 is contained in K and K is locally nilpotent, Q_1 is nilpotent. Thus $Q_1 = (Q^P)^{Q_1}$ implies that $Q_1 = Q^P$, which in turn implies that $R = \langle P, Q \rangle \leqq H$. Since x belongs to R but not to H, we have a contradiction and the theorem is proved.

As a simple consequence of Theorem A we have

COROLLARY 1. *If G is a locally Noetherian, radical group and*

H is a maximal subgroup of G, there exists a chief factor U/V of G such that $HU = G$ and $H \cap U = V$.

Here we are using the term *radical* in the sense of Plotkin [12]; a group G is radical if every nontrivial homomorphic image has a nontrivial locally nilpotent normal subgroup.

To prove the corollary suppose that H is a maximal subgroup of G and let V be the core of H in G. Since G is a radical group, there is a nontrivial locally nilpotent normal subgroup U/V of G/V. Application of Theorem A to G/V shows that $H \cap U = V$. Suppose that $L \lhd G$ and $V < L \leqq U$. Then certainly $L \not\leqq H$, so that $G = HL$. Hence $U = U \cap (HL) = VL = L$. It follows that U/V is a chief factor of G and that $G = HU$.

We remark that the index of H in G equals the order of U/V. Therefore, if all the chief factors of G are finite, then so are the indices of all the maximal subgroups of G. This allows us to prove

COROLLARY 2. *Suppose that G is a locally polycyclic group. If the chief factors of every finitely generated subgroup of G all have rank at most r, then every maximal subgroup of G has index dividing the r-th power of some prime.*

Since chief factors of supersoluble groups have prime order, it follows from this that any maximal subgroup of a locally supersoluble group has prime index. This answers a question raised by R. E. Phillips in [11; p. 350].

To prove this corollary it is enough by the first corollary and a theorem of McLain [10; p. 104] to show that the group G is radical. This can be done by using the theorem of Zassenhaus [13; p. 294] which asserts the existence of an integer r^*, depending only on r, such that any soluble linear group of degree r has derived length at most r^*.

Suppose that X is any finitely generated subgroup of G—so that X is polycyclic—and let $Y = X^{(r^*)}$, the r^*th term of the derived series of X. If Y^n denotes the subgroup of Y generated by all nth powers of elements of Y, then Y/Y^n is finite and consequently has a series the factors of which are chief factors of X. All these must be centralized by Y, so Y/Y^n is nilpotent and hence every finite homomorphic image of Y is nilpotent. We deduce from a theorem of K. A. Hirsch [6; p. 190] that Y is nilpotent.

It follows easily that $G^{(r^*)}$ is locally nilpotent and therefore that G is a radical group.

2. The question arises whether there exist generalized soluble

groups which have maximal subgroups of infinite index but all of whose chief factors are finite. We shall show that such groups do exist by considering a cyclic extension of one of the groups discussed by McLain in [7].

Let F be a field with a prime number of elements and suppose that V is a vector space over F with basis elements v_n, $n = 0, \pm 1, \pm 2, \cdots,$. For integers $m < n$ we shall write $\tau_{m,n}$ for the linear transformation of V determined by

$$v_m \tau_{m,n} = v_m + v_n \quad \text{and} \quad v_r \tau_{m,n} = v_r \quad \text{if} \quad r \neq m .$$

The group M generated by all the $\tau_{m,n}$ is the McLain group in question. The linear transformation t of V which sends each v_n onto v_{n+1} transforms $\tau_{m,n}$ into $\tau_{m+1,n+1}$ and therefore normalizes M. We define G to be the group generated by M and t. We shall prove

THEOREM B. *Every chief factor of G is finite and yet G has a maximal subgroup of infinite index.*

We show first that the chief factors of G are all finite. Since those of G/M obviously are, it suffices to consider chief factors U/V of G with $U \leqq M$. By an argument of McLain [8], every nontrivial normal subgroup of M contains one of the generators $\tau_{i,j}$. If the normal subgroup is invariant under $\langle t \rangle$, then it must clearly contain one of the subgroups

$$(1) \qquad \gamma_n(M) = \langle \tau_{i,i+n} \mid i = 0, \pm 1, \pm 2, \cdots \rangle$$

of M. However it is well known and easy to see that $\gamma_n(M)$ is the nth term of the lower central series of M and also that $\bigcap_n \gamma_n(M)$ is trivial. This shows that V is nontrivial and therefore M/V is nilpotent. It follows that U/V is a central factor of M and consequently an irreducible representation space for $\langle t \rangle$ over F. By Theorem 3.1 of P. Hall [5] or a simple direct argument we deduce that U/V is finite, as required.

