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COMPACT CONVEX SETS WITH THE EQUAL SUPPORT PROPERTY

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Simplexes may be characterized as follows: (C) X is a simplex if and only if each $x \in X$ has a unique $\prec$-maximal representing measure, where $\prec$ denotes the Choquet ordering on the set $M^+(X)$ of positive regular Borel measures on X. In this paper, we study compact convex sets which satisfy a condition which is similar to that given in (C). Definition: X has the equal support property if, for each $x \in X$, any two $\prec$-maximal representing measures for x have the same support. Some of our theorems are extensions to sets with the equal support property of results which hold for simplexes. Other results given here are analogous of theorems which hold for simplexes. We are especially interested in the relationships between the equal support property and a topology, called the structure topology, which was first defined for the set of extreme points of a simplex, but also makes sense for a wider class of compact convex sets.

The background material for §§ 1 through 4 of this paper may be found in [12]. The equal support property was first considered by Feinberg in [8]. The expression "compact convex set" will always refer to a nonempty compact convex subset of a locally convex Hausdorff linear space. Let X be a compact convex set. $ex\ X$ will denote the set of extreme points of X. ∂X will denote the closure $\overline{ex\ X}$ of $ex\ X$. Consider a point $x \in X$ and a closed subset F of X. R_x^F will denote the set of representing measures for x which are supported by F, i.e., vanish on $X - F$. (We will not distinguish between measures on F and measures on X which are supported by F.) It is known [12, p. 5] that $y \in \overline{\text{cov}}\ F$ if and only if $R_y^F \neq \emptyset$. Recall tha $X = \overline{\text{cov}}\ (ex\ X)$ (Krein-Milman Theorem, KMT), or, equivalently, $R_x^{\partial X} \neq \emptyset$ for each $x \in X$. We will make use of Milman's converse to the Krein-Milman Theorem, henceforth refered to as MT, which states that: $X = \overline{\text{cov}}\ S$ implies $ex\ X \subseteq \bar{S}$.

1. Extreme sets. In this section, X will denote a compact convex set. A fuction $f: X \to Z$ where Z is a convex set is called *affine* if $f(cx + (1 - c)y) = cf(x) + (1 - c)f(y)$ for every

$$(x, y, c) \in X \times X \times [0, 1] .$$

A function g: $X \to (-\infty, \infty]$ is called *concave* if

$$cg(x) + (1 - c)g(y) \leqq g(cx + (1 - c)y)$$

for all $(x, y, c) \in X \times X \times [0, 1]$.

DEFINITION 1.1. A set $S \subseteqq X$ will be called *extreme* if

$$(u, v, b) \in X \times X \times (0, 1)$$

and $bu + (1 - b)v \in S$ imply $u, v \in S$. A convex extreme set will be called a *face*.

Note the union of any collection of closed faces is extreme. In particular, if $S \subseteqq ex\ X$, then S is extreme.

LEMMA 1.2. *A closed subset E of X is extreme if and only if $x \in E$ implies supp $\rho \subseteqq E$ for each $\rho \in R_x^X$ (supp ρ denotes the closed support of the measure ρ).*

The proof is a slight modification of the proof of [12, prop. 1.4]. See also, [1, p. 100].

Let $\mathscr{F}(X)$ denote the collection of closed faces of X. Note that, if $\mathscr{C} \subseteqq \mathscr{F}(X)$, then $\cap \{C \mid C \in \mathscr{C}\} \in \mathscr{F}(X)$. Thus, $\mathscr{F}(X)$ is a complete lattice in the containment ordering. Let $F', F'' \in \mathscr{F}(X)$. We will denote $\cap \{F \mid F \in \mathscr{F}(X)$ and $F \supseteqq F' \cup F''\}$ by $F' \vee F''$. The following gives some elementary but useful information about members of $\mathscr{F}(X)$:

PROPOSITION 1.3. *Let $F, F' \in \mathscr{F}(X)$. Then:* (a) $ex\ F = ex\ X \cap F$. (b) $ex\ (\text{cov}\ (F \cup F')) = ex\ F \cup ex\ F' = ex\ X \cap \text{cov}\ (F \cup F')$.

Proof. (a) is clear. By MT, $ex\ (\text{cov}\ (F \cup F')) \subseteqq F \cup F'$. (b) now follows from (a).

It is natural to ask, whether $F, F' \in \mathscr{F}(X)$ imply cov $(F \cup F') \in \mathscr{F}(X)$? A more general question is: If E is closed and extreme, is $\overline{\text{cov}}\ E$ extreme? It follows from a result of Effros that, if X is a simplex and $E \subseteqq ex\ X$, then the answer to the last question is yes (see [7, Th. 3.3]). The following theorem shows that the answer to the above question is affirmative if X has the equal support property, furthermore, the proof given here may be used to obtain Th. 3.3 of [7]:

THEOREM 1.4. *Suppose that X has the equal support property. Let E be a closed extreme set of X. Then $\overline{\text{cov}}\ E$ is extreme, and hence is a face.*

In the proof of Theorem 1.4, we will need the following propositions:

PROPOSITION 1.5. *Let Y be a compact convex set. Suppose $\mu, \nu \in M^+(Y)$ and $\mu \prec \nu$. Then supp $\mu \subseteqq \overline{\text{cov}}$* (supp ν).

Proof. Suppose $y \in \text{supp}\,\mu - \overline{\text{cov}}$ (supp ν). Then, there is a continuous affine function u on X such that $u(y) = -1$ and inf $u(\overline{\text{cov}}$ (supp ν)) $= 0$. Let $w = \min\{u, 0\}$. Then w is concave. Hence, $\int wd\,\nu \leqq \int wd\,\mu < 0$. But, $w = 0$ on supp ν – *a* contradiction.

