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TAYLOR'S THEOREM

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Taylor's theorem requires here the continuity of only those mixed partials actually involved in the remainder term.

Taylor's Theorem seems always to require, at the very least, the continuity of all mixed partials of order $n+1$. In marked contrast the remainder involves only *naturally* arranged mixed partials of order $n+1$. It turns out in Theorem 10 that the natural conjecture is valid. Theorem 10 follows routinely from Theorem 0 and 2.

We agree that ω is the set of integers n for which $n \geqq 0$, that 0 is the empty set, that

$$A \sim B = \{x\colon x \in A \text{ and } x \notin B\} ,$$

that

$$\text{end } n = \{j \in \omega\colon 1 \leqq j \leqq n\} ,$$

that R is the set of real finite numbers, that

$$R^p = \{x\colon p \in \omega \text{ and } x \text{ is on end } p \text{ to } R\} .$$

For $x \in R^p$ it is customary to put

$$x(j) = x_j .$$

We note that

$$R^p \cap R^q = 0 \text{ whenever } p \in \omega \text{ and } p \neq q \in \omega ,$$

and we assume that no member of R is a nonvacuous function.

We agree that if r is a function on S then.

$$\text{sum } r = \textstyle\sum j \in S\, r(j) .$$

To simplify printing we use this notation instead of the traditional notation

$$\text{sum } r = \sum_{j \in S} r(j) .$$

Thus in particular if $1 \leqq p \in \omega$ and r is on end p to R, then

$$\text{sum } r = \textstyle\sum j \in \text{end } p \;\; r_j = \displaystyle\sum_{j=1}^{p} r_j \in R .$$

We agree that:

$$R_n^p = \{r \in R^p \colon r \text{ is to } \omega \text{ and sum } r \leqq n \in \omega\} \; ;$$
$$\bar{R}_n^p = \{r \in R_n^p \colon \text{sum } r = n\} \; .$$

For each k we agree that Kk is that function ψ on ω for which:

$$\psi(j) = 1 \text{ whenever } k = j \in \omega \; ;$$
$$\psi(j) = 0 \text{ whenever } k \neq j \in \omega \; .$$

For each k and each function f we understand: $D_k f$ is such a function that for each x

$$D_k f(x) = \lim_{h \to 0} \frac{f(x + h \cdot Kk) - f(x)}{h} \; ;$$
$$D_k^0 f = f \; ;$$

for each $n \in \omega$

$$D_k^{n+1} f = D_k D_k^n f \; ;$$

if $1 \leqq p \in \omega$ and r is a function on end p to ω, then

$$D^r f = D_1^{r_1} D_2^{r_2} \cdots D_p^{r_p} f \; .$$

At this point we urge the reader to notice that the subscripts appear only in their natural order and that $D^r f$ is built inductively from right to left.

Some comments and illustration follow.

If $1 \leqq p \in \omega$ and f is a real valued function on some subset S of R^p, then $D_k f$ is a real valued function on some subset T_k of S; the set T_k may be 0, in which event, $D_k f = 0$.

If f is that function on R^2 for which

$$f(x) = x_1^2 \cdot x_2 \text{ whenever } x \in R^2 \; ,$$

then, for each k, $D_k f$ is such a function on R^2 to R that for each $x \in R^2$:

if $k = 1$ then $D_k f(x) = 2 \cdot x_1 \cdot x_2$;

if $k = 2$ then $D_k f(x) = x_1^2$;

if $k \notin \text{end } 2$, then $D_k f(x) = 0$.

Let us agree now that g is *trivial* if and only if g is a function for which $g(t) = 0$ whenever t is in the domain of g.

For example, if $p \in \omega$, then the origin of R^p is trivial.

Now suppose A is the characteristic function of the rationals and B is the characteristic function of the irrationals. Let f be that function on R^2 to R for which

$$f(x) = A(x_1) \cdot A(x_2) + B(x_1) \cdot B(x_2) \text{ whenever } x \in R^2 \; .$$

Here $D_k f$ is on some subset T_k, of R^2; in fact

$$D_1 f = D_2 f = 0 = T_1 = T_2 ;$$

if $k \in \text{end } 2$ then $T_k = R^2$ and $D_k f$ is trivial. In particular $D_1 f$ is both trivial and vacuous while $D_3 f$ is trivial but certainly nonvacuous.

