PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 2, 1971

UNIVERSAL COEFFICIENT THEOREMS FOR GENERALIZED HOMOLOGY AND STABLE COHOMOTOPY

PAUL C. KAINEN

We show that if h is a nice (e.g. representable) homology functor and G is an Abelian group, then there is a cohomology functor $k(X; G)$ which is a "quasi-functor" of G and a short exact sequence

$$0 \longrightarrow \operatorname{Ext}(h(\Sigma X), G) \longrightarrow k(X; G) \longrightarrow \operatorname{Hom}(h(X), G) \longrightarrow 0$$

which is natural in X, "strongly quasi-natural" in G, and split if two additional conditions are satisfied.

If, for example, $h(X) = H_n(X)$, then $k(X; G) = H^n(X; G)$, and we obtain a proof of the ordinary Universal Coefficient Theorem which does not descend to the chain level but which does make heavy use of Brown's Representability Theorem [2]. After setting up the machinery and proving some technical results in § 1, we derive in § 2 quasi-naturality and, with suitable restrictions, splitting of the sequence.

The construction of $k(X; G)$ involves an injective resolution of G. We show (2.8) that $k(X; G)$ is independent (up to *non*-canonical isomorphism) of the resolution chosen and we remark (in 2.12) that there is a particular injective resolution $\Gamma(G)$ which is even functorial.

In § 3 we prove a corresponding Universal Coefficient Theorem for stable cohomotopy. We construct (3.8) the following short exact sequence for finitely generated G and finite dimensional X

$$0 \longrightarrow \operatorname{Ext}_Z(G, \pi_S^{n-1}X) \longrightarrow \{X, L(G, n)\} \longrightarrow \operatorname{Hom}_Z(G, \pi_S^n X) \longrightarrow 0$$

which is natural in X, strongly quasi-natural in G, and split if $\{X, L(G, n)\}$ is a functor of G. $L(G, n)$ denotes the co-Moore space of type (G, n), $\{X, Y\}$ = stable homotopy classes of maps, and $\pi_S^q(X) = \{X, S^q\}$. In § 4 we present some examples and a conjecture.

Let us recall from [5] the definition of a quasi-functor. Suppose $\mathscr{A}$ and $\mathscr{B}$ are categories and $S: |\mathscr{B}| \to |\mathscr{A}|$ is a function from the objects of $\mathscr{B}$ to the objects of $\mathscr{A}$. We call S a *quasi-functor* if given any morphism $\beta: B \to B'$ in $\mathscr{B}$ there is a nonempty set $S(\beta)$ of morphisms in $\mathscr{A}$ satisfying

(a) $S(\beta) \subset \mathscr{A}(SB, SB')$;

(b) $\beta: B \to B'$ and $\beta': B' \to B''$ imply

$$S(\beta'\beta) \supset \{\alpha'\alpha \mid \alpha' \in S(\beta'), \alpha \in S(\beta)\};$$

(c) $1_{SB} \in S(1_B)$.

Now if S, U: $\mathscr{B} \to \mathscr{A}$ are quasi-functors, we say that ν is a strong quasi-natural transformation from S to U provided that ν associates to each $B \in |\mathscr{B}|$ a morphism ν_B: $S(B) \to U(B)$ and if β: $B \to B'$ then the following diagram is commutative for all $s \in S(\beta)$ and all $u \in U(\beta)$

$$\begin{array}{ccc} S(B) & \xrightarrow{\nu_B} & U(B) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle u} \\ S(B') & \xrightarrow{\nu_{B'}} & U(B')\,. \end{array}$$

We call ν quasi-natural if for every $s \in S(\beta)$ there exists $u \in U(\beta)$ such that the above diagram commutes, and symmetrically, if for every u there exists s making the diagram commute. Note that if S is a quasi-functor which is not a functor and if ν: $S \to S$ is the identity, then ν is quasi-natural but not strongly quasi-natural.

Early versions of these results comprised a portion of the author's doctoral dissertation written at Cornell University under the direction of Professor Peter Hilton. I am grateful to Professor Hilton for pointing out a number of substantial improvements. I should also like to thank the referee for his very helpful suggestions.