PPOPOSITION 1.6. *Let F be a closed extreme set of the compact convex set Y. Suppose $\nu_1 \prec \nu_2$ and supp $\nu_1 \subseteqq F$. Then supp $\nu_2 \subseteqq F$.*

Proof. Let J_F denote the function which is 0 on F and 1 on $Y - F$. Then J_F is l.s.c. and concave. Let D be the set of continuous concave functions on Y which are strictly dominated (pointwise) by J_F. By a result of Mokobodski [11, p. 222], D is directed upward and $J_F = \sup\{g \mid g \in D\}$. Since $\int gd\,\nu_2 \leqq \int gd\,\nu_1$ for each $g \in D$, it follows that $\int J_F d\,\nu_2 \leqq \int J_F d\,\nu_1$. (For a proof that

$$\int J_F d\,\nu_2 = \sup\left\{\int gd\,\nu_2 \mid g \in D\right\},$$

see, e.g., [4. p. 8].) Thus, $\nu_2\,(Y - F) = \nu_1\,(Y - F) = 0$.

Proof of Theorem 1.4. Let $x \in \overline{\text{cov}}\ E$. It will be shown that supp $\mu \subseteqq \overline{\text{cov}}\ E$ for all $\mu \in R_x^X$. By [12, Lemma 4.1] and proposition 1.5, it is enough to prove that $\overline{\text{cov}}\ E$ contains the support of every $\prec$-maximal measure in R_x^X. Since X has the equal support property, it is only necessary to find one $\prec$-maximal representing measure which is supported by E. Let $\nu \in R_x^E$ (see [12, p. 5]). There is a $\prec$-maximal measure μ such that $\nu \prec \mu$ [12, Lemma 4.1]. By proposition 1.6, it follows that supp $\mu \subseteqq E$.

In [1], Alfsen proved that if Z is an r-simplex, i.e., a simplex whose set of extreme points is closed, then $\overline{\text{cov}}\ B$ is a face for every $B \subseteqq ex\ Z$. We will show that X satisfies the conclusion of Alfsen's result if and only if X has the equal support property and $ex\ X$ is closed. We claim that X has the equal support property and $ex\ X = \partial X$ if and only if for each $x \in X$, all measures in $R_x^{\partial X}$ have the same support. The "only if" part of the previous statement follows from the fact that if $\operatorname{supp}\mu \subseteqq ex\, X$, then μ is $\prec$-maximal [12, pp. 26-27]. Suppose that $z \in X$ and that all measures in $R_z^{\partial X}$ have the same support. Let ν be a $\prec$-maximal measure with $\varepsilon_z \prec \nu$. Since all $\prec$-maximal measures are supported by X [12, p. 30], it follows that $\varepsilon_z = \nu$. Thus, $z \in ex\, X$ [12, p. 27]. The proof of the above statement is complete. ($\varepsilon_z =$ the Dirac measure at z.)

DEFINITION 1.7. X has the *strong equal support property* if, for each $x \in X$, all measures in $R_x^{\partial X}$ have the same support.

For the sake of brevity, we will use the abbreviation "s.e.s.p." to indicate the "strong equal support property".

THEOREM 1.8. *The following are equivalent*:

(i) X *has the* s.e.s.p.

(ii) $\overline{\operatorname{cov}}\ B$ *is extreme for every* $B \subseteqq \partial X$.

Proof. That (i) implies (ii) follows from Theorem 1.4.

If (ii) holds, then $\operatorname{cov}\{x\} = \{x\}$ is extreme for each $x \in X$. Thus, $\partial X \subseteqq ex\ X$. It must be shown that X has the equal support property. Let $y \in X$ and let μ and ν be $\prec$-maximal representing measures for y. Since $y \in \overline{\operatorname{cov}}\ (\operatorname{supp}\mu)$,it follows, by Proposition 1.2, that $\operatorname{supp}\nu \subseteqq \overline{\operatorname{cov}}\ (\operatorname{supp}\mu)$. By MT and Proposition 1.3, $\operatorname{supp}\nu \subseteqq \overline{\operatorname{cov}}\ (\operatorname{supp}\mu) \cap ex\ X = \operatorname{supp}\mu$. Similarly, $\operatorname{supp}\mu \subseteqq \operatorname{supp}\nu$.

In [8] Feinberg gave a proof of Theorem 1.8 which is independent of Theorem 1.4.

We are now able to give an example of a compact convex set which has the s.e.s.p., but is not a simplex.

EXAMPLE 1.9. Let $M([0, 1])$ denote the space of real valued regular Borel measures on $[0, 1]$. Assume that $M([0, 1])$ is equipped with the weak* topology (recall that $M([0, 1])$ is the dual of the space of real valued continuous functions on $[0, 1]$). In [9, Remark 4] Lazar considers the quotient space $M([0, 1])/V$, where V is the linear subspace of $M([0, 1])$ spanned by a certain measure μ. μ possesses the following properties: (P1) $\mu([0, 1]) = 0$ and (P2) every open subinterval of $[0, 1]$ contains sets of positive μ-measure and sets of negative μ-measure. Let T be the restriction to $P([0, 1])$ ($P([0, 1]) =$ the probability measures on $[0, 1]$) of the quotient map of $M([0, 1])$ onto $M([0, 1])/V$ and let $Z = T(P([0, 1]))$. Then T maps $ex\ \ P([0, 1])$ homeomorphically onto $ex\ Z$ and satisfies: $T^{-1}(T(F)) = F$ for each $F \in \mathscr{F}(P([0, 1]))$. Furthermore, Z is not a simplex.

It is claimed that Z has the s.e.s.p. By Theorem 1.8, it is enough to show that $\overline{\text{cov}}\ E \in \mathscr{F}(Z)$ whenever E is a closed subset of $ex\ Z$. Let $E' = ex\ P([0, 1]) \cap T^{-1}(E)$. Then, $T(\overline{\text{cov}}\ E') = \overline{\text{cov}}\ E$. By Theorem 1.8, it follows that $\overline{\text{cov}}\ E'$ is a face. Thus,

$$T^{-1}(\overline{\text{cov}}\ E) = \overline{\text{cov}}\ E' .$$

A straightforward argument shows that $\overline{\text{cov}}\ E$ is a face.

Rogalski [13, Prop. 13] has given an example of a compact convex set Y which does not have the equal support property but which satisfies the following: If E is a closed subset of Y with $E \subseteqq ex\ Y$, then $\overline{\text{cov}}\ E$ is a face of Y.

