PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

ALGEBRAIC STRUCTURE FOR A SET OF NONLINEAR INTEGRAL OPERATIONS

DAVID LOWELL LOVELADY

A generalized addition is introduced for a set of generators, and a generalized multiplication is introduced for a set of evolution systems. Then the mapping which takes a generator to the corresponding evolution system becomes an isomorphism. Necessary and sufficient conditions are found for the generalized addition to reduce to addition, and hence, under these conditions, we are able to write a formula for the evolution system generated by the sum of two generators.

Preliminaries. Let $S = [0, \infty)$, and let $(G, +)$ be a complete normed abelian group with norm N_1. Let H be the set to which A belongs only in case A is a function from G to G, $A[0] = 0$, and there is a number b so that $N_1[A[p] - A[q]] \leqq bN_1[p - q]$ whenever (p, q) is in $G \times G$. If A is in H, let $N_2[A]$ be the least number b so that $N_1[A[p] - A[q]] \leqq bN_1[p - q]$ whenever (p, q) is in $G \times G$, and let $N_3[A]$ be the least number b so that $N_1[A[p]] \leqq bN_1[p]$ whenever p is in G.

Let OA^+, OM^+, and $\mathscr{E}^+$ be as in [8]. Let OA be the set to which V belongs only in case V is a function from $S \times S$ to H so that

(i) $V(x, y) + V(y, z) = V(x, z)$ whenever (x, y, z) is in $S \times S \times S$ and y is between x and z, and

(ii) there is a member α of OA^+ so that

$$N_2[V(a, b)] \leqq \alpha(a, b)$$

whenever (a, b) is in $S \times S$.
If α and V are related as in (ii), α will be said to dominate V.

Let OM be the set to which W belongs only in case W is a function from $S \times S$ to H so that

(i) $W(x, y)W(y, z) = W(x, z)$ whenever (x, y, z) is in $S \times S \times S$ and y is between x and z, where the multiplication is composition, and

(ii) there is a member μ of OM^+ so that

$$N_2[W(a, b) - I] \leqq \mu(a, b) - 1$$

whenever (a, b) is in $S \times S$, where I in H is given by $I[p] = p$.

The following theorem is due to Mac Nerney [9].

THEOREM 1. *There is a bijection $\mathscr{E}$ from OA onto OM so that if V is in OA and W is in OM, then* (i), (ii), (iii), (iv), *and* (v) *are* *equivalent.*

(i) $W = \mathscr{E}[V]$.

(ii) $W(a, b)[p] = {}_a\Pi^b[I + V][p]$ *whenever* (a, b, p) *is in* $S \times S \times G$.

(iii) $V(a, b)[p] = {}_a\Sigma^b[W - I][p]$ *whenever* (a, b, p) *is in* $S \times S \times G$.

(iv) *There is* (α, μ) *in* $\mathscr{E}^+$ so that

$$N_3[W(a, b) - I - V(a, b)] \leqq \mu(a, b) - 1 - \alpha(a, b)$$

whenever (a, b) *is in* $S \times S$.

(v) *If* (a, p) *is in* $S \times G$, *and* h *is given by* $h(t) = W(t, a)[p]$, *then* h *has bounded* N_1*-variation on each bounded interval of* S, *and is the only such function such that*

$$h(t) = p + (R)\int_t^a V[h]$$

whenever t *is in* S.

REMARK 1. The notions of Π, Σ, and $(R)\int$ are to be taken as in [9].

Let OAI be that subset of OA to which V belongs only in case each of $I + V(t, t^+)$, $I + V(t, t^-)$, $I + V(t^+, t)$, and $I + V(t^-, t)$ has inverse in H whenever t is in S. The following theorem is due to Herod [6] (see also [4] and [5]).

THEOREM 2. *Let* (V, W) *be in* $\mathscr{E}$. *Then* (i) *and* (ii) *are equivalent.*

(i) V *is in* OAI.

(ii) *Each value of* W *has inverse in* H.

Furthermore, there is a bijection $\mathscr{G}$ *from* OAI *onto* OAI *such that if* V *is in* OAI, *then each of* (iii), (iv), (v), *and* (vi) *is true.*

(iii) $\mathscr{G}[\mathscr{G}[V]] = V$.

(iv) $\mathscr{G}[V](a, b) = -V(b, a)$ *for each* (a, b) *in* $S \times S$ *only in case* ${}_a\Sigma^b N_3[V[I - V] - V] = 0$ *whenever* (a, b) *is in* $S \times S$.

(v) $\mathscr{E}[\mathscr{G}[V]](a, b) \cdot \mathscr{E}[V](b, a) = \mathscr{E}[V](b, a) \cdot \mathscr{E}[\mathscr{G}[V]](a, b) = I$ *whenever* (a, b) *is in* $S \times S$.