One may view this paper as an alternative to Adams' approach (see [1]).

1. The machinery. Let us recall that a homology functor on the category $\mathscr{W}_*^\omega$ of based connected CW complexes is a covariant functor h: $\mathscr{W}_*^\omega \to Ab$, the category of abelian groups, satisfying the following two conditions:

(i) if $A \xrightarrow{f} X \xrightarrow{g} C$ is a cofiber sequence, then

$$h(A) \xrightarrow{h(f)} h(X) \xrightarrow{h(g)} h(C)$$

is exact;

(ii) the natural map

$$\coprod_{\alpha \in \Gamma} h(X_\alpha) \longrightarrow h(\bigvee_{\alpha \in \Gamma} X_\alpha)$$

is an isomorphism for any index set Γ, where $\coprod$ and $\bigvee$ denote coproducts in Ab and $\mathscr{W}_*^\omega$, respectively.

A contravariant functor k: $\mathscr{W}_*^\omega \to Ab$ is a cohomology functor provided that it satisfies the duals of (i) and (ii).

DEFINITION 1.1. We say that a homology functor is *special* provided that for every pair (X, A) of spaces in $|\mathscr{W}_*^\omega|$

$$\zeta: \lim_{\overrightarrow{n}} h(X^n \cup A) \longrightarrow h(X)$$

is a monomorphism, where X^n is the n-skeleton of X and ζ is induced by the inclusions $\iota_n\colon X^n \cup A \to X$. For example, h is special if it is representable in the sense of Whitehead [7]. We call a cohomology functor $k\colon \mathscr{W}^{\omega}_{*} \to Ab$ *special* if it satisfies the dual condition-that is, the natural map

$$\rho\colon k(X) \longrightarrow \varprojlim_{n} k(X^n \cup A)$$

is epic.

For the remainder of this section, let h be a fixed but arbitrary special homology functor on $\mathscr{W}^{\omega}_{*}$.

LEMMA 1.2. *Let I be an injective Abelian group. Then there is a based CW complex $\hat{B}(I)$ and a natural equivalence*

$$\hat{\eta}_I\colon [-, \hat{B}(I)] \longrightarrow \operatorname{Hom}(h(-), I) \tag{1.3}$$

of cohomology functors on $\mathscr{W}^{\omega}_{*}$, where $[-, -]$ denotes homotopy classes of maps.

Proof. Since $\operatorname{Hom}(-, I)$ is an exact functor, $\operatorname{Hom}(h(-), I)$ is a special cohomology functor on $\mathscr{W}^{\omega}_{*}$. Hence, by the Representability Theorem of E. H. Brown [2], the conclusion follows.

LEMMA 1.4. *$\hat{B}$ is a functor on injective Abelian groups.*

Proof. Let I and J be injective and let $\psi\colon I \to J$. Let $\hat{B}(\psi)\colon \hat{B}(I) \to \hat{B}(J)$ be the unique (up to homotopy) map which makes the diagram below commutative.

$$\begin{array}{ccc} [-, \hat{B}(I)] & \xrightarrow{\hat{\eta}_I} & \operatorname{Hom}(h(-), I) \\ \big\downarrow{\scriptstyle \hat{B}(\psi)_\sharp} & & \big\downarrow{\scriptstyle \psi_\sharp} \\ [-, \hat{B}(J)] & \xrightarrow{\hat{\eta}_J} & \operatorname{Hom}(h(-), J) \end{array} \tag{1.5}$$

where the vertical arrows are induced by $\hat{B}(\psi)$ and ψ, respectively. (The existence and uniqueness of a *map* $\hat{B}(\psi)$ inducing the natural transformation $\hat{\eta}_J^{-1}\psi_\sharp\hat{\eta}_I$ follows from the Yoneda Lemma of category theory.)

For brevity, we shall write $\hat{\psi}$ instead of $\hat{B}(\psi)$. Let $\Gamma\colon 0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ be a short exact sequence in which I and J are injective.

