NUCLEAR SPACES, SCHAUDER BASES,
AND CHOQUET SIMPLEXES

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We prove that a Frechet space E with a Schauder basis is
nuclear if and only if every bounded subset of E is contained
in a bounded Choquet simplex. We then show how this theorem
relates to the basis problem for nuclear Frechet spaces and
to the classical Banach-Dieudonné theorem.

In the class of all infinite dimensional Banach spaces the Hilbert
spaces (and their isomorphs) seem to be the natural generalization of
the finite dimensional normed linear spaces. Indeed many authors
have characterized Hilbert spaces by extending results from Euclidean
geometry (see [4, Chap. VII, § 3 p. 115–121] for an excellent discussion).

However, in the class of all infinite dimensional Fréchet spaces
(complete, metrizable, locally convex) this is no longer the case. It
now appears that the nuclear spaces of A. Grothendieck [9], [26] are
the proper generalizations of the finite dimensional spaces.

Indeed, Grothendieck [9] has shown that a Frechet space E is
nuclear if and only if E has the Dvoretzky-Rogers property: *every
unconditionally convergent series in E is absolutely convergent.*

This property is known to characterize the finite dimensional
normed linear spaces in the class of all normed linear spaces [7].

Also, it is well known that nuclear spaces can be nicely approxi-
mated by finite dimensional spaces (Kolmogorov diameters, diametric
and approximative dimensions) see [1], [22] and [26]. Moreover, the
Dynin-Mitiagin theorem [8] and a recent result of Wojtynski [29]
asserts that a Fréchet space E with a Schauder basis is nuclear if
and only if every basis of E is absolute. Since pelczynski and Singer
[23] have shown that every infinite dimensional Banach space with a
basis has a conditional basis, the above result can be viewed as a
generalization to the class of Fréchet spaces with bases of a well-
known property of finite dimensional spaces.

Aside from these characterizations and a few related results, there
seems to be few instances where nuclear spaces have been characterized
by some property common to finite dimensional normed linear spaces.
The purpose of this paper is to present such a characterization.

It is obvious that a bounded set in a finite dimensional normed
linear space lies in a bounded simplex. Our goal is to prove

THEOREM 1. *A Fréchet space E with a Schauder basis is nuclear
if and only if every bounded subset of E is contained in a bounded*

Choquet simplex. (Moreover, this simplex can be taken to be compact).

Part of this result, without the basis requirement, has appeared in [18].

2. **Terminology.** All linear-topological prerequisites can be found in Köthe [15] and Day [4]. Throughout this paper E denotes a real Fréchet space. We will use the word neighborhood to mean "closed convex circled neighborhood".

If U is a neighborhood of the origin in E with Minkowski functional ρ_U then by E_U we denote the quotient space $E/\rho_U^{-1}(0)$, on which ρ_U is a norm. The space E is *nuclear* [26] if for each neighborhood U of 0 in E there is a neighborhood of 0, V, absorbed by U, such that the canonical map

$$I_{VU} : E_V \longrightarrow E_U$$

is nuclear, i.e.

$$I_{VU}(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, f_n \rangle \rho_n , \qquad x \in E_V$$

where $\sum_{n=1}^{\infty} |\lambda_n| < +\infty$, $(f_n) \subset E_V'$ is equicontinuous and $(\rho_n) \subset E_U$ is bounded.

(For equivalent ways to define nuclearity see [26].) We will use conv A and $\Gamma(A)$ to denote, respectively, the convex hull and convex circled hull of a set A in E. As usual, $\bar{A}$ denotes the closure of A.

A sequence (x_n) in E is *topologically free* if no x_n is in the closed linear span of $(x_m : m \neq n)$; equivalently, (x_n) is topologically free if there are functionals (f_n) in E' such that

$$f_n(x_m) = \delta_{nm}, \text{ the Kronecker delta.}$$

(See [5] for a discussion of topologically free sequences in locally convex spaces.) A topologically free sequence (x_n) with associated functionals (f_n) is a *Schauder basis* for E if for each $x \in E$,

$$x = \sum_{n=1}^{\infty} f_n(x) x_n$$

convergence in the topology of E.