2. Extremally concave functions.

In this section X will denote a compact convex set and A will denote the set of real valued continuous affine functions on X. Consider the following abstract "Dirichlet" problem: (D) Given $f \in C(\partial X)$ ($C(\partial X)$ = space of real continuous functions on ∂X), find $a_f \in A$ such that $a_f | \partial X = f$. In [2] Bauer characterized r-simplexes as follows: X is an r-simplex if and only if (D) is solvable for each $f \in C(\partial X)$. In view of the definition of the equal support property, it is natural to expect that there is a characterization of compact convex sets with the s.e.s.p. which is similar to Bauer's result. In § 4 we will characterize the s.e.s.p. in terms of a problem which is the same as (D) except that A is replaced by a certain collection $\mathscr{E}(X)$ of functions on X. This section is concerned with investigating $\mathscr{E}(X)$.

Suppose X has the s.e.s.p. Let $f \in C(ex\ X)$. For each $c \in (-\infty, \infty]$, let $F_c = \overline{\text{cov}}\ (f^{-1}(-\infty, c])$. By Theorem 1.8, $F_c \in \mathscr{F}(X)$ for each $c \in (-\infty, \infty]$. Define a function $f^*: X \to (-\infty, \infty]$ as follows: for each $x \in X$, let $f^*(x) = \inf\{c \mid x \in F_c\}$. Then f^* extends f and satisfies: $(f^*)^{-1}(-\infty, d] \in \mathscr{F}(X)$ for each $d \in (-\infty, \infty]$.

DEFINITION 2.1. A function $g: X \to (-\infty, \infty]$ will be called *extremally concave* if $g^{-1}(-\infty, c] \in F(X)$ for each $c \in (-\infty, \infty]$. The set of all extremally concave functions on X will be denoted by $\mathscr{E}(X)$.

PROPOSITION 2.2. (a) *Extremally concave functions are concave and* l.s.c. (b) *For each* $x \in X$ *and each* $\mu \in R_x^X$, $\int g d\mu \leqq g(x)$ *for every* $g \in \mathscr{E}(X)$.

Proof. Let g be an extremally concave function. That g is l.s.c. is clear. The fact that g is concave is a special case of (b). Suppose $\mu \in R_x^X$. Let $\lambda = g(x)$. Since $x \in g^{-1}(\infty, \lambda]$, it follows that $\operatorname{supp} \mu \subseteqq g^{-1}(-\infty, \lambda]$ (Prop. 1.2). Thus $\int g d\mu \leqq \lambda \mu(X) = g(x)$.

PROPOSITION 2.3. *Suppose* $f, g \in \mathscr{E}(X)$. *Then:* (a) $f \leqq g$ *if and only if* $f \mid ex\ X \leqq g \mid ex\ X$. (b) $-f \leqq g$ *if and only if* $(-f) \mid ex\ X \leqq g \mid ex\ X$. *It follows from* (a) *that extremally concave functions are uniquely determined by their values on* $ex\ X$.

Proof. Suppose $f \mid ex\, X \leqq g \mid ex\, X$. Let $x \in X$ and let $b = g(x)$. Clearly, $g^{-1}(-\infty, b] \cap ex\, X \subseteqq f^{-1}(-\infty, b] \cap ex\, X$. By Proposition 1.3 and the KMT, $g^{-1}(-\infty, b] \subseteqq f^{-1}(-\infty, b]$. Thus, $f(x) \leqq b = g(x)$.

Suppose $-f \mid ex\, X \leqq g \mid ex\, X$ and $(f + g)^{-1}(-\infty, 0) \neq \emptyset$. Then there are real numbers s, t with $s < t$ and

$$g^{-1}(-\infty, s] \cap (-f)^{-1}[t, \infty) \neq \emptyset .$$

Since $g^{-1}(-\infty, s] \cap (-f)^{-1}[t, \infty) \in \mathscr{F}(X)$, it follows by Proposition 1.3 and the KMT that $g^{-1}(-\infty, s] \cap (-f)^{-1}[t, \infty) \cap ex\, X \neq \emptyset$ — a contradiction.

The next theorem is crucial in the proofs of some later theorems.

THEOREM 2.4. *Let* $\mathscr{D} \subseteqq \mathscr{E}(X)$. *Then* $\sup \{g \mid g \in \mathscr{D}\}$ (pointwise) *is in* $\mathscr{E}(X)$. *Consequently,* $\mathscr{E}(X)$ *is a complete lattice in the pointwise ordering.*

Proof. Let $k = \sup \{g \mid g \in D\}$. Suppose $c \in (-\infty, \infty]$. Then $k^{-1}(-\infty, c] = \cap \{g^{-1}(-\infty, c] \mid g \in D\}$. Since $\mathscr{F}(X)$ is closed under arbitrary intersections, it follows that $k \in \mathscr{E}(X)$.

Let $f, g \in \mathscr{E}(X)$. $f \wedge g$ will denote $\sup \{k \mid k \in \mathscr{E}(X)$ and $k \leqq \min \{f, g\}\}$.

We will complete this section by giving another property of extremally concave functions and some simple examples.

PROPOSITION 2.5. *Let* $g \in \mathscr{E}(X)$, $x \in X$, *and* $\mu \in R_x^X$. *Then* $g(x) = \sup\ g(\operatorname{supp} \mu)$. *In particular, if* $x = cu + (1 - c)v$ *where* $u, v \in X$ *and* $c \in (0, 1)$, *then* $g(x) = \max \{g(u), g(v)\}$.

Proof. Since $x \in g^{-1}(-\infty, g(x)]$, it follows from Proposition 1.2 that $\operatorname{supp} \mu \subseteqq g^{-1}(-\infty, g(x)]$. Thus, $\sup g(\operatorname{supp} \mu) \leqq g(x)$. Let $Z = \overline{\text{cov}}$ $(\operatorname{supp} \mu)$. Then $x \in Z$. Note that $g \mid Z \in \mathscr{E}(Z)$. (This fact follows from the fact that if S is an extreme subset of X then $S \cap Z$ is an extreme subset of Z.) By MT, $ex\ Z \subseteqq \text{supp}\,\mu$. By Proposition 2.3, $g|Z \leqq \sup g(\text{supp}\,\mu)$.