DEFINITION 1.6. We define $B(\Gamma)$ to be the mapping kernel of $\hat{\psi}$, so $B(\Gamma)$ fits into the following pull-back square

$$
(1.7)\qquad
\begin{array}{ccc}
B(\Gamma) & \longrightarrow & E\hat{B}(J) \\
{\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\
\hat{B}(I) & \xrightarrow{\hat{\psi}} & \hat{B}(J)
\end{array}
$$

where $E\hat{B}(J)$ is the (contractible) space of paths in $\hat{B}(J)$ starting at the base point, $p(\omega) = \omega(1)$, and the fibre of the fibration p is $\Omega\hat{B}(J)$. Note that $\hat{B}(I)$ and $\hat{B}(J)$ are homotopy associative and homotopy commutative H-spaces, and $\hat{\psi}$ is an H-map, so that $B(\Gamma)$ is also a homotopy associative and commutative H-space.

By Eckmann-Hilton duality, the map $\hat{\psi}$ fits into a co-Puppe sequence $P(\Gamma)$:

$$
(1.8)\qquad
\begin{array}{l}
\cdots \longrightarrow \Omega B(\Gamma) \xrightarrow{\Omega j} \Omega\hat{B}(I) \xrightarrow{\Omega\hat{\psi}} \Omega\hat{B}(J) \\
\longrightarrow B(\Gamma) \xrightarrow{j} \hat{B}(I) \xrightarrow{\hat{\psi}} \hat{B}(J) \,.
\end{array}
$$

LEMMA 1.9. *B and P are quasi-functors on injective resolutions Γ and morphisms of short exact sequences.*

Proof. Let $\Gamma: 0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ and $\Gamma': 0 \longrightarrow G' \xrightarrow{\varphi'} I' \xrightarrow{\psi'} J' \longrightarrow 0$ be injective resolutions, and let μ be a morphism from Γ to Γ'

$$
\mu = (e, f, g):\quad
\begin{array}{ccccccccc}
0 & \longrightarrow & G & \xrightarrow{\varphi} & I & \xrightarrow{\psi} & J & \longrightarrow & 0 \\
 & & {\scriptstyle e}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\
0 & \longrightarrow & G' & \xrightarrow{\varphi'} & I' & \xrightarrow{\psi'} & J' & \longrightarrow & 0 \,.
\end{array}
$$

Now we may choose a map $m: B(\Gamma) \to B(\Gamma')$ so that the diagram of homotopy classes of maps

$$
(1.10)\qquad
\begin{array}{ccccccccccc}
\cdots \longrightarrow & \Omega\hat{B}(I) & \longrightarrow & \Omega\hat{B}(J) & \longrightarrow & B(\Gamma) & \longrightarrow & \hat{B}(I) & \longrightarrow & \hat{B}(J) \\
 & \downarrow{\scriptstyle \Omega\hat{f}} & & \downarrow{\scriptstyle \Omega\hat{g}} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \hat{g}} \\
\cdots \longrightarrow & \Omega\hat{B}(I') & \longrightarrow & \Omega\hat{B}(J') & \longrightarrow & B(\Gamma') & \longrightarrow & \hat{B}(I') & \longrightarrow & \hat{B}(J')
\end{array}
$$

is commutative. Thus, m induces a morphism $\bar{m}$ from $P(\Gamma)$ to $P(\Gamma')$. However, the homotopy class of m is not uniquely determined. We now define $B(\mu)$ to be the set of all such homotopy classes m and $P(\mu)$ to be the set of all corresponding morphisms $\bar{m}$ from $P(\Gamma)$ to $P(\Gamma')$. B and P are quasi-functors because the composite of commutative diagrams is a commutative diagram.

DEFINITION 1.11. We define for any injective resolution Γ the cohomology functor $k(-;\Gamma) = [-, B(\Gamma)]$. By the preceding lemma, $k(-;\Gamma)$ is a quasi-functor of Γ.

2. The sequence.

Now we are ready to state and prove our main result.

THEOREM 2.1. *Let h be any special homology functor, let $X \in |\mathscr{W}_*^\omega|$, and let $\Gamma: 0 \longrightarrow G \xrightarrow{\varphi} I \xrightarrow{\psi} J \longrightarrow 0$ be an injective resolution. Then there is a short exact sequence*

$$\sigma(X;\Gamma): 0 \longrightarrow \operatorname{Ext}(h(\Sigma X)G) \longrightarrow k(X;\Gamma) \longrightarrow \operatorname{Hom}(h(X), G) \longrightarrow 0$$

in which the arrows are natural in X and strongly quasi-natural in Γ.