If A is a set of positive sequences $a = (a_n)$ (i.e., all $a_n \geqq 0$) satisfying
  (i) for each n there is an $a \in A$ such that $a_n \neq 0$; and,
  (ii) for $a^1, \cdots, a^n \in A$ there is an $a \in A$ and $M > 0$ such that

$$a_n^i \leqq M a_n \text{ for all } i \text{ and } n, \text{ then}$$

$$\lambda(A) = \left\{ x = (x_n) : \rho_a(x) = \sum_{n=1}^{\infty} a_n |x_n| < +\infty \text{ for all } a \in A \right\}$$

with the topology generated by the seminorms $\{\rho_a : a \in A\}$ is called a *Köthe sequence space*. It follows from the Dynin-Mitiagin theorem above that a nuclear Fréchet space E with a Schauder basis (x_n) is isomorphic to the Köthe sequence space $\lambda(A)$, $A = \{|x_n|_m\}$, where $|\cdot|_1 \leq |\cdot|_2 \leq \cdots$ is an increasing family of seminorms generating the topology of E. Thus, nuclear Fréchet Köthe sequence spaces coincide with the class of nuclear Fréchet spaces with bases. (The above discussion is the main theme of the work [1]; for general relations between bases and sequence spaces see [28]).

In general, $\lambda = \lambda(A)$ is nuclear if and only if for each $x \in \lambda$ there is a $y \in \lambda$ and $t \in l_1$ with $x_n \leq t_n y_n$ for each n. This property is known as the *Grothendieck-Pietsch criterion* [14].

Grothendieck has conjectured that every nuclear Fréchet space has a Schauder basis. Our Theorem 2 below sheds some new light on this (still unsolved) conjecture.

Finally we need the notion of a Choquet simplex.

Let S be a convex subset of E. Passing to $E \times R$, R the scalar field, if necessary, we may suppose that S lies in a hyperplane of E which misses the origin. The set S is a *Choquet simplex* [3], [25] if the cone

$$C = \{\alpha s : \alpha \geq 0, s \in S\}$$

generated by S induces a lattice order in C-C. Choquet has observed that in finite dimensional spaces a Choquet simplex coincides with the usual notion of simplex. (A word of caution is necessary. A Choquet simplex S may not, strictly speaking, generate a cone in E, e.g. if the origin is in the core of S. More precisely, S is a Choquet simplex if S is affinely homeomorphic to a set S' with the above properties. In particular, a translate of a Choquet simplex is again a Choquent simplex.)

3. **Three lemmas.** In this section we develop the machinery used to prove Theorems 1 and 2. Lemma 1, purely technical in nature, appears in [18]. We reproduce it here for completeness. Lemmas 2 and 3 are of independent interest.

LEMMA 1. *Let (T, Σ, μ) be a measure space and $(g_n)_{n=0}^{\infty}$ a sequence in $L_1(T, \Sigma, \mu)$. Suppose there is a constant M such that $g_0 \geq 0$, $g_0 + \sum_{n \in \sigma} g_n \geq 0$ and $\int_T g_0 d\mu + \sum_{n \in \sigma} \int_T g_n d\mu \leq M$ for any finite set σ of positive integers. Then*

$$\sum_{n=0}^{\infty} \int_T |g_n| d\mu < +\infty .$$

Proof. Since $L_1(T, \Sigma, \mu)$ is weakly complete and

$$\left\| \sum_{n \in \sigma} g_n \right\| \leqq M + \| g_0 \|$$

it follows that $\sum_{n=0}^{\infty} g_n$ is unconditionally convergent [4, p. 60], [21], say to g. For any positive integer m let $(E_i)_{i=1}^{p(m)}$ be a partition of T into disjoint measurable sets such that each g_n, $1 \leqq n \leqq m$ has constant sign on E_i, $1 \leqq i \leqq p(m)$.