(vi) $\mathscr{G}[V](a, b)[p] = -{}_b\Sigma^a V[I + V]^{-1}[p]$ *whenever* (a, b, p) *is in* $S \times S \times G$.

The $\oplus$ Operation.

LEMMA 1. *If each of* α *and* β *is in* OA^+, *and* (a, b) *is in* $S \times S$, *then* ${}_a\Sigma^b\alpha[1 + \beta]$ *exists and is the greatest lower bound of the set to which* r *belongs only in case there is a chain* $(t_k)_{k=0}^n$ *from* a *to* b *so that* $r = \Sigma_{k=1}^n \alpha(t_{k-1}, t_k)[1 + \beta(t_{k-1}, t_k)]$.

Proof. It suffices to show that if (a, b, c) is in $S \times S \times S$, and b is between a and c, then

$$\alpha(a, c)[1 + \beta(a, c)] \geqq \alpha(a, b)[1 + \beta(a, b)] + \alpha(b, c)[1 + \beta(b, c)] .$$

But $\alpha(a, c) \geqq \alpha(a, b)$ and $\alpha(a, c) \geqq \alpha(b, c)$, so

$$\begin{aligned} \alpha(a, c)\beta(a, c) &= \alpha(a, c)\beta(a, b) + \alpha(a, c)\beta(b, c) \\ &\geqq \alpha(a, b)\beta(a, b) + \alpha(b, c)\beta(b, c) , \end{aligned}$$

and the proof is complete.

THEOREM 3. *If each of V_1 and V_2 is in OA, and (a, b, p) is in $S \times S \times G$, then ${}_a\Sigma^b V_1[I + V_2][p]$ exists. If, for $i = 1, 2$, α_i in OA^+ dominates V_i, then*

$$\begin{aligned} &N_3[V_1(a, b)[I + V_2(a, b)] - {}_a\Sigma^b V_1[I + V_2]] \\ &\qquad \leqq \alpha_1(a, b)[1 + \alpha_2(a, b] - {}_a\Sigma^b \alpha_1[1 + \alpha_2] \end{aligned}$$

whenever (a, b) is in $S \times S$. Furthermore, if U is given by $U(a, b)[p] = {}_a\Sigma^b V_1[I + V_2][p]$, then U is in OA.

Proof. Let (a, b, c, p) be in $S \times S \times S \times G$, with b between a and c. Now

$$\begin{aligned} &N_1[V_1(a, c)[I + V_2(a, c)][p] - V_1(a, b)[I + V_2(a, b)][p] \\ &\qquad - V_1(b, c)[I + V_2(b, c)][p]] \\ &\quad = N_1[V_1(a, b)[I + V_2(a, c)][p] - V_1(a, b)[I + V_2(a, b)][p] \\ &\qquad + V_1(b, c)[I + V_2(a, c)][p] - V_1(b, c)[I + V_2(b, c)][p]] \\ &\quad \leqq [\alpha_1(a, b)\alpha_2(b, c) + \alpha_1(b, c)\alpha_2(a, b)]N_1[p] \\ &\quad = N_1[p](\alpha_1(a, c)[1 + \alpha_2(a, c)] - \alpha_1(a, b)[1 + \alpha_2(a, b)] \\ &\qquad - \alpha_1(b, c)[1 + \alpha_2(b, c)]) . \end{aligned}$$

The theorem is now clear.

DEFINITON 1. If each of V_1 and V_2 is in OA, then $V_1 \oplus V_2$ is that member U of OA given by

$$U(a, b)[p] = V_2(a, b)[p] + {}_a\Sigma^b V_1[I + V_2][p] .$$

DEFINITION 2. If V is in OA, V^* will be that member of OA given by $V^*(a, b) = V(b, a)$.

THEOREM 4. *If each of V_1, V_2, and V_3 is in OA, then*

$$V_1 \oplus (V_2 \oplus V_3) = (V_1 \oplus V_2) \oplus V_3 ,$$

and consequently $(OA, \oplus)$ *is a semigroup.* $(OAI, \oplus)$ *is a subgroup of* $(OA, \oplus)$*, each subgroup of* $(OA, \oplus)$ *is contained in* OAI*, and if* V *is in* OAI*, then*

$$V \oplus \mathscr{G}[V]^* = \mathscr{G}[V]^* \oplus V = 0 .$$

Proof. Let U be given by

$$U(a, b)[p] = V_3(a, b)[p] + {}_a\Sigma^b V_2[I + V_3][p] + {}_a\Sigma^b V_1[I + V_2][I + V_3][p] .$$

A moment's reflection shows

$$V_1 \oplus (V_2 \oplus V_3) = U = (V_1 \oplus V_2) \oplus V_3 ,$$

so the first part of the theorem is clear.