REMARK 2.2. A word is necessary here to describe the second and fourth terms of $\sigma(X;\Gamma)$ as functors of Γ. If Γ is an injective resolution of G, Γ' is an injective resolution of G', and $\mu = (e, f, g)$: $\Gamma \rightarrow \Gamma'$, then the corresponding morphisms from $\operatorname{Ext}(h(\Sigma X), G)$ to $\operatorname{Ext}(h(\Sigma X), G')$ and from $\operatorname{Hom}(h(X), G)$ to $\operatorname{Hom}(h(X), G')$ are, respectively, $\operatorname{Ext}(1, e)$ and $\operatorname{Hom}(1, e)$.

Proof of 2.1. Applying the functor $[X, -]$ to 1.8 and using the adjointness of Ω and Σ, we obtain the exact sequence

$$(2.3)\qquad \begin{aligned} &[\Sigma X, \hat{B}(I)] \xrightarrow{\hat{\psi}_\sharp(\Sigma X)} [\Sigma X, \hat{B}(J)] \longrightarrow [X, B(\Gamma)] \\ &\qquad\longrightarrow [X, \hat{B}(I)] \xrightarrow{\hat{\psi}_\sharp(X)} [X, \hat{B}(J)] \end{aligned}$$

and so, by homological algebra, a short exact sequence

$$(2.4)\qquad 0 \longrightarrow \operatorname{cok}(\hat{\psi}_\sharp(\Sigma X)) \longrightarrow k(X;\Gamma) \longrightarrow \ker(\hat{\psi}_\sharp(X)) \longrightarrow 0$$

which is natural in X and strongly quasi-natural in Γ.

But by 1.5 there are isomorphisms

$$(2.5)\qquad \begin{aligned} s&: \operatorname{cok}(\hat{\psi}_\sharp(\Sigma X)) \cong \operatorname{cok}(\psi_\sharp(h(\Sigma X))) , \\ t&: \ker(\hat{\psi}_\sharp(X)) \cong \ker(\psi_\sharp(h(X))) ; \end{aligned}$$

and these isomorphisms are natural in X and Γ. (Note that the above groups are *functor* of Γ.) Moreover, there are also isomorphisms, well-known from homological algebra,

$$(2.6)\qquad \begin{aligned} u&: \operatorname{cok}(\psi_\sharp(h(\Sigma X))) \cong \operatorname{Ext}(h(\Sigma X), G) , \\ v&: \ker(\psi_\sharp(h(X))) \cong \operatorname{Hom}(h(X), G) , \end{aligned}$$

which are natural in X and Γ. There isomorphisms simply express the independence of Hom and Ext of the resolution of G. Now the composite isomorphisms us and vt transform 2.4 into $\sigma(X; G)$ and preserve naturality in X and strong quasi-naturality in Γ.

The following lemma is well-known.

LEMMA 2.7. *Let $e: G \to G'$ be any homomorphism and let Γ and Γ' be injective resolutions of G and G', respectively. Then e extends* (non-uniquely) *to a morphism $(e, f, g): \Gamma \to \Gamma'$ of resolutions.*

Now we can state a corollary to Theorem 2.1.

COROLLARY 2.8. *Let Γ and Γ' be two injective resolutions of the same group G, let h be a special homology theory, and let $X \in |\mathscr{W}_*^\omega|$. Then there is a* (non-unique) *isomorphism $\sigma(X; \Gamma) \cong \sigma(X; \Gamma')$.*

Proof. By 2.7, $1: G \to G$ extends to $(1, f, g): \Gamma \to \Gamma'$ which yields a morphism $M: \sigma(X; \Gamma) \to \sigma(X; \Gamma')$. Neither process is unique. But M induces the identity on the second and fourth terms, and therefore M must be an isomorphism by the 5-lemma.