For fixed i let

$$P_i = \{n : \ g_n \,|\, E_i \geqq 0, 1 \leqq n \leqq m\} ,$$

and

$$Q_i = \{1, 2, \cdots, m\} - P_i .$$

Then

$$\sum_{n \in P_i} \int_{E_i} |g_n| \, d\mu = \int_{E_i} \left(\sum_{n \in P_i} g_n \right) d\mu \leqq \int_{E_i} g \, d\mu$$

and

$$\sum_{n \in Q_i} \int_{E_i} |g_n| \, d\mu = - \int_{E_i} \left(\sum_{n \in Q_i} g_n \right) d\mu \leqq \int_{E_i} g_0 \, d\mu .$$

Thus

$$\sum_{n=1}^{m} \int_{E_i} |g_n| \, d\mu \leqq \int_{E_i} (g_0 + g) \, d\mu$$

and so,

$$\sum_{n=1}^{m} \int_{T} |g_n| \, d\mu \leqq \int_{T} (g_0 + g) \, d\mu < +\infty .$$

The next lemma is the main tool needed in the proofs of Theorems 1 and 2.

LEMMA 2. *Let (y_n) be a topologically free null sequence in a locally convex space. If $K = \overline{\mathrm{conv}} \, (y_n)$ is compact, then K is a Choquet simplex.*

Proof. By Milman's theorem [15, p. 332] the set of extreme points of K, ext K, is included in $\{y_n : \ n \geqq 1\} \cup \{0\}$. By the Choquet-Meyer uniqueness theorem [25, p. 66] we must show that every point $k \in K$ admits a unique maximal (in Choquet's order) representing measure. The set ext K is closed since (y_n) is a null sequence and so the maximal measures on K are exactly those measures supported by ext K [25, p. 27, p. 30 and Prop. 9.3, p. 69], i.e. they are atomic

measures supported by $\{y_n: n \geqq 1\} \cup \{0\}$.   Let μ and ν be two maximal measures representing $k \in K$.   Let $\mu(\{y_n\}) = \mu_n$, $\mu(\{0\}) = \mu_0$ and similarly for ν_n and ν_0.   Then $\mu_n, \nu_n \geqq 0$ for $n = 0, 1, 2, \cdots$,

$$(+) \qquad \sum_{n=0}^{\infty} \mu_n = \sum_{n=0}^{\infty} \nu_n = 1 \, ,$$

and

$$k = \sum_{n=1}^{\infty} \mu_n y_n = \sum_{n=1}^{\infty} \nu_n y_n \, .$$

Let (f_n) be the functionals associated with (y_n).   Then $f_n(k) = \mu_n = \nu_n$ and from $(+)$ it follows that $\mu_0 = \nu_0$, i.e. $\mu = \nu$ and K is a Choquet simplex.

It is very likely that the conclusion of Lemma 2 holds without the assumption that K is compact.   In the case of a Fréchet space, our interest here, of course K is compact.   Thus we have not checked the general result.

For our final lemma we need the notation of the normal hull of a set of sequences.   If M is a set of real sequences $b = (b_n)$, the *normal hull* of M is the set of all real sequences $y = (y_n)$ with $|y_n| \leqq |b_n|$ for all n and some $b \in M$.

Köthe [14] has shown that in a nuclear sequence space λ, each bounded set B is contained in the normal hull of a single element $\rho = (\rho_n)$.

Geometrically Lemma 3 says that in a nuclear sequence space a "parallelopiped" can be placed in a simplex.

LEMMA 3.   *Let λ be a nuclear Köthe sequence space and let $\rho = (\rho_n) \in \lambda$, $\rho_n \geqq 0$ for each n.   Then $B = \{x \in \lambda: |x_n| \leqq \rho_n\}$ is contained in a compact Choquet simplex.*

Proof.   Considering $B + \rho$ instead of B we may assume $B = \{x \in \lambda: 0 \leqq x_n \leqq \rho_n\}$.   By the Grothendieck-Pietsch criterion there is a $\sigma = (\sigma_n) \in \lambda$ and $(t_n) \in l_1$ such that $\rho_n \leqq t_n \sigma_n$.   Dividing by a constant if necessary we can assume that $\Sigma t_n \leqq 1$, $t_n > 0$.   Also we can suppose that no σ_n is zero.   If (e_n) is the unit vector basis of λ then $(\sigma_n e_n)$ is a topologically free null sequence.   Thus if $x \in B$, $x = \Sigma x_n e_n = \Sigma t_n (\sigma_n t_n)^{-1} x_n \sigma_n e_n$ and $\Sigma |t_n(\sigma_n t_n)^{-1} x_n| \leqq \Sigma t_n \leqq 1$ i.e. $B \subset \overline{\Gamma(\sigma_n e_n)}$.   By the Dynin-Mitiagin theorem $\Sigma \sigma_n e_n$ is absolutely convergent to σ.   Let $S = \overline{\mathrm{conv}}\,(2\sigma_n e_n)$.   By Lemma 2, S and hence any translate of S, is a compact Choquet simplex.   Thus $B + \sigma \subset \overline{\Gamma(\sigma_n e_n)} + \sigma \subset S$, i.e..