We agree that f is *n fold smooth* on α if and only if there is a p for which:

$1 \leqq p \in \omega$; $n \in \omega$; α is a nonvacuous open subset of R^p; f is to R ;

$D^r f$ is continuous on α whenever $r \in R^p_n$.

THEOREM 0. *If $1 \leqq p \in \omega$, $\alpha \subset R^p$, f is n fold smooth on α, $j \in \text{end } p$, $r \in R^p_{n-1}$, and $x \in \alpha$, then*

$$D_j D^r f(x) = D^{r+Kj} f(x) .$$

Theorem 0, which is a consequence of Theorem 7, can be proved directly by induction in sum r. This direct proof, based on well known Theorem 5, is so simple and straightforward that we are amazed that Theorem 0 seldom, if ever, appears in books.

Because of Theorem 0 alone, we can usually[1] diminish the continuity requirements in Taylor's Theorem. We need only require that

$$D^r f \text{ is continuous whenever } r \in R^p_{n+1} ,$$

where the number of points in R^p_{n+1} is

$$\binom{n+p+1}{p} .$$

Since these functions are all involved in Taylor's Formula we can usually check their continuity, at a glance, as we needfully compute. Ordinarily, in applying Taylor's Theorem, we should check continuity, quite possibly in an intuitive flash, in at least p^{n+1} cases.

A fairly obvious companion to Theorem 0 is

THEOREM 1. *If $1 \leqq p \in \omega$, $\alpha \subset R^p$, f is n fold smooth on α, $r \in R^p_n$, $s \in R^p_n$, $r + s \in R^p_n$, and $x \in \alpha$, then*

$$D^r D^s f(x) = D^{r+s} f(x) .$$

The question which intrigues us here is answered by Theorem 2 below. Before coming to Theorem 2 let us agree that f is *n smooth* on α if and only if there is a p for which: $1 \leqq p \in \omega$; $n \in \omega$; α is a

[1] Exceptions: $n = 0$; $p = 1$; $n + p \leqq 4$.

nonvacuous open subset of R^p; f is to R; $D^r f$ is continuous on α whenever $r \in \bar{R}^p_n$.

THEOREM 2. *If f is n smooth on α then f is n fold smooth on α.*

Because of Theorem 2 we can still further diminish the continuity requirements in Taylor's Theorem. We need only require that

$$D^r f \text{ is continuous whenever } r \in \bar{R}^p_{n+1}$$

where the number of points in $\bar{R}^p_{n+1}$ is

$$\binom{n+p}{p-1}.$$

If nothing else we find Theorem 2 and especially Theorem 0 computationally reassuring. We use these, in turn, to verify our computationally pleasant Theorem 10.

We now start our attack on Theorems 0 and 2. We capture the essence of the conclusion of Theorem 2 by agreeing that C^n consists of such functions f that, for some α, f is on α and f is n fold smooth on α. We capture the essence of the premise of Theorem 2 by agreeing that $\bar{C}^n$ consists of such functions f that, for some α, f is on α and f is n smooth on α.

We feel that our rather tangled inductive attack can be clarified by the introduction of some technical (left, center, right) pivot concepts and their preliminary analysis in Lemmas 4, 5, and 6. This material is only of momentary interest and is to be forgotten as soon as Theorems 7 and 8 have been proved.

If $f \in \bar{C}^n$, $1 \leqq p \in \omega$, $\alpha \subset R^p$, and f is on α, then we agree that:

Lpivot $fn = \{j \in \text{end } p\colon D_j D^r f = D^{r+Kj} f$ whenever $r \in R^p_{n-1}\}$;

Cpivot $fn = \{j \in \text{end } p\colon D^{r+Kj} f$ is continuous whenever $r \in R^p_{n-1}\}$;

Rpivot $fn = \{j \in \text{end } p\colon D^r D_j f = D^{r+Kj} f$ whenever $r \in R^p_{n-1}\}$.

If $f \notin \bar{C}^n$ then we agree that

Lpivot $fn =$ Cpivot $fn =$ Rpivot $fn = 0$.