Select for every Abelian group G an injective resolution $\Gamma(G)$ and define $\sigma(X; G) = \sigma(X; \Gamma(G))$. By 2.7, $\Gamma(G)$ is a quasi-functor of G and so $\sigma(X; G)$ is strongly quasi-natural in G. By 2.8, $\sigma(X; G)$ is independent, up to noncanonical isomorphism, of the resolution chosen. We shall fix, for definiteness, a particular $\Gamma(G)$ in 2.12.

Now we need a lemma.

LEMMA 2.9. *Let $G = G_1 \oplus G_2$ and let $\iota_j: G_j \to G$ denote the canonical injection $(j = 1, 2)$. Let $X \in |\mathscr{W}_*^\omega|$ be fixed but arbitrary. Choose $m_j \in k(X; \iota_j)$ so that by strong quasi-naturality we have the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}\,(h(\Sigma X), G_j) & \longrightarrow & k(X; G_j) & \longrightarrow & \mathrm{Hom}\,(h(X), G_j) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Ext}\,(1, \iota_j)} & & \downarrow{\scriptstyle m_j} & & \downarrow{\scriptstyle \mathrm{Hom}\,(1, \iota_j)} & & \\ 0 & \longrightarrow & \mathrm{Ext}\,(h(\Sigma X), G) & \longrightarrow & k(X; G) & \longrightarrow & \mathrm{Hom}\,(h(X), G) & \longrightarrow & 0\,. \end{array} \tag{2.10}$$

Then

$$m_1 \oplus m_2: k(X; G_1) \oplus k(X; G_2) \longrightarrow k(X; G)$$

is an isomorphism.

Proof. Ext and Hom are additive and, therefore, by the 5-lemma, $m_1 \oplus m_2$ is an isomorphism.

This lemma permits us to apply an elegant theorem of Hilton [3] to the sequence $\sigma(X; G)$.

THEOREM 2.11. (Universal Coefficient Theorem). *Let h be any special homology theory, let $X \in |\mathscr{W}_*^\omega|$, and let G be an Abelian group.*

(a) *Then there is a representable cohomology functor $k(X; G)$ which is a quasi-functor of G and a short exact sequence*

$$\sigma(X; G)\colon 0 \longrightarrow \operatorname{Ext}(h(\Sigma X), G) \xrightarrow{\tau_{XG}} k(X; G) \xrightarrow{\eta_{XG}} \operatorname{Hom}(h(X), G) \longrightarrow 0$$

in which τ_{XG} and η_{XG} are natural in X and strongly quasi-natural in G.

(b) *Moreover, if for some fixed $X \in |\mathscr{W}_*^\omega|$ we have*

(i) *$k(X; G)$ is a* functor *of G and*

(ii) Hom $(h(X, G)$ *is a direct sum of cyclic groups, then $\sigma(X; G)$ splits for that X and every G.*

Proof. Part (a) is simply 2.1 with $\Gamma = \Gamma(G)$. Part (b) follows from [3] since Hom is a left-exact functor and, by (i) and 2.9, $k(X; G)$ is an additive functor of G so that $\sigma(X; G)$ is pure. Condition (ii) yields splitting.

2.12 Construction of $\Gamma(G)$

The following construction of $\Gamma(G)$ was related to me by Peter Hilton. Let G be any Abelian group. Then G has a canonical free resolution $0 \longrightarrow RG \xrightarrow{\lambda} FG \xrightarrow{\rho} G \longrightarrow 0$, where FG = free Abelian group on underlying set of G and RG = kernel $(FG \to G)$. Let $QG = \coprod_{g \in G} Q_g$ where $Q_g = Q$, the rationals, for every $g \in G$, and define π: $FG \to QG$ by $\pi(\hat{g}) = 1 \in Q_g$ where $\hat{g}$ is the generator of FG corresponding to g. Then setting $\pi\lambda = \bar{\lambda}\colon RG \to QG$, we have the following commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & RG & \xrightarrow{\lambda} & FG & \xrightarrow{\rho} & G & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \pi} & & \vdots{\scriptstyle \varphi'_G} & & \\
0 & \longrightarrow & RG & \xrightarrow{\bar{\lambda}} & QG & \longrightarrow & I'G & \longrightarrow & 0\\
 & & \downarrow & & & & \downarrow{\scriptstyle \psi_G} & & \\
 & & 0 & & & & J'G & & \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array} \tag{2.13}$$

where $I'G = \operatorname{cok}(\bar{\lambda})$, φ'_G is induced by $(1, \pi)$, and $J'G = \operatorname{cok}(\varphi'_G)$ with $\psi'_G\colon I'G \to J'G$ the canonical map. Put $\Gamma(G)$ = right-hand column in