The "In particular $\cdots$" part of the proposition follows from the fact that $c\varepsilon_u + (1-c)\varepsilon_v \in R_x^X$.

Examples. Let $X = [0, 1]$. Then $f \in \mathscr{E}(X)$ if and only if f is in one of the following forms: (i) f is a constant $b \in (-\infty, \infty]$ on $(0, 1]$ and $f(0) \leqq b$. (ii) $f(x) = g(1-x)$ where g is of form (i).

Let $X = \{(a, b) \mid a, b \in [0, 1] \text{ and } a + b \leqq 1\}$. Let F_1, F_2, and F_3 be the "edges" of X. We will regard the F_i's as copies of $[0, 1]$. Then $f \in \mathscr{E}(X)$ if and only if f is in one of the forms $f(x) = b \in (-\infty, \infty]$ for $x \in X - F_j$ and $f(x) = g(x) \leqq b$, where $g \in \mathscr{E}(F_j)$, when $x \in F_j$ $j = 1, 2, 3$.

3. The structure topology. Let Z be a simplex. Effros [6, p. 117] has defined a topology for $ex\ Z$ called the *structure topology*. In this section, we extend Effros' definition to a larger class of compact convex sets. Alfsen and Andersen [2] have defined a topology called the *facial topology*, for the set of extreme points of an arbitrary compact convex set. The facial topology is a generalization of Effros' structure topology. At the end of this section, we will make a comparison between the structure and facial topologies.

For the rest of this section, X will denote a nonempty compact convex set. Let $\mathscr{T}_X = \{F \cap ex\ X \mid F \in \mathscr{F}(X)\}$. Note that $\varnothing$, $ex\ X \in \mathscr{T}_X$ and that the intersection of any sub-collection of $\mathscr{T}_X$ is in $\mathscr{T}_X$. Suppose X has the e.s.p. Consider $F, F' \in \mathscr{F}(X)$. By Theorem 1.4, $\text{cov}\,(F \cup F') \in \mathscr{F}(X)$. By Proposition 1.3,

$$(F \cap ex\ X) \cup (F' \cap ex\ X) = ex\ X \cap \text{cov}\,(F \cup F')\ .$$

It follows that the union of any finite sub-collection of $\mathscr{T}_X$ is in $\mathscr{T}_X$. Thus, $\mathscr{T}_X$ is the collection of closed sets for a topology on $ex\ X$, whenever X has the e.s.p.

DEFINITION 3.1. If $\mathscr{T}_X$ is closed under finite unions, then the topology on $ex\ X$ for which $\mathscr{T}_X$ is the collection of closed sets will be called the *structure topology*. We will use the adjective "structurally" to replace "$\cdots$ in the structure topology", e.g., "structurally compact" means "compact in the structure topology".

THEOREM 3.2. *The following are equivalent:*
(i) *The structure topology exists on* $ex\ X$.
(ii) $\text{cov}\,(F \cup F') \in \mathscr{F}(X)$ *for all* $F, F' \in \mathscr{F}(X)$.
(iii) $\text{cov}\,(F \cup F') = F \vee F'$ *for all* $F, F' \in (X)$.
(iv) $\mathscr{F}(X)$ *is a distributive lattice.*

Proof. The equivalence of (i) and (ii) follows from Proposition 1.3. The equivalence of (ii) and (iii) is clear.

Let $F, F'' \in \mathscr{F}(X)$. Suppose (iii) holds. To show that

$$F \cap (F' \vee F'') = (F \cap F') \vee (F \cap F'') ,$$

it is enough to prove that

$$(1) \qquad F \cap \text{cov } (F' \cup F'') = \text{cov } ((F \cap F') \cup (F \cap F'')) .$$

By the *KMT*, equation (1) is equivalent to

$$(2) \qquad ex\ (F \cap \text{cov } (F' \cup F'')) = ex\ (\text{cov } ((F \cap F') \cup (F \cap F''))) .$$

Equation (2) follows from Proposition 1.3. Suppose (iv) holds and $F \vee F'$ properly contains cov $(F \cup F')$. By the *KMT* and Proposition 1.3, there is a point $x_1 \in ex\ X \cap (E \vee F' - \text{cov } (F \cup F'))$. Since $\{x_1\} \in \mathscr{F}(X)$ we have

$$\begin{aligned}\{x_1\} &= \{x_1\} \cap (F \vee F') \\ &= (\{x_1\} \cap F) \vee (\{x_1\} \cap F') \\ &= \varnothing \vee \varnothing = \varnothing .\end{aligned}$$

The next lemma indicates that compact convex sets X for which $\mathscr{F}(X)$ is distributive are similar to simplexes.

LEMMA 3.3. *Suppose $\mathscr{F}(X)$ is distributive. Let $S \subseteqq ex\ X$. Then* cov S *is extreme and each point in* cov S *has a unique $\prec$-maximal representing measure.*

Proof. To prove that cov S is extreme, it is enough to consider the case where S is finite. cov S is surely extreme if S is a singleton. Suppose that, for sets S of cardinality $<$ the integer n, cov S is extreme. Let B be a subset of $ex\ X$ with n elements. Choose some $y \in B$. Then cov $B = \text{cov } (\{y\} \cup \text{cov } (B - \{y\}))$. By Theorem 3.2, cov B is extreme.