$$B \subset S - \sigma \, .$$

4.   Proof of theorem 1.   We first suppose that E is a nuclear Fréchet space with a basis.   If $B \subset E$ is bounded then by the result

of Köthe above B is contained in the normal hull of some $\rho = (\rho_n) \in \lambda$, the sequence space representation of E. By Lemma 3, B is contained in a compact Choquet simplex.

Now suppose each bounded subset of E is contained in a bounded Choquet simplex. Let $\sum_{n=1}^{\infty} x_n$ be an unconditionally convergent series in E and let $A = \{\sum_{n \in \sigma} x_n : \sigma \text{ finite}\} \cup \{0\}$. It is well known that A is precompact, hence by hypothesis, contained in a bounded Choquet simplex B. Consider in $E \times R$ the cone C generated by $B \times \{1\}$. Then $F = C-C$ is a linear lattice with the order induced by C. The Minkowski functional of $\Gamma(B \times \{1\})$ is a norm on F. This norm and the order are related by the following two properties:

(i)   if $x \wedge y = 0$ then $\|x + y\| = \|x - y\|$; and
(ii) if $x \geqq 0, y \geqq 0$ then $\|x + y\| = \|x\| + \|y\|$.

It follows from the Kakutani theorem [16], [4, p. 98–100] that the completion of F is an abstract L-space (with order determined by $\bar{C}$).

Let P be the restriction to F of the natural projection from $E \times R$ onto E. Since B is bounded, P is cotinuous from F to E. Let $y_n = (x_n, 1)$, $n = 0, 1, 2, \cdots$ where $x_0 = 0$. Clearly $y_0 \in B \times \{1\}$ and $y_0 + \sum_{n \in \sigma} (y_n - y_0) \in B \times \{1\}$ for each finite set of positive integers σ. But $B \times \{1\}$ is part of the positive face of the unit ball in an abstract L-space. By Lemma 1 it follows that $\sum_{n=1}^{\infty} \|y_n - y_0\| < +\infty$. Since $Py_n = x_n$ it follows from the continuity of P that Σx_n is absolutely convergent, i.e. E has the Dvoretzky-Rogers property. By the result of Grothendieck quoted in the introduction, E is nuclear.

5. Cones and the existence of bases. In any separable Fréchet space E (in particular a nuclear Fréchet space) there is a topologically free null sequence (z_n) whose closed linear span is E, i.e. (z_n) is *fundamental*. (This result is due essentially to Markuchevitch [19]; see also Klee [11]). Thus $S = \overline{\text{conv}}\,(z_n)$ is, by Lemma 2, a Choquet simplex. It is not hard to see that $C = \{\alpha s : \alpha \geqq 0, s \in S\}$ is always a proper cone. There are important cases in which $\bar{C}$ is not only a cone but makes $\bar{C}-\bar{C}$ a vector lattice (see [24] for terminology). We will call a simplex S a *VL-simplex* if $\bar{C}$ is a cone making $\bar{C}-\bar{C}$ a vector lattice.

Let (x_n) be an unconditional basis for E with coefficient functionals (f_n) and let $K = \{x \in E : f_n(x) \geqq 0 \text{ for all } n\}$. Then K is called the *basis cone of (x_n).*

Ceitlin [2] has shown that K is normal (for any two nets (x_β), (y_β) in E with $0 \leqq x_\beta \leqq y_\beta$, if $\lim_\beta x_\beta = 0$ then $\lim x_\beta = 0$) and $K-K = E$. Ceitlin observes that in this situation the induced lattice operations are continuous. The work of Ceitlin above makes it easy to prove the following result.

PROPOSITION 1.   *If (x_n) is an unconditional basis in a Fréchet*

space E with $\lim_n x_n = 0$ then $S = \overline{\text{conv}}\,(x_n)$ is a VL-simplex.