2.13. Then $\Gamma(G)$ is an injective resolution of G since injective Abelian groups are closed under coproducts and quotients. Moreover, $\Gamma(G)$ is even functorial in G.

REMARK 2.14. The epimorphism η_{XG} of 2.11(a) can be interpreted as providing a weak adjunction from h to $B(-)$, where $B(G)$ is the space which represents $k(-;G)$. Thus, $B(-): Ab \to \mathscr{W}_*^\omega$ is a weak right adjoint (in the sense of [5]) to $h: \mathscr{W}_*^\omega \to Ab$, just as $K(-, n)$: $Ab \to \mathscr{W}_*^\omega$, which associates to a group G the Eilenberg-MacLane space $K(G, n)$, is a weak right adjoint to $H_n: \mathscr{W}_*^\omega \to Ab$, the ordinary homology functor.

REMARK 2.15. The results of this section hold for theories as well as functors. Moreover, they can also be modified to hold for other categories than $\mathscr{W}_*^\omega$. Finally, there is nothing special about using Ab as a target; we could just as well do everything for R-module-valued homology and cohomology functors where R is a (commutative) ring of cohomological dimension 1.

3. The universal coefficient theorem for stable cohomotopy. Let G be a finitely generated Abelian group. Then there is a standard projective resolution $\rho(G)$ of G

$$0 \longrightarrow RG \xrightarrow{\sigma_G} FG \xrightarrow{\tau_G} G \longrightarrow 0 \tag{3.1}$$

where FG is the free Abelian group on a set SG of generators of G, τ_G is the canonical projection, RG is the kernel of τ_G, and σ_G is the canonical injection of RG into FG. As in Lemma 2.7 $\rho(G)$ is a quasi-functor of G. Define

$$\tilde{F}_n G = \bigvee_{t \in SG} S^n_{(t)},\ S^n_{(t)} = S^n,\ t \in SG,\ n \geqq 0\ , \tag{3.2}$$

and, similarly, define

$$\tilde{R}_n G = \bigvee_{q \in \Gamma} S^n_{(q)},\ S^n_{(q)} = S^n,\ n \geqq 0,\ q \in \Gamma = \text{set of generators of } RG\ . \tag{3.3}$$

LEMMA 3.4. *Let $n \geqq 1$. Then there exists a map $\tilde{\sigma}^n_G: \tilde{F}_n G \to \tilde{R}_n G$ (unique up to homotopy) which induces σ_G upon applying $H^n(-;Z)$.*

Proof. If $\varphi: Z \to Z$, then φ is just multiplication by some integer m ($m = 0$ is not excluded), and we write $\varphi = m$. Then any map f of degree m from S^n to S^n induces φ in nth cohomology, and we can write $\tilde{\varphi}^n = m$.

Thus, by stable additivity, $[\tilde{F}_n G, \tilde{R}_n G]$ is in one-to-one correspondence with integer matrices (m_{tq}), and the set Hom (RG, FG) of homomorphisms is in one-to-one correspondence with integer matrices (m_{qt}). Moreover, (m_{qt}) is induced by its transpose (m_{tq}) so we let

$$\tilde{\sigma}_G^n = (m_{tq}) \ , \tag{3.5}$$

where (m_{qt}) is the matrix corresponding to σ_G.