Now if A is in H, and $I + A$ has inverse in H, then

$$\begin{aligned} &-A[I + A]^{-1} + A[I - A[I + A]^{-1}] \\ &\quad = -A[I + A]^{-1} + A[[I + A] - A][I + A]^{-1} = 0 . \end{aligned}$$

This, with (vi) of Theorem 2, says that if V is in OAI, then $V \oplus \mathscr{G}[V]^* = 0$. Similarly, $\mathscr{G}[V]^* \oplus V = 0$, so $(OAI, \oplus)$ is a group.

To complete the proof it suffices to show that if U and V are in OA, and $U \oplus V = V \oplus U = 0$, then U is in OAI and $V = \mathscr{G}[U]^*$. If t is in S, then $[U \oplus V](t, t^+) = 0$, so

$$\begin{aligned} &U(t, t^+)[I + V(t, t^+)] + V(t, t^+) = 0 , \\ &U(t, t^+)[I + V(t, t^+)] + [I + V(t, t^+)] = I , \\ &[I + U(t, t^+)][I + V(t, t^+)] = I . \end{aligned}$$

Similarly, since $[V \oplus U](t, t^+) = 0$, we have

$$[I + V(t, t^+)][I + U(t, t^+)] = I .$$

Similar computations for (t, t^-), (t^+, t), and (t^-, t) show that each of U and V is in OAI. Also, it is clear that V is given by

$$V(a, b)[p] = -{}_a\Sigma^b U[I + U]^{-1}[p] = \mathscr{G}[U]^*(a, b)[p] ,$$

so the proof is complete.

LEMMA 2. *Let each of* α_1 *and* α_2 *be in* OA^+*, and let* β *be a continuous member of* OA^+*. Suppose* $\beta(a, b) \leqq {}_a\Sigma^b \alpha_1\alpha_2$ *whenever* (a, b) *is in* $S \times S$*. Then* $\beta = 0$.

REMARK 2. Lemma 2 is immediate, and we shall not prove it here.

THEOREM 5. *Let each of* V_1 *and* V_2 *be in* OA. *Then* (i) *and* (ii) *are equivalent, and* (iii) *and* (iv) *are equivalent.*

(i) $V_1 \oplus V_2 = V_1 + V_2$.

(ii) $V_1[I + V_2] - V_1 = 0$ *at all "pairs" of the forms* (t, t^+), (t, t^-), (t^+, t), *and* (t^-, t) *for* t *in* S.

(iii) $V_1 \oplus V_2 = V_2 \oplus V_1$.

(iv) $V_1 - V_2 = V_1[I + V_2] - V_2[I + V_1]$ *at all "pairs" of the forms* (t, t^+), (t, t^-), (t^+, t), *and* (t^-, t) *for* t *in* S.

Proof. We shall indicate the first equivalence, and leave the second to the reader. Since $[V_1 \oplus V_2] - [V_1 + V_2] = \Sigma V_1[I + V_2] - V_1$, it is clear that (i) implies (ii). Now suppose (ii). For $i = 1, 2$, let α_i in OA^+ dominate V_i. Let β in OA^+ be given by $\beta(a, b) = {}_a\Sigma^b N_3[V_1[I + V_2] - V_1]$. Now, by (ii), β is continuous, and clearly $\beta(a, b) \leqq {}_a\Sigma^b \alpha_1\alpha_2$ whenever (a, b) is in $S \times S$. Thus $\beta = 0$, (i) follows, and the proof is complete.

The $\otimes$ Operation and the Exponential Identity.

THEOREM 6. *Let each of* (V_1, W_1) *and* (V_2, W_2) *be in* $\mathscr{E}$, *and let* (a, b, p) *be in* $S \times S \times G$. *Then each of*

$$ {}_a\Pi^b[I + V_1][I + V_2][p] \quad and \quad {}_a\Pi^b W_1 W_2[p] $$

exists, and they are equal. Furthermore, if M *is given by*

$$ M(a, b)[p] = {}_a\Pi^b W_1 W_2[p] , $$

then M *is in* OM.