We will now prove the second assertion. Let $z \in \text{cov } S$. Then z may be written in the form (†) $z = \sum_{i=1}^{p} c_i x_i$ where the x_i's are distinct elements of S, $c_i > 0$ for $i = 1, 2 \cdots, p$ and $\sum_{i=1}^{p} c_i = 1$. By Lemma 1.2, all representing measures for z are supported by cov $\{x_1, \cdots, x_p\}$. Thus, to show that $\sum_{i=1}^{p} c_i \varepsilon_{x_i}$ is the only maximal representing measure for z, it is enough to show that the representation (†) is unique (see [18, p. 26]). Let $z = \sum_{j=1}^{q} b_j y_j$ be another representation for z of the form (†). Since cov $\{x_1, \cdots, x_p\}$ is extreme,

$$\begin{aligned}\{y_1, \cdots, y_q\} &\subseteqq \text{cov } \{x_1, \cdots, x_p\} \cap ex\ X \\ &\subseteqq \{x_1, \cdots, x_p\} .\end{aligned}$$

Similarly, $\{x_1, \cdots, x_p\} \subseteqq \{y_1, \cdots, y_q\}$. Assume $y_i = x_i$ $i = 1, 2, \cdots, p$. Suppose $b_1 > c_1$. Then

$$\left(\frac{b_1 - c_1}{1 - c_1}\right)x_1 + \frac{b_2}{1 - c_1}x_2 + \cdots + \frac{b_p}{1 - c_1}x_p$$

$\in \operatorname{cov}\{x_2, \cdots, x_p\}$. Thus $x_1 \in \operatorname{cov}\{x_2, \cdots, x_p\}$ — a contradiction. Hence, $c_1 \geqq b_1$. Similarly, $b_1 \geqq c_1$. By the same argument, $c_i = b_i$, $i = 2, \cdots, p$.

THEOREM 3.4. *Suppose X is a compact convex subset of a finite dimensional linear space. Then the following are equivalent:*

(i) *For some integer n, there is an affine homeomorphism of X onto the standard n-simplex.*

(ii) *X is a simplex.*

(iii) *$\mathscr{F}(X)$ is distributive.*

(iv) *X has the e.s.p.*

Proof. For a proof of the equivalence of (i) and (ii) see [18, Prop. 9. 11]. At the beginning of this section we proved that (iv) implies (iii), and it is clear that (ii) implies (iv). Suppose (iii) holds. By a theorem of Minkowski [18, p. 1], $X = \operatorname{cov}(ex\ X)$. By the previous lemma, X is a simplex.

In [10], Feinberg proved the equivalence of (ii) and (iii). Theorem 3.4 shows that compact convex sets X for which $\mathscr{F}(X)$ is distributive and those which have the equal support property are generalizations of finite dimensional simplexes. The following example shows that they are distinct generalizations:

EXAMPLE 3.5. The technique used in constructing this example is due to Alfsen [1]. Let N^* be the one-point compactification of the space of positive integers. Then $C(N^*)$ is the space of real sequences of the form $(a_1, a_2 \cdots; a_\infty)$ where $a_\infty = \lim a_n$. Let

$$M(N^*) = M^+(N^*) - M^+(N^*) ,$$

then $M(N^*)$ is the dual space of $C(N^*)$ and may be identified with the space of real sequences of the form $(x_1, x_2, \cdots; x_\infty)$ where $\sum_{i=1}^{\infty}|x_i| < \infty$. It will be assumed that $M(N^*)$ is equipped with the weak* topology. $P(N^*)$ will denote $\{(x_1, \cdots; x_\infty) \mid x_n \geqq 0\ n = 1, 2, \cdots \infty$ and $x_\infty + \sum_{n=1}^{\infty} x_n = 1\}$. Then $P(N^*)$ is a compact convex subset of $M(N^*)$, indeed $P(N^*)$ is a simplex (see, e.g., [2, Satz 13]). The extreme points of $P(N^*)$ are those of the form ε_p where ε_p is 1 at the p-th and 0 at every other position $p = 1, 2, \cdots, \infty$ (see [4, p. 441]).

Consider the sequences $u = (1/2, 0, 1/4, 0, 1/8, 0, \cdots -1)$ and $v = (0, 1/2, 0, 1/4, 0, 1/8, \cdots -1)$. Let L be the subspace of $M(N^*)$ generated by u and v. Consider the quotient space $V = M(N^*)/L$ and let T be the cannonical projection of $M(N^*)$ onto V. Let $Z = T(P(N^*))$. It is claimed that $ex\ Z = \{T(\varepsilon_n) \mid 1 \leqq n < \infty\}$. Suppose $y \in ex\ Z$. Then $T^{-1}(y) \cap P(N^*)$ is a closed convex extreme subset of $P(N^*)$. By Proposition 1.3, and the *KMT*, it follows that $y = T(\varepsilon_n)$ for some $1 \leqq n \leqq \infty$. It must be shown that $n \neq \infty$. Let δ_n denote the Dirac measure at $T(\varepsilon_n)$ for $n = 1, 2, \cdots, \infty$. The series $\sum_{j=1}^{\infty} 1/2^j\, \delta_{2j-1}$ converges in the norm topology of the dual of $C(Z)$ to a measure μ_1 which is $\prec$-maximal and represents $T(\varepsilon_\infty)$ (see [18, p. 26]). It follows that $T(\varepsilon_\infty) \notin ex\ Z$.

Next, it is asserted that $F \in \mathscr{F}(Z)$ if and only if $F = X$ or $F = \operatorname{cov} E$ where E is a finite subset of $ex\ Z$. Suppose $F \in \mathscr{F}(Z)$ and $ex\ F$ is infinite. Then $T^{-1}(F) \cap ex\ P(N^*)$ is infinite. It follows that $\varepsilon_\infty \in T^{-1}(F)$. The measure $\mu_2 = \sum_{i=1}^{\infty} 1/2^j\, \delta_{2j-2}$ is a $\prec$-maximal representing measure for $T(\varepsilon_\infty)$. By Lemma 1.2, $\operatorname{supp} \mu_2 \cup \operatorname{supp} \mu_1 \subseteqq F$. But $\operatorname{supp} \mu_1 \cup \operatorname{supp} \mu_2 \supseteqq ex\ Z$. By the *KMT*, it follows that $F = Z$. Conversely, suppose $F = \operatorname{cov}\{x_1, \cdots, x_n\}$ where $\{x_1, \cdots, x_n\} \subseteqq ex\ Z$. To prove $F \in \mathscr{F}(Z)$, it is enough to show $T^{-1}(F) \cap P(N^*) \in \mathscr{F}(P(N^*))$. There are integers $k_1, k_2, \cdots, k_n$ such that $T(\varepsilon_{k_j}) = x_j$, $j = 1, 2, \cdots, n$. Clearly, $\operatorname{cov}\{\varepsilon_{k_1}, \cdots, \varepsilon_{k_n}\} \subseteqq T^{-1}(F) \cap P(N^*)$ and $T(\operatorname{cov}\{\varepsilon_{k_1}, \cdots, \varepsilon_{k_n}\}) = F$. It follows that, if $w = (a_1, \cdots; a_\infty) \in T^{-1}(F) \cap P(N^*)$, then there is an $s \in \operatorname{cov}\{\varepsilon_{k_1}, \cdots, \varepsilon_{k_n}\}$ such that $w - s = cu + dv$ where c, d are real numbers. Hence, $a_\infty = -(c + d) \geqq 0$. Also, there integers p and q such that $a_{2p-1} = 1/2^p c$ and $a_{2q} = 1/2^q d$. Thus, $c = d = 0$. It follows that $\operatorname{cov}\{\varepsilon_{k_1}, \cdots, \varepsilon_{k_n}\} = T^{-1}(F) \cap P(N^*)$.