THEOREM 3. *If $1 \leqq p \in \omega$, $k \in \text{end } p$, 0 is the origin of R^p, and g is to R, then:*

.0 $D^0 g = g$;

.1 *if r is on* end p *to ω and*

$$r_i = 0 \textit{ whenever } i \in \text{end}\,(k-1)\,,$$

then

$$D_k D^r g = D^{r+Kk} g\ ;$$

.2 $D_k g = D^{q+Kk} g$;
.3 *if* $m \in \omega$, $j \in \omega$, $0 \leqq j \leqq m$, *and*

$$D_k^m g(x) \in R \ ,$$

then

$$D_k^j g(x) \in R \ .$$

Conclusions .0 and .1 and .3 are fairly obvious; Conclusion .2 follows from .0 and .1.

LEMMA 4. *If* $f \in \bar{C}^n$, $1 \leqq p \in \omega$, $\alpha \subset R^p$, *and* f *is on* α, *then*:
.0 *if* $r \in R_n^p$ *then* $D^r f$ *is on* α *to* R;
.1 *if* $j \in \operatorname{end} p$ *and*

$$\operatorname{end} p \sim \operatorname{end} j \subset \operatorname{Lpivot} fn \ ,$$

then

$$D^r D_j f = D^{r+Kj} f \ \textit{whenever} \ r \in R_{n-1}^p \ ;$$

.2 *if* $j \in \operatorname{end} p$ *and*

$$\operatorname{end} p \sim \operatorname{end} j \subset \operatorname{Lpivot} fn \ ,$$

then

$$j \in \operatorname{Rpivot} fn \ ;$$

.3 *if* $1 \leqq n$ *and* $j \in \operatorname{Rpivot} fn$, *then*

$$D_j f \in \bar{C}^{n-1} \ ;$$

.4 *if* $j \in \operatorname{Rpivot} fn$ *and* $D_j f \in C^{n-1}$, *then*

$$j \in \operatorname{Cpivot} fn \ ;$$

.5 *if* $\operatorname{end} p \subset \operatorname{Cpivot} fn$, *then* $f \in C^n$.

If in 3.3 we take $k = 1$ and

$$g = D_2^{r_2} D_3^{r_3} \cdots D_p^{r_p} f$$

then we see that .0 is at hand.

Conclusion .1, which strikes us as intuitively evident, can be proved by induction in $\operatorname{sum} r$. We shall give the details later.

Conclusion .1 implies .2. Conclusions .3 and .4 are easy.

We now turn to .5, let θ be the origin of R^p, and note we may as well assume $1 \leqq n$. Clearly if

$$\theta \neq r \in R_n^p$$

then $D^r f$ is continuous. On the other hand if $r = \theta$, then:

$\theta \neq \theta + Kk \in R_n^p$ whenever $k \in \operatorname{end} p$;

$D^{\theta + Kk} f$ is continuous whenever $k \in \operatorname{end} p$; because of this, .0, and 3.2,

$D_k f$ is continuous on α whenever $k \in \operatorname{end} p$; accordingly

$$f \text{ is continuous },$$

and, because of 3.0,

$$D^r f \text{ is continuous }.$$

Consequently $D^r f$ is continuous whenever $r \in R_n^p$, and hence, because of .0, $f \in C^n$.

We now tackle .1 by verifying the

Statement. *If*

$$N = \{m \in \omega\colon \textit{for some } r \in R_{n-1}^p,\ \operatorname{sum} r = m \textit{ and } D^r D_j f \neq D^{r+Kj} f\},$$

then,

$$N = 0 .$$

Proof. We suppose $N \neq 0$, and use 3.0 and 3.2 to learn that N is a nonvacuous set of positive integers. We let ν be the smallest integer in N and then so choose $s \in R_{n-1}^p$ that