Since $\sum \tilde{F}_n G = \tilde{F}_{n+1} G$, $\sum \tilde{R}_n G = \tilde{R}_{n+1} G$, and $\sum \tilde{\sigma}_G^n = \tilde{\sigma}_G^{n+1}$, we have the following Puppe sequence $\tilde{\rho}_G^n$ for $\tilde{\sigma}_G^n$, $n \geqq 1$

$$\tilde{F}_n G \xrightarrow{\tilde{\sigma}_G^n} \tilde{R}_n G \longrightarrow L(G, n+1) \longrightarrow \tilde{F}_{n+1} G \xrightarrow{\tilde{\sigma}_G^{n+1}} \tilde{R}_{n+1} G \tag{3.6}$$

where $L(G, n+1) =$ (reduced) mapping cone of $\tilde{\sigma}_G^n$. Thus, $L(G, n+1)$ is just the co-Moore space of type $(G, n+1)$; i.e. $H^q(L(G, n+1); Z) = 0$ $q \neq n+1$, $H^{n+1}(L(G, n+1); Z) = G$, and $\pi_1(L(G, n+1)) = 0$ by Van Kampen when $n \geqq 2$. Since $\rho(G)$ is a quasi-functor of G, so is $\tilde{\rho}^n(G)$ and, hence, $L(G, n+1)$.

Let $\mathscr{W}_*^\infty$ denote the category of based connected finite-dimensional CW complexes. If $X \in | \mathscr{W}_*^\infty |$ and $Y \in | \mathscr{W}_*^\omega |$, then we define

$$\{X, Y\} = \varinjlim_k [\Sigma^k X, \Sigma^k Y] \ ,$$

and we recall that $\{X, -\}$ is a special homology functor on $\mathscr{W}_*^\omega$.

Therefore, applying $\{X, -\}$ to 3.6, we obtain an exact sequence

$$\begin{aligned} \{X, \tilde{F}_n G\} &\xrightarrow{\tilde{\sigma}_{G*}^n} \{X, \tilde{R}_n G\} \longrightarrow \{X, L(G, n+1)\} \\ &\longrightarrow \{X, \tilde{F}_{n+1} G\} \xrightarrow{\tilde{\sigma}_{G*}^{n+1}} \{X, \tilde{R}_{n+1} G\} \ . \end{aligned} \tag{3.7}$$

But clearly $\{X, \tilde{F}_n G\} \cong \mathrm{Hom}\,(FG, \pi_S^n(X))$ by an isomorphism which is natural in X and also natural in $G(\pi_S^n(X) = \{X, S^n\})$. Therefore, as in § 2 we obtain the following theorem.

THEOREM 3.8. *Let G be a finitely generated Abelian group. Let $n \geqq 2$ and let $X \in | \mathscr{W}_*^\infty |$. Then there is a short exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Ext}\,(G, \pi_S^{n-1}(X)) &\longrightarrow \{X, L(G, n)\} \\ &\longrightarrow \mathrm{Hom}\,(G, \pi_S^n(X)) \longrightarrow 0 \end{aligned} \tag{3.9}$$

which is natural in X and strongly quasi-natural in G. The sequence splits if, for some fixed X, $\{X, L(G, n)\}$ is a functor of G.

As a corollary of this theorem, we have the following result of Hilton-Olum-see [4].

COROLLARY 3.10. *Let G_1 and G_2 be finitely generated Abelian* *groups and* $n \geqq 4$. *Then there is a short exact sequence*

$$(3.11)\qquad \begin{aligned} 0 \longrightarrow T(G_1)^* \otimes G_2 \otimes Z_2 \longrightarrow [L(G_2, n), L(G_1, n)] \\ \longrightarrow \mathrm{Hom}\,(G_1, G_2) \longrightarrow 0 \end{aligned}$$

which is strongly quasi-natural in G_1 *and* G_2, *where* $T(G) =$ *torsion subgroup of* G *and* $G^* = \mathrm{Hom}\,(G, Q/Z)(\cong G$ if G is finite).