It is now clear that $\operatorname{cov}\ (F \cup E) \in \mathscr{F}(Z)$ for all $F, E \in \mathscr{F}(Z)$. By Theorem 3.2, $\mathscr{F}(X)$ is distributive. Z does not, however, have the e.s.p. for the measures μ_1 and μ_2 are both $\prec$-maximal and represent $T(\varepsilon_\infty)$ but $\operatorname{supp} \mu_1 \neq \operatorname{supp} \mu_2$.

It is interesting to note that the structure topology on $ex\ Z$ is the smallest T_1 topology on $ex\ Z$.

The following theorem extends [7, Corollary 4.6].

Theorem 3.6. *If the structure topology is defined on ex X, then ex X is structurally compact.*

Proof. Let $\mathscr{J}$ be a collection of structurally closed subsets of $ex\ X$ having the finite intersection property. For each $C \in \mathscr{J}$, there is a $F_C \in \mathscr{F}(X)$ such that $F_C \cap ex\ X = C$. Hence, the family $\{F_C \mid C \in \mathscr{J}\}$ has the finite intersection property. Thus, $\cap \{F_C \mid C \in \mathscr{J}\}$ is nonempty. By the *KMT* and Proposition 1.3, it follows that $\cap \{C \mid C \in \mathscr{J}\} \neq \varnothing$.

THEOREM 3.7. *Suppose the structure topology is defined on ex X. Let $f\colon ex\ X \to (-\infty, \infty]$ be structurally l.s.c., then there is an $f^* \in \mathscr{E}(X)$ such that $f^* | ex\ X = f$.*

Proof. For each $b \in (-\infty, \infty]$, let $F_b = \overline{\mathrm{cov}}\ f^{-1}(-\infty, b]$. By Proposition 1.3, and the *KMT*, it follows that $F_b \in \mathscr{F}(X)$ for all $b \in (-\infty, \infty]$. Define f^* by: $f^*(x) = \inf\{b \mid x \in F_b\}$.

NOTATION. Suppose $f\colon ex\ X \to (-\infty, \infty]$ ($[-\infty, \infty)$). If there is an extension of f to $\mathscr{E}(X)$ $(-\mathscr{E}(X))$, denote it by $f^*(f_*)$.

When the structure topology is defined, it is stronger than the facial topology. If the facial topology is Hausdorff, then the facial, structure, and relative topologies coincide (see [2, Th. 6.2]).

Consider Example 1.9. We will identify *ex* $P([0, 1])$ and $[0, 1]$ and regard each $\rho \in P([0, 1])$ as its own unique $\prec$-maximal representing measure. Note that, on *ex* Z, "closed" and "structurally closed" mean the same thing. Let E be a proper closed subset of *ex* Z and let $E' = [0, 1] \cap T^{-1}(E)$. It is known that the set $F = \overline{\mathrm{cov}}\ E$ is a face of Z. Furthermore, $T^{-1}(F) = \mathrm{cov}\ E' = \{\rho \mid \mathrm{supp}\,\rho \subseteqq E'\}$ is a face of $P([0, 1])$. We claim that E is closed in the facial topology if and only if $|\mu|(E') = 0$, where $|\mu|$ denotes the total variation of μ. (It will be convenient to assume that $|\mu|([0, 1]) = 2$.)

Suppose that $|\mu|(E') = 0$. It is enough to show that F is a *split face* ([2, p. 9]). Let $G = \{x \mid x = T(\nu)$ where $\nu(E') = 0$. Then G is a face of Z and, since $T^{-1}(F) = \overline{\mathrm{cov}}\ E'$, G is disjoint from F. Since every $\rho \in P([0, 1])$ can be written as a convex combination of a measure in $\overline{\mathrm{cov}}\ E'$ and a measure in $P([0, 1])$ which vanishes on E', it follows that $Z = \mathrm{cov}\ (F \cup G)$. Thus, any face of Z which is disjoint from F must be contained in G. Hence, G is the *complementary set* ([2]) of F. Suppose that $w \in Z - F \cup G$ and

$$w = c_1u_1 + (1 - c_1)v_1 = c_2u_2 + (1 - c_2)v_2\ ,$$

where $0 < c_i < 1$, $u_i \in F$, and $v_i \in G$ for $i = 1, 2$. For $i = 1, 2$ choose α_i such that $T(\alpha_i) = u_i$, and β_i such that $\beta_i(E') = 0$ and $T(\beta_i) = v_i$. Then there is a real number k such that

$$c_1\alpha_1 + (1 - c_1)\beta_1 - c_2\alpha_2 - (1 - c_2)\beta_2 = k\mu\ .$$

Since $|\mu|(E') = 0$, it follows that $c_1 = c_2$ and $\alpha_1 = \alpha_2$. Hence, F is a split face.

Assume that $|\mu|(E') > 0$. It will be shown that F is not split. Let $\mu = \mu^+ - \mu^-$ be the Hahn decomposition of μ. By Pl, we have $\mu^+, \mu^- \in P(0, 1)$.