.6 $$\operatorname{sum} s = \nu , \quad D^s D_j f \neq D^{s+Kj} f .$$

Since $0 < \operatorname{sum} s$ we choose k to be the smallest integer in

$$\{i \in \operatorname{end} p\colon s_i > 0\} .$$

Thus we have

.7 $$s_i = 0 \text{ whenever } i \in \operatorname{end} (k-1) .$$

We let

.8 $$u = s - Kk , \quad v = u + Kj$$

and note that

.9 $$u \in R_{n-1}^p,\ v \in R_{n-1}^p,\ u + Kk = s,\ v + Kk = s + Kj .$$

Since

$$\operatorname{sum} u = \nu - 1$$

we see, with the help of .8, that our choice of ν insures that

.10 $$D^u D_j f = D^v f .$$

We infer from .7 and .8 that

.11 $$u_i = 0 \text{ whenever } i \in \operatorname{end} (k-1) .$$

We use 3.1, .11, .10, and .9 in checking that

.12 $$D^s D_j f = D_k D^v f \, .$$

We must have

.13 $$1 \leqq k \leqq j$$

since otherwise we would know that

$$k \in \operatorname{Lpivot} fn$$

and then could use .12 and .9 to infer that

$$D^s D_j f = D^{s+Kj} f$$

in contradiction to .6. We infer from .11, .13, and .8 that

.14 $$v_i = 0 \text{ whenever } i \in \operatorname{end}(k-1) \, .$$

We use 3.1, .14, .12, and .9 to conclude

$$D^s D_j f = D^{s+Kj} f$$

in contradiction to .6.

The following well known Theorem of Interchange is ample for our needs.

THEOREM 5. *If* $1 \leqq p \in \omega$, *α is an open subset of* R^p, *g is on α to R*, $k \in \operatorname{end} p$, $j \in \operatorname{end} p$,

$$D_k g \text{ is on } \alpha \text{ to } R, \; D_k D_j g \text{ is continuous on } \alpha,$$

then

$$D_j D_k g = D_k D_j g \, .$$

Our next lemma is crucial.

LEMMA 6. $\operatorname{Cpivot} fn \subset \operatorname{Lpivot} fn$

Proof. We suppose the contrary, choose

$$j \in \operatorname{Cpivot} fn \sim \operatorname{Lpivot} fn \, ,$$

note that

$$f \in \bar{C}^n \, ,$$

and so choose p and α that

$$1 \leqq p \in \omega, \; \alpha \text{ is an open subset of } R^p, \; f \text{ is on } \alpha.$$

Letting

$$N = \{m \in \omega\colon \text{ for some } r \in R^p_{n-1} \text{ sum } r = m \text{ and } D_j D^r f \neq D^{r+Kj} f\} \, ,$$

we use 3.2 to learn that N is a nonvacuous set of positive integers.

We choose ν to be the smallest integer in N and then so choose $s \in R^p_{n-1}$ that

.0 $$\operatorname{sum} s = \nu \,, \quad D_j D^s f \neq D^{s+Kj} f \,.$$

Since $0 < \operatorname{sum} s$ we choose k to be the smallest integer in

$$\{i \in \operatorname{end} p \colon s_i > 0\} \,.$$

Thus we see

.1 $$s_i = 0 \text{ whenever } i \in \operatorname{end} (k-1) \,.$$

We must have

.2 $$k < j \leqq p$$

since otherwise we could use .1 to see that

$$s_i = 0 \text{ whenever } i \in \operatorname{end} (j-1)$$

and then could use 3.1 to infer that

$$D_j D^s f = D^{s+Kj} f$$

in contradiction to .0. We let

.3 $$u = s - Kk \,, \quad v = u + Kj$$

and note that

.4 $$u \in R^p_{n-1}, \, v \in R^p_{n-1}, \, u + Kk = s, \, v + Kk = s + Kj \,.$$

Since

$$\operatorname{sum} u = \nu - 1$$

we see, with the help of .3, that our choice of ν insures that

.5 $$D_j D^u f = D^v f \,.$$

We infer from .1 and .3 that

.6 $$u_i = 0 \text{ whenever } i \in \operatorname{end} (k-1)$$

and then from .6 and .2 that

.7 $$v_i = 0 \text{ whenever } i \in \operatorname{end} (k-1) \,.$$

We use 3.1, .7, .5, and .4 in checking that

.8 $$D_k D_j D^u f = D^{s+Kj} f \,.$$

We use 3.1, .6, and .4 in checking that

.9 $$D_k D^u f = D^s f \,.$$

We let

$$g = D^u f \,,$$

and from 4.0, .9, .8, and the fact that

$$j \in \operatorname{Cpivot} fn \,,$$

we infer that:

$$g \text{ is on } \alpha \text{ to } R;$$

.10 $$D_k g = D^s f;$$

$$D_k g \text{ is on } \alpha \text{ to } R;$$

.11 $$D_k D_j g = D^{s+Kj} f \,;$$

$$D_k D_j g \text{ is continuous on } \alpha \,.$$

Because of Theorem 5 we know

.12 $$D_j D_k g = D_k D_j g \,.$$

From .10, .12, and .11 we now conclude

$$D_j D^s f = D_j D_k g = D_k D_j g = D^{s+Kj} f$$

in contradiction to .0.