Proof. By Lemma 2 and the above remarks we need only observe that $\bar{C} = K$ where $C = \{\alpha s : \alpha \geq 0,\ s \in S\}$ and K is the basis one of (x_n). The inclusion $\bar{C} \subset K$ is obvious and since any finite sum $\sum_{i=1}^n a_i x_i$, $a_i \geq 0$ is in C, $K \subset \bar{C}$.

Our next theorem shows the difficulties in trying to remove the basis hypothesis in Theorem 1.

THEOREM 2. *Let E be a nuclear Fréchet space. The following are equivalent*:

(a) *E has a Schauder basis*;

(b) *each bounded subset of E is contained in a translate of a bounded VL-simplex*; *and*

(c) *there is a fundamental bounded set containing 0 and contained in a translate of a bounded VL-simplex.*

Proof. (a)→(b): Let (e_n) be a basis for E, B a bounded subset of E and let $\rho = (\rho_n)$, $\sigma = (\sigma_n)$ have the meaning of Lemma 3.

Then by Lemma 3, $B \subset -\rho + S$ where $S = \overline{\text{conv}}\,(2\sigma_n e_n)$. Since we may assume that each $\sigma_n \neq 0$ $(2\sigma_n e_n)$ is, by the Dynin-Mitiagin theorem, an unconditional basis for E. Also $(2\sigma_n e_n)$ is a null sequence. By Proposition 1, S is a VL-simplex.

(b)→(c) is obvious from [19] since E is separable.

(c)→(a). Let B be the fundamental bounded set of (c). By hypothesis there is a $\rho \in E$ and a VL-simplex S such that $B - \rho \subset S$. If C is the cone generated by S then, since $0 \in B$

$$B \subset B - B = (B - \rho) - (B - \rho) \subset \bar{C} - \bar{C} \quad \text{i.e.}$$

$\bar{C} - \bar{C}$ is dense in E. Since E is nuclear and S is a VL-simplex $\bar{C} - \bar{C}$ is a nuclear vector lattice and so [13] the lattice operations can be extended to E in such a way that E is a nuclear vector lattice. However, from the generalized Grinblyum K-condition [20] and the results of Kōmura-Koshi [13] it follows that every Fréchet nuclear vector lattice has a basis. (See [1] for a complete discussion of the Kōmura-Koshi result and its relation to basis theory.)

6. **The strong Banach-Dieudonné property.** Let us recall the Banach-Dieudonné theorem for Fréchet spaces: *every precompact subset of E is contained in the closed convex circled hull of a null sequence* (see e.g. [15, p. 273]).

The following proposition allows one to strengthen this result.

PROPOSITION 2. *Let (x_n) be a null sequence in a locally convex space F. Then there is either a linearly independent null sequence or a finitely nonzero sequence (z_n) in F such that $\overline{\mathrm{conv}}\,(x_n) \subset \overline{\mathrm{conv}}\,(z_n)$.*

Proof. If there is no infinite dimensional bounded set in F then $\overline{\mathrm{conv}}\,(x_n)$ is bounded and finite dimensional and the assertion is obvious. Thus suppose (y_n) is a bounded linearly independent sequence in F. Construct an infinite array in the following manner: If $x_1, \cdots, x_i$ are linearly independent and $x_{i+1} = \sum_{j=1}^{i} \lambda_j x_j$ put $x_j, 1 \leq j \leq i$ in the first place of the j-th row. Choose n_1 such that $n_1^{-1} y_{n_1} + x_{i+1}$ is linearly independent from the elements in the array and put this new element in the first place of the $i + 1$-row. If x_{i+2} is linearly independent from the elements of the array put it in the first place of the $i + 2$-row. If not, put in this place $n_2^{-1} y_{n_2} + x_{i+2}$ where $n_2 > n_1$ and $n_2^{-1} y_{n_2} + x_{i+2}$ is linearly independent from the elements of the array. Next choose $n_3 > n_2$ such that $n_2^{-1} y_{n_3} + x_{i+1}$ is linearly independent from the elements of the array and put it in the second place of the $i + 1$-row. The first element of the $i + 3$-row will be x_{i+3} or of the form $n_4^{-1} y_{n_4} + x_{i+3}$ (where $n_4 > n_3$) depending on whether x_{i+3} is linearly independent or not from the elements of the array. Next put $n_5^{-1} y_{n_5} + x_{i+1}, n_6^{-1} y_{n_6} + x_{i+2}$ with $n_6 > n_5 > n_4$ in the third place of the $i + 1$-row and the second place of the $i + 2$-row (if the row does not begin with x_{i+2}), where these elements are chosen to be linearly independent from the array. Continue the process and enumerate the array diagonally to obtain the sequence (z_n). Then (z_n) is a null sequence and $(x_n) \subset \overline{\mathrm{conv}}\,(z_n)$.