We turn now to the existence of maximal subgroups of G which are of infinite index. It will be enough to show that the subgroup

$$K = \langle \tau_{0,1}\tau_{0,2}, t \rangle$$

does not contain the element $\tau_{0,1}$. For suppose this has been done. It is clear that G is generated by $\tau_{0,1}$ and t. Let H be a subgroup of G containing K which is maximal with respect to excluding $\tau_{0,1}$. Clearly H is a maximal subgroup of G and, by (1), $G = H\gamma_2(M)$. Therefore, by a well-known argument, $G = H\gamma_n(M)$ for each positive integer n. However M is locally nilpotent, so a subgroup of finite index must contain a term of the lower central series of M. Therefore $[M\colon M \cap H] = [G\colon H]$ is infinite.

$$\text{Now} \quad K = N\langle t\rangle\,, \quad \text{where} \quad N = (\tau_{0,1}\tau_{0\,2})^{\langle t\rangle} = M \cap K\,.$$

We must show that

$$(2) \qquad \tau_{0,1} \notin N$$

and in order to achieve this we shall write

$$M^{-} = \langle \tau_{m,n} \mid m < 0\rangle$$

and

$$M^{+} = \langle \tau_{m,n} \mid m \geqq 0\rangle\,.$$

It is easy to see that M^{-} is a normal subgroup of M and that

$$(3) \qquad M = M^{+}M^{-}\,,$$

$$(4) \qquad 1 = M^{+} \cap M^{-}\,.$$

We define

$$\xi_n = \tau_{0\,n}\tau_{0,n+1}\,, \qquad (n \geqq 1)$$

and

$$\eta_m = \tau_{m,m+1}\tau_{m,m+2}\,, \qquad (m = 0, \pm 1, \pm 2, \cdots)\,,$$

and let $Z = \langle \xi_n, \eta_n \mid n \geqq 1\rangle$.

It is clear that $Z \leqq M^{+}$, and that

$$(5) \qquad N = \langle \xi_1, \eta_n \mid n = \pm 1, \pm 2, \cdots\rangle \leqq ZM^{-}\,.$$

(3), (4) and (5) together show that (2) will follow from showing that

$$(6) \qquad \tau_{0,1} \notin Z\,.$$

Let $X = \langle \xi_n \mid n \geqq 1\rangle$ and $Y = \langle \eta_n \mid n \geqq 1\rangle$. For $n \geqq 1$, the element ξ_n commutes with every η_r, $r \geqq 1$, except η_n and η_{n+1}. Moreover, $[\xi_n, \eta_n] = \xi_{n+1}$ and $[\xi_n, \eta_{n+1}] = \xi_{n+2}$. This shows that Y normalizes X. Therefore $Z = XY$. Suppose $\tau_{0\,1} \in Z$; then $\tau_{0,1} = \xi\eta$ for some $\xi \in X$ and some $\eta \in Y$. But then $\xi^{-1}\tau_{0,1} = \eta$ and this element fixes v_n for $n \neq 0$ since both ξ and $\tau_{0.1}$ do, and also fixes v_0 because η does. Hence $\eta = 1$ and $\xi = \tau_{0,1}$ belongs to X. However, the elements $\tau_{0,1}, \tau_{0,2}, \cdots, \tau_{0,n} \cdots$ form a basis of the elementary abelian group which they generate, and therefore $\tau_{0,1}$ cannot belong to X. This establishes (6).

We note that this group G is a 2-generator radical group. Since M is generated by its abelian normal subgroups $\langle L^{M}_{n,m}\rangle$, the group G is even subsoluble in the sense of Baer [2: p. 421]. Moreover, the group G satisfies Max-n, the maximal condition for normal subgroups. To see this, we remark that for each n the lower central factor $\gamma_n(M)/\gamma_{n+1}(M)$ is isomorphic as an $F\langle t\rangle$-module with the group algebra

$F\langle t\rangle$ itself and so, according to a theorem of P. Hall [4: p. 429], is a Noetherian module. It follows that each of the groups $G/\gamma_n(M)$ satisfies Max-n. Since every nontrivial normal subgroup of G must contain some $\gamma_n(M)$, it follows that G has Max-n.

Thus even with these stringent additional hypotheses the finiteness of all the chief factors of a group does not imply that each maximal subgroup is of finite index. On the other hand, it is easy to prove that a hyperabelian (or SI^*-) group with finite chief factors has every maximal subgroup of finite index. Hyperabelian groups, of course, form a rather special class of subsoluble groups.

3. We conclude by observing that the same situation prevails in the opposite direction. More precisely we shall establish

THEOREM C. *There exists a metabelian group which has no maximal subgroups and yet has infinite chief factors.*

For let H be a quasicyclic group of type C_{p^∞} and K a group of type C_{q^∞} where p and q are distinct primes. The group which will demonstrate Theorem C is the restricted standard wreath product $W = H \wr K$. Let $B = H^W$, the base group of W. Since $W/B \cong K$, no maximal subgroup of W can contain B. Thus if L is a maximal subgroup of W, it follows that $W = BL$ and $B/B \cap L$ is a chief factor of W. But B is a radicable abelian p-group, so this is impossible. Hence W has no maximal subgroups.

On the other hand W does have infinite chief factors: for if x is an element of H with order p, then $\langle x\rangle^W$ is isomorphic as an FK-module with the group algebra FK where F is a field with p elements, and it has been shown by Čarin [3] that K has an infinite dimensional irreducible representation over F.

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