From Lemma 6 we have at once

THEOREM 7. *If $f \in C^n$, $1 \leqq p \in \omega$, $\alpha \subset R^p$, f is on α, and $r \in R^p_{n-1}$, then*

$$D_j D^r f = D^{r+Kj} f \text{ whenever } j \in \operatorname{end} p \,.$$

THEOREM 8. $\bar{C}^n = C^n$.

Proof. Suppose $\bar{C}^n \neq C^n$, let

$$N = \{m \in \omega \colon \bar{C}^m \neq C^m\} \,,$$

note that

$$0 \neq N \subset \omega$$

and choose ν to be the smallest integer in N, and then choose

.0 $$f \in \bar{C}^\nu \sim C^\nu \,.$$

Note that

.1 $$1 \leqq \nu ,$$

let

$$M = \text{end } p \sim \text{Cpivot} f\nu$$

and observe that

$$M \neq 0$$

since otherwise 4.5 would demand that

$$f \in C^{\nu}$$

in contradiction to .0. Let then j be the largest integer in M. Note that

.2 $$j \in \text{end } p \sim \text{Cpivot} f\nu$$

and, because of Lemma 6, that

$$\text{end } p \sim \text{end } j \subset \text{Cpivot} f\nu \subset \text{Lpivot} f\nu .$$

From this and 4.2 infer that

.3 $$j \in \text{Rpivot} f\nu .$$

From .1, .3, 4.3, and the choice of ν, infer that

.4 $$D_j f \in \bar{C}^{\nu-1} = C^{\nu-1} .$$

From .3, .4, and 4.4 conclude, in contradiction to .2, that

$$j \in \text{Cpivot} f\nu .$$

We now infer Theorem 0 from Theorem 7 and Theorem 2 from Theorem 8.

In order to formulate and prove Theorem 10 we shall understand that if $1 \leqq p \in \omega$, $h \in R^p$, and r is on end p to ω then:

$$r! = r_1! \cdot r_2! \cdot \cdots \cdot r_p! ;$$
$$h^r = h_1^{r_1} \cdot h_2^{r_2} \cdot \cdots \cdot h_p^{r_p} .$$

THEOREM 9. *If* $n \in \omega$, $1 \leqq p \in \omega$, $S = \text{end } p$, *and* A *is on* $\bar{R}^p_{n+1}$ *to* R, *then*

$$\sum r \in \bar{R}^p_n \sum j \in S \frac{A(r + Kj)}{r!} = (n + 1) \cdot \sum r \in \bar{R}^p_{n+1} \frac{A(r)}{r!} .$$

Proof. Let B and H be such functions on R^p that:

$$B(r) = A(r) \text{ whenever } r \in \bar{R}^p_{n+1};$$
$$B(r) = 0 \text{ whenever } r \in R^p \sim \bar{R}^p_{n+1};$$
$$H(r) = \frac{1}{r!} \text{ whenever } r \in R^p \text{ and } r \text{ is to } \omega\ ;$$
$$H(r) = 0 \text{ whenever } r \in R^p \text{ and } r \text{ is not to } \omega\ .$$