In particular if E is metrizable, Proposition 2 and the Banach-Dieudonné theorem show that *every precompact subset of E is contained in the closed convex circled hull of a linearly independent null sequence.*

This motivates the following definition. We say that a locally convex space F has the *strong Banach-Dieudonné property* if each bounded subset of F is contained in the closed convex circled hull of a topologically free null sequence.

REMARK 1. If F is barrelled and has the strong Banach-Dieudonné property then F is a Montel [15] space. Indeed, in this case, bounded sets are necessarily precompact. In particular, if a Banach space E has the strong Banach-Dieudonné property then E is finite dimensional.

If one replaces "null sequence" in the above definition with "absolutely convergent sequence" then the proof of Lemma 2 and [26] show that a space F with this property must be nuclear.

We do not know if a Fréchet space E with the strong Banach-

Dieudonné property is necessarily nuclear. Lemma 3 shows that a nuclear Fréchet space with a Schauder basis (e_n) has the strong Banach-Dieudonné property. Indeed if

$$B = \{x \in \lambda : |x_n| \leq \rho_n\}$$

then by the proof of Lemma 3

$$A = \{y \in \lambda : 0 \leq y_n \leq 2\rho_n\} \subset \overline{\Gamma(\sigma_n e_n)} \ .$$

Also,

$$-A \subset \overline{\Gamma(\sigma_n e_n)} \quad \text{and} \quad B \subset \overline{\text{conv}} \, (A \cup -A)$$

since $\Sigma x_n e_n$ is we conclude with two examples.

EXAMPLE 1. *A nuclear space F which lacks the strong Banach-Dieudonné property.*

Let F be l_1 with the $\sigma(l_1, l_\infty)$-topology. Since F has a neighborhood basis $\{U\}$ such that for each $U \in \{U\}$, F_U is finite dimensional, F is nuclear. By Shur's lemma sequential convergence in F is norm convergence in l_1. If B is the unit ball of l_1 there is no null sequence (y_n) in F with $B \subset \Gamma(y_n)$.

EXAMPLE 2. *A locally convex space F with the strong Banach-Dieudonné property which is not nuclear.*

Let F be l_1 with the bounded weak*-topology (i.e. the topology of uniform convergence on null sequences in c_0). Since a functional f is bw^*-continuous if and only if it is w^*-continuous [6], a bounded subset of F is w^*-bounded, hence norm bounded. Thus to show that F has the strong Banach-Dieudonné property it suffices to show that the unit ball of l_1 is contained in the closed convex circled hull of a topologically free F-null sequence. Let (e_n) be the unit vector basis of l_1. Clearly $B \subset \overline{\Gamma(e_n)}$ and (e_n) is topologically free. Also w^*-$\lim e_n = 0$ and so by [4, p. 42] (e_n) is F-null.

To see that F is not nuclear it suffices [26] to produce a series Σx_n in F which is unconditionally but not absolutely, convergent.

By the proof of the Dvoretzky-Rogers theorem [7], [4] there is a sequence (x_n) in l_1 such that Σx_n is unconditionally convergent (in norm) and $\|x_n\| = 1/n$. For each n let $y_n \in c_0$, $\|y_n\| = 1$ and $x_n(y_n) > 1/2n$. Let y be the null sequence $((1/ln \, m)y_m)$. Then

$$\rho_y(x_n) = \sup_m \left| \frac{1}{ln \, m} x_n(y_m) \right| \geq \frac{1}{2n \, ln \, n} \ .$$

Thus $\Sigma \rho_y(x_n)$ diverges. But ρ_y is a continuous seminorm on F and so Σx_n is not absolutely convergent in F.

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