Case 1. $\mu^+(E') > 0$ and $\mu^-(E') > 0$. By P2, we have $1 > \mu^+(E)$,

$\mu^-(E')$. Let $c_1 = \mu^+(E')$ and $c_2 = \mu^-(E')$. Define measures $\alpha_1, \alpha_2, \beta_1, \beta_2$ as follows: $\alpha_1(B) = c_1^{-1}\mu^+(E' \cap B)$, $\alpha_2(B) = c_2^{-1}\mu^-(E' \cap B)$, $\beta_1(B) = (1-c_1)^{-1}\mu^+(B-E')$, and $\beta_2(B) = (1-c_2)^{-1}\mu^-(B-E')$. Then $c_1x_1 + (1-c_1)y_1 = c_2x_2 + (1-c_2)y_2$, where $x_i = T(\alpha_i)$ and $y_i = T(\beta_i)$ for $i = 1, 2$. A straightforward argument shows that either $c_1 \neq c_2$, $x_1 \neq x_2$, or $y_1 \neq y_2$. Since α_1, α_2 are supported by E', it follows that $x_1, x_2 \in F$. Suppose it can be shown that y_1 and y_2 are in the complementary set of F. Then, by definition, F is not a split face.

The set $S = \{u \in Z \mid cu + (1-c)v = y_1$ for some $(c, v) \in (0, 1) \times Z\}$. is the smallest face containing y_1. Suppose that $cu + (1-c)v = y_1$. Choose ν_1, ν_2 such that $u = T(\nu_1)$ and $v = T(\nu_2)$. Then there is a real number k such that $c\nu_1 + (1-c)\nu_2 - \beta_1 = k\mu$. It follows that $k\mu$ is non-negative on subsets of E'. Thus, $k = 0$. It follows that $u, v \notin F$. Hence, $S \cap F = \varnothing$. i.e., y_1 is in the complementary set of F. Similarly, y_2 is in the complementary set of F.

Case 2. $\mu^+(E') > 0$ and $\mu^-(E') = 0$. Let I denote the characteristic function of the set F. The pointwise inf of the set of all continuous affine functions strictly dominating I will be denoted by $\hat{I}$. Suppose F is a split face then, by [2, Th. 3.5], $\hat{I}$ is affine. Define the measure $\mu^+ \circ T^{-1}$ by $\mu^+ \circ T^{-1}(B) = \mu^+(T^{-1}(B))$. Define $\mu^- \circ T^{-1}$ similarly. Then $\mu^+ \circ T^{-1}$ and $\mu^- \circ T^{-1}$ are $\prec$-maximal representing measures for the point $T(\mu^+) = T(\mu^-)$. It follows by [12, Lemma 9.7] that

$$\int \hat{I} d\, \mu^+ \circ T^{-1} = \int \hat{I} d\, \mu^- \circ T^{-1} .$$

Since I is u.s.c. on Z, we have $I = \hat{I}$ on $ex\ Z$ (see [12, p. 27]). It follows that $\mu^+ \circ T^{-1}(E) = \mu^- \circ T^{-1}(E)$. Therefore, $\mu^+(E') = \mu^-(E') = 0$ – a contradiction. Hence, F is not split.

The case in which $\mu^+(E') = 0$ and $\mu^-(E') > 0$ is handled in the same way as Case 2.

By P2, a proper subset of $ex\ Z$ which is closed in the facial topology, is nowhere dense in the structure topology.

4. Characterizations of the strong equal support property.

THEOREM 4.1. *Let X be a nonempty compact convex set. Then the following are equivalent:*

(i) *X has the s.e.s.p.*

(ii) *For each $f \in C(\partial X)$, there is a function $f^* \in \mathscr{E}(X)$ with $f^* \mid \partial X = f$*

(iii) *$\mathscr{E}(X)$ is distributive and separates the points of $ex\ X$ in the following sense: if x and y are distinct extreme points of X, then there are $g, f \in E(X)$ such that $f(x) > 0$, $g(y) > 0$ and $f \wedge g \leqq 0$*

(iv) *$\mathscr{F}(X)$ is distributive and ex X is structurally Hausdorff.*

(v) *For each continuous affine function a on X, there is an $a^* \in \mathscr{E}(X)$ such that $a | ex\ X = a^* | ex\ X$.*

Proof. For a proof that (i) implies (ii), see the discussion preceding Definition 2.1.

(ii) $\rightarrow$ (iii). Suppose x and y are distinct elements of $ex\ X$. There are functions $h, k \in C(\partial X)$ such that $h(x) > 0$, $k(y) > 0$, and $\min\{h, k\} \leqq 0$. Since $h^*, k^* | ex\ X \leqq 0$, it follows by Proposition 2.3 that $h^* \wedge k^* \leqq 0$.

To prove that $\mathscr{E}(X)$ is distributive, it is enough to show that, for all $f, g \in \mathscr{E}(X)$,

$$(\dagger) f \wedge g | \partial X = \min\{f, g\} | \partial X .$$

Since f and g are l.s.c., it follows that $f | \partial X = \sup\{u | u < f | \partial X$ and $u \in C(\partial X)\}$ and $g | \partial X = \sup\{v | v < g | \partial X$ and $v \in C(\partial X)\}$. Let $h = \sup\{u^* \wedge v^* | u, v \in C(\partial X), u \leqq f | \partial X, v \leqq g | \partial X\}$. By Theorem 2.4, it follows that $h \in \mathscr{E}(X)$. Note that, if $u, v \in C(\partial X)$, $u^* \wedge v^* = (\min\{u, v\})^*$ (Prop. 2.3). Thus, $h | \partial X = \min\{f | \partial X, g | \partial X\}$. It follows by Proposition 2.3 that $h = f < g$.

(iii) $\rightarrow$ (iv). First, it will be shown that $\mathscr{F}(X)$ is distributive. We will use J_S to denote the function which is 0 on S and 1 elsewhere. Let $F, E \in \mathscr{F}(X)$. Then $J_F, J_E \in \mathscr{E}(X)$. Note that $J_F \wedge J_E = J_{F \vee E}$. Suppose $x \in ((F \vee E) \cap ex\ X) - \operatorname{cov}(F \cup E)$. Then

$$\begin{aligned} J_{\{x\}} &= \max\{J_{\{x\}}, J_F \wedge J_E\} \\ &= (\max\{J_{\{x\}}, J_F\}) \wedge (\max\{J_{\{x\}}, J_E\}) \\ &= 1 \wedge 1 = 1 . \end{aligned}$$

Thus, $((F \vee E) \cap ex\ X) - \operatorname{cov}(F \cup E) = \varnothing$. By Proposition 1.3 and the *KMT*, $F \vee E = \operatorname{cov}(F \cup E)$. It follows by Theorem 3.2 that $\mathscr{F}(X)$ is distributive.