We notice that if $j \in S$ then:
if $r \in R^p$ and r is to ω then

$$r \in \bar{R}^p_n \text{ if and only if } r + Kj \in \bar{R}^p_{n+1}\ ;$$

if $r \in R^p \sim \bar{R}^p_n$ then

$$H(r) \cdot B(r + Kj) = 0\ ;$$

if $r \in R^p$ then

$$H(r - Kj) = r_j \cdot H(r)\ .$$

Because of all this we now infer

$$\begin{aligned}
\sum r \in \bar{R}^p_n \sum j \in S \frac{A(r + Kj)}{r!} &= \sum r \in \bar{R}^p_n \sum j \in S[H(r) \cdot B(r + Kj)] \\
&= \sum r \in R^p \sum j \in S[H(r) \cdot B(r + Kj)] \\
&= \sum j \in S \sum r \in R^p[H(r) \cdot B(r + Kj)] \\
&= \sum j \in S \sum r \in R^p[H(r - Kj) \cdot B(r)] \\
&= \sum r \in R^p \sum j \in S\, [H(r - Kj) \cdot B(r)] \\
&= \sum r \in R^p \sum j \in S\, [r_j \cdot H(r) \cdot B(r)] \\
&= \sum r \in R^p[(\textstyle\sum j \in S\ r_j) \cdot H(r) \cdot B(r)] \\
&= \sum r \in \bar{R}^p_{n+1}[(\text{sum } r) \cdot H(r) \cdot B(r)] \\
&= \sum r \in \bar{R}^p_{n+1}[(\text{sum } r) \cdot H(r) \cdot B(r)] \\
&= \sum r \in \bar{R}^p_{n+1}[(n + 1) \cdot H(r) \cdot B(r)] \\
&= (n + 1) \cdot \sum r \in \bar{R}^p_{n+1}[H(r) \cdot B(r)] \\
&= (n + 1) \cdot \sum r \in \bar{R}^p_{n+1} \frac{A(r)}{r!}\ .
\end{aligned}$$

TAYLOR'S THEOREM 10. *If $1 \leqq p \in \omega$, α is an open subset of R^p, $c \in \alpha$, $h \in R^p$,*

$$c + t \cdot h \in \alpha \text{ whenever } 0 \leqq t \leqq 1\ ,$$

$n \in \omega$, f is $n + 1$ smooth on α, then there are Δ and θ for which:

$$0 < \theta < 1 ;$$
$$\Delta = \sum r \in \bar{R}^p_{n+1} \frac{D^r f(c + \theta \cdot h) \cdot h^r}{r!} ;$$
$$f(c + h) = \sum r \in R^p_n \frac{D^r f(c) \cdot h^r}{r!} + \Delta .$$

Proof. In the usual way we let F be such a function that

$$F(t) = f(c + t \cdot h) \text{ whenever } t \in R ,$$

and then use 2, 0, and 9 to check by induction in k that

$$\frac{F^{(k)}(t)}{k!} = \sum r \in \bar{R}^p_k \frac{D^r f(c + t \cdot h) \cdot h^r}{r!}$$

whenever

$$k \in \omega , \quad 0 \leqq k \leqq n + 1 , \quad 0 \leqq t \leqq 1 .$$

Also in the usual way we so choose Δ and θ that:

$$0 < \theta < 1 ;$$
$$\Delta = \frac{F^{(n+1)}(\theta)}{(n + 1)!} ;$$
$$F(1) = \sum_{k=0}^{n} \frac{F^{(k)}(0)}{k!} + \Delta .$$

The desired conclusion is now at hand.

Without resorting to Banach space integration we can verify the following

TAYLOR'S THEOREM FOR A LINEAR NORMED SPACE 11. *If X is a linear normed space, $1 \leqq p \in \omega$, α is an open subset of R^p, $c \in \alpha$, $h \in R^p$,*

$$c + t \cdot h \in \alpha \text{ whenever } 0 \leqq t \leqq 1 ,$$

$n \in \omega$, f is $n + 1$ smooth on α to X, then there is a $\Delta \in X$ for which:

$$|| \Delta || \leqq (n + 1) \cdot \int_0^1 \left\| (1 - t)^n \cdot \sum r \in \bar{R}^p_{n+1} \frac{D^r f(c + t \cdot h) \cdot h^r}{r!} \right\| dt ;$$
$$f(c + h) = \sum r \in R^p_n \frac{D^r f(c) \cdot h^r}{r!} + \Delta .$$

By completing the space X of Theorem 11 we obtain an alternative proof of Theorem 11 based upon

TAYLOR'S THEOREM FOR A BANACH SPACE 12. *If X is a Banach* *space,* $1 \leqq p \in \omega$, *α is an open subset of* R^p, $c \in \alpha$, $h \in R^p$,

$$c + t \cdot h \in \alpha \text{ whenever } 0 \leqq t \leqq 1 \ ,$$

$n \in \omega$, *f is $n+1$ smooth on α to X, and*

$$\varDelta = (n+1) \cdot \int_0^1 (1-t)^n \cdot \sum r \in \bar{R}^p_{n+1} \frac{D^r f(c + t \cdot h) \cdot h^r}{r!} dt \ ,$$

then

$$f(c+h) = \sum r \in R^p_n \frac{D^r f(c) \cdot h^r}{r!} + \varDelta \ .$$

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