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ON k-SHRINKING AND k-BOUNDEDLY COMPLETE BASIC SEQUENCES AND QUASI-REFLEXIVE SPACES

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A Banach space X is called quasi-reflexive (of order n) if $\mathrm{codim}_{X^{**}}\pi(X) < +\infty$ ($\mathrm{codim}_{X^{**}}\pi(X) = n$), where π denotes the canonical embedding of X into its second conjugate X^{**}. R. Herman and R. Whitley have shown that every quasi-reflexive space contains an infinite dimensional reflexive subspace. In this paper this result is extended by showing that if X is quasi-reflexive of order n and $0 \leqq k \leqq n$ then X contains a subspace which is quasi-reflexive of order k.

1. Preliminaries. Throughout this paper X will denote a Banach space, X^* its first conjugate and X^{**} its second conjugate.

The sequence $\{x_i\}$ in X is said to be *basic* if $\{x_i\}$ is a basis for $[x_i]$ (where $[x_i]$ denotes the closed span of $\{x_i\}$). The sequence of functionals $\{f_i\}$ in $[x_i]^*$ defined by $f_i(x_j) = \delta_{ij}$ (where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$) are called the functionals biorthogonal to $\{x_i\}$. We will write $\{x_i, f_i\}$ is a basic sequence. It is well known [10] that the sequence $\{x_i\}$ in X, $x_i \neq 0$ $(i = 1, 2, \cdots)$, is basic if and only if there exists $K > 0$ such that

$$(1) \qquad \left\|\sum_{i=1}^{n} a_i x_i\right\| \leqq K \left\|\sum_{i=1}^{m} a_i x_i\right\|$$

for $1 \leqq n \leqq m < +\infty$ and any choice of scalars $a_1, a_2, \cdots, a_m$.

If $\{x_i\}$ is a basic sequence we call the sequence $\{z_n\}$, $z_n \neq 0$ $(n = 1, 2, \cdots)$, a block basic sequence [1] of