Proof. Applying 3.9 to $G = G_1$ and $X = L(G_2, n)$, we get

$$(3.12)\qquad \begin{aligned} 0 \longrightarrow \mathrm{Ext}\,(G_1, \pi_S^{n-1}(L(G_2, n)) \longrightarrow \{L(G_2, n), L(G_1, n)\} \\ \longrightarrow \mathrm{Hom}(G_1, \pi_S^n(L(G_2, n)) \longrightarrow 0\,. \end{aligned}$$

But for $n \geqq 4$

$$\pi_S^{n-1}(L(G_2, n)) \cong G_2 \otimes Z_2$$

$$\{L(G_2, n), L(G_1, n)\} \cong [L(G_2, n), L(G_1, n)]\,,$$

and

$$\pi_S^n(L(G_2, n)) \cong G_2\,, \quad \text{so we have for } n \geqq 4$$

$$(3.13)\qquad \begin{aligned} 0 \longrightarrow \mathrm{Ext}\,(G_1, G_2 \otimes Z_2) \longrightarrow [L(G_2, n), L(G_1, n)] \\ \longrightarrow \mathrm{Hom}\,(G_1, G_2) \longrightarrow 0\,. \end{aligned}$$

Now we are done since $\mathrm{Ext}\,(G_1, -) \cong T(G_1)^* \otimes -$ as functors on the category of finitely generated Abelian groups.

4. Some examples and a conjecture. The general problem of computing $k^*(X; G)$, for a given homology theory h_* and group G, is very difficult, even when the group is injective. For example, if $h_q = \pi_q^S = \{S^q, -\}$ and $G = Q$, then

$$(4.1)\qquad k^q(X; Q) \cong H^q(X; Q)$$

by an easy argument based on Serre's result [6] that $\pi_q^S(S^r)$ is finite for $r \neq q$. With h_* as above and $G = Q/Z$ it is easy to establish

$$(4.2)\qquad k^q(S^r; Q/Z) \cong \begin{cases} \pi_q^S(S^r), & r \neq q \\ Q/Z, & r = q\,. \end{cases}$$

Thus computing $k^*(X; Q/Z)$ in this case amounts to knowing the stable homotopy groups of spheres!

If the homology theory h_* is represented by a spectrum B, then the spectrum $B(G)$ which represents $k^*(-; G)$ can be thought of as obtained from B by introducing G coefficients. The spectrum B also represents a cohomology theory, and we have the following

CONJECTURE 4.3. If $\pi_* B$ is a ring of cohomological dimension 1, then there is a homotopy equivalence of spectra $B \simeq B(Z)$.

This conjecture simply says that our method and Adams' [1] coincide over rings of cohomological dimension 1-where his spectral sequence collapses to a Universal Coefficient Sequence.

REMARK 4.4. It is *not* true in general that $k^*(-; Z)$ is the cohomology theory associated to the spectrum B which represents h_*. For example, if, as above, $B =$ sphere spectrum and $h_* =$ stable homotopy is the homology theory represented by B, then

$$(4.5) \qquad k^n(S^q; Z) = 0 \qquad \text{for all } q > n .$$

But the cohomology functor associated to the sphere spectrum is stable cohomotopy, and certainly

$$(4.6) \qquad \pi_S^n(S^q) \neq 0 \qquad \text{for all } q > n .$$

In particular, $k^n(S^{n+1}; Z) = 0 \not\cong Z_2 = \pi_S^n(S^{n+1})$.

REFERENCES

1. J. F. Adams, Lectures on Generalized Cohomology, in *Category Theory, Homology Theory and Their Applications III*, Springer Lecture Notes 99, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

2. E. H. Brown, *Cohomology Theories*, Ann. Math., **75** (1962), 467-484.

3. A. Deleanu and P. J. Hilton, *On the Splitting of Universal Coefficient Sequences*, to appear in Proc. Symp. Alg. Top., Aarhus, 1970.

4. P. J. Hilton, *Putting Coefficients into a Cohomology Theory*, Proc. Kon. Ned. Akad. Wer., **73** (3) (1970), 196-216.

5. P. C. Kainen, *Weak Adjoint Functors*, to appear in Math. Zeit.

6. J. P. Serre, *Homologie Singulière des Espaces Fibrés*, Ann. Math., **54** (1951), 425-505.

7. G. W. Whitehead, *Generalized Homology Theories*, Trans. Amer. Math. Soc., **102** (1962), 227-283.

Received October 23, 1970 and in revised form February 24, 1971.

CASE WESTERN RESERVE UNIVERSITY