Proof. Let $U = V_1 \oplus V_2$. Let α be a member of OA^+ which dominates each of U, V_1, and V_2, and let $\mu = \mathscr{E}^+[\alpha]$. Let (a, b, p) be in $S \times S \times G$, and let $(t_k)_{k=0}^n$ be a chain from a to b. Now, by [7, Lemma 4],

$$
\begin{aligned}
N_1[\Pi_{k=1}^n[I + U(t_{k-1}, t_k)][p] - \Pi_{k=1}^n[I + V_1(t_{k-1}, t_k)][I + V_2(t_{k-1}, t_k)][p]] \\
\leqq N_1[p]\mu(a, b)^2 \Sigma_{k=1}^n N_3[V_1(t_{k-1}, t_k)[I + V_2(t_{k-1}, t_k)] \\
- {}_{t_{k-1}}\Sigma^{t_k} V_1[I + V_2]] \\
\leqq N_1[p]\mu(a, b)^2[\Sigma_{k=1}^n \alpha(t_{k-1}, t_k)[1 + \alpha(t_{k-1}, t_k)] - {}_a\Sigma^b\alpha[1 + \alpha]] .
\end{aligned}
$$

It is now clear that ${}_a\Pi^b[I + V_1][I + V_2][p]$ exists and equals ${}_a\Pi^b[I + U][p]$ whenever (a, b, p) is in $S \times S \times G$. Now [9, Lemma 1.2] tells us that ${}_a\Pi^b W_1 W_2[p] = {}_a\Pi^b[I + V_1][I + V_2][p]$ whenever (a, b, p) is in $S \times S \times G$. Since these products describe $\mathscr{E}[U]$, it is clear that M is in OM and the proof is complete.

DEFINITION 3. If each of W_1 and W_2 is in OM, $W_1 \otimes W_2$ is that member M of OM given by $M(a, b)[p] = {}_a\Pi^b W_1 W_2[p]$.

There emerges from the proof of Theorem 6 a fact which we now record.

THEOREM 7. *If each of V_1 and V_2 is in OA, then*

$$\mathscr{E}[V_1 \oplus V_2] = \mathscr{E}[V_1] \otimes \mathscr{E}[V_2] .$$

REMARK 3. Theorem 7, together with the first equivalence of Theorem 5, includes and extends Theorem 6 of [7].

THEOREM 8. *Let V_1 be in OA, V_2 in OAI. Let U in OA be given by*

$$U(a, b)[p] = {}_a\Sigma^b V_1[I + V_2]^{-1}[p] .$$

Then

$$\mathscr{E}[V_1 + V_2] = \mathscr{E}[U] \otimes \mathscr{E}[V_2] .$$

Proof. Let (a, b, p) be in $S \times S \times G$. Now

$$\begin{aligned}[\mathscr{E}[U] \otimes \mathscr{E}[V_2]](a, b)[p] &= {}_a\Pi^b \mathscr{E}[U]\mathscr{E}[V_2][p] \\ &= {}_a\Pi^b[I + U][I + V_2][p] \\ &= {}_a\Pi^b[I + V_1[I + V_2]^{-1}][I + V_2][p] \\ &= {}_a\Pi^b[I + V_1 + V_2][p] \\ &= \mathscr{E}[V_1 + V_2](a, b)[p] .\end{aligned}$$

This completes the proof.

REMARK 4. Note that by using Theorems 5, 7, and 8 we can compute, under two different sets of hypotheses, $\mathscr{E}[V_1 + V_2]$ in terms of the $\otimes$ operation.

REMARK 5. The notion of continuously multiplying solutions for generators in order to construct the solution for a sum of generators has been used by Trotter [11] and Chernoff [1], [2] for the case of autonomous linear differential equations with discontinuous linear operators, by Helton [3] for the case of linear Stieltjes integral equations, and by Mermin [10] for the case of autonomous nonlinear differential equations with accretive operators.

References

1. P. R. Chernoff, *Note on product formulas for operator semigroups*, J. Functional Analysis, **2** (1968), 238-242.
2. ———, *Semigroup product formulas and addition of unbounded operators*, Bull. Amer. Math. Soc., **76** (1970), 395-398.
3. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math., **16** (1966), 297-322.
4. J. V. Herod, *Multiplicative inverses of solutions for Volterra-Stieltjes integral equations*, Proc. Amer. Math. Soc., **22** (1969), 650-656.
5. ———, *Coalescence of solutions for nonlinear Volterra equations*, Notices Amer. Math. Soc., **16** (1969), 834.
6. ———, *Coalescence of solutions for nonlinear Stieltjes equations*, J. Reine Angew. Math., (to appear).
7. D. L. Lovelady, *Perturbations of solutions of Stieltjes integral equations*, Trans. Amer. Math. Soc., **155** (1971), 175-188.
8. J. S. Mac Nerney, *Integral equations and semigroups*, Illinois J. Math., **7** (1963), 148-173.
9. ———, *A nonlinear integral operation*, Illinois J. Math., **8** (1964), 621-638.
10. J. L. Mermin, *Accretive operators and nonlinear semigroups*, Ph. D. Thesis, University of California, Berkeley (1968).
11. H. F. Trotter, *On the product of semigroups of operators*, Proc. Amer. Math. Soc., **10** (1959), 545-551.

Received June 4, 1970.

GEORGIA INSTITUTE OF TECHNOLOGY
AND
UNIVERSITY OF SOUTH CAROLINA