To prove that $ex\ X$ is structurally Hausdorff, it is enough to show that, for each pair $x, y \in ex\ X, x \neq y$, there are $F, E \in \mathscr{F}(X)$ such that $x \notin F$, $y \notin E$ and $F \cup E \supseteqq ex\ X$. By hypothesis, there are functions $f, g \in \mathscr{E}(X)$ such that $f(x) > 0, g(y) > 0$ and $f \wedge g \leqq 0$. Let $F = f^{-1}(-\infty, 0]$ and $E = g^{-1}(-\infty, 0]$. Since $\min\{f | ex\ X, g | ex\ X\}$ is structurally l.s.c., it follows by Theorem 3.7 and Proposition 2.3 that $f \wedge g | ex\ X = \min\{f, g\} | ex\ X$. Thus, $(F \cup E) \cap ex\ X = ex\ X$.

(iv) $\rightarrow$ (i). Suppose it can be shown that $ex\ X$ is closed. Then, since the structure topology is weaker than the relative topology, every closed subset of $ex\ X$ is structurally closed. Thus, if $E \subseteqq ex\ X$ is closed, then there is an $F \in \mathscr{F}(X)$ with $E = F \cap ex\ X$. By Proposition 1.3, $\overline{\operatorname{cov}}\ E = F$. It follows by Theorem 1.8 that X has the s.e.s.p.

It remains to be shown that $ex\ X$ is closed. Let $C_s(ex\ X)$ be the space of structurally continuous real-valued functions on $ex\ X$. Suppose $z_1 \in \partial X$. Then there is a net $\{z_\alpha\}$ of elements in $ex\ X$ which converges to z_1 and converges structurally to some $z \in ex\ X$. Hence, by Theorem 3.7 and the proof of Theorem 3.8, $f^*(z_1) = f(z)$ for each $f \in C_s(ex\ X)$. Note that $J_{\{z\}} | ex\ X$ is structurally l.s.c. Since $ex\ X$ is structurally compact and Hausdorff, it follows that

$$J_{\{z\}} | ex\ X = \sup \{f \in C_s(ex\ X) | f < J_{\{z\}} | ex\ X\} .$$

By Theorem 2.4 and Proposition 2.3, it follows that

$$J_{\{z\}} = \sup \{f^* | f \in C_s(ex\ X), f < J_{\{z\}} | ex\ X\} .$$

Therefore, $J_{\{z\}}(z_1) = J_{\{z\}}(z)$. Thus, $z = z_1$.

That (ii) implies (v) is clear.

(v) $\rightarrow$ (iv). First it will be shown that $\mathscr{F}(X)$ is distributive. Let $F, E \in \mathscr{F}(X)$. Let $K = \{a | a$ is continuous affine and $a^{-1}(-\infty, 0] \supseteqq$ cov $(F \cup E)\}$. It is claimed that

$$\text{cov}\ (F \cup E) = \cap \{(a^*)^{-1}(-\infty, 0] | a \in K\} .$$

If $a \in K$, then $(a^*)^{-1}(-\infty, 0] \supseteqq ex$ (cov $(F \cup E)$) by Proposition 1.3. Suppose $a^*(x) \leqq 0$ for every $a \in K$. If h is any real continuous affine function on X, then $h^* - h$ is concave and l.s.c. Since $(h^* - h) | ex\ X \geqq 0$, it follows that $h^* \geqq h$. Thus, $a(x) \leqq 0$ for every $a \in K$. A simple separation argument shows that $x \in$ cov $(F \cup E)$.

To prove that $ex\ X$ is structurally Hausdorff, it is enough to note that $C_s(ex\ X)$ contains a point-separating subspace.

Many of the results contained in this paper can be generalized to the following setting: Let Y be a compact Hausdorff space. Suppose that S is a cone of real-valued continuous functions on Y which satisfies: f, g $\mathscr{S}$ implies that $\min \{f, g\} \in \mathscr{S}$. A notion of "convexity with respect to $\mathscr{S}$" can be introduced on Y. The concepts of: extreme point, extreme set, face, and structure topology can be given meaning in the above context. For details, see [10].

REFERENCES

1. E. M. Alfsen, *On the geometry of Choquet simplexes*, Math. Scand., **15** (1964), 97-110.

2. E. M. Alfsen and T. B. Andersen, *Split faces of compact convex sets*, Aarhus Universitet, preprint series 1968/1969 No. 32.

3. H. Bauer, *Shilovscher Rand und Dirichletsches Problem*, Ann. Inst. Fourier (Grenoble) **11** (1961), 89-136.

4. M. Brelot, *Elements de la theorie classique du potential*, 3rd ed., Centre Documentation Univ., Paris, 1965.

5. N. Dunford and J. T. Schwartz, *Linear Operators* I, Interscience, New York, 1957.

6. E. Effros, *Structure in simplexes*, Acta. Math., **117** (1967), 103-121.

7. ———, *Structure in simplexes II*, J. of Functional Analysis, **1** (4) (1967), 379-391.

8. I. Feinberg, *Henry Rutgers thesis*, Rutgers-The State University (1968).

9. A. J. Lazar, *On affine functions on simplexes*, Ditoed notes, Hebrew University, Jerusalem.

10. J. N. Mc Donald, *Thesis*, Rutgers-The State University (1969).

11. P. A. Meyer, *Probability and potentials*, Blaisdell, Waltham, Mass. (1966).

12. R. R. Phelps, *Lectures on Choquet's theorem*, Van Nostrand, Princeton, N. J. (1966).

13. M. Rogalski, *Quelques problèmes concernant une caracterisation des simplexes*, C. R. Acad. Sci., Paris, Oct. (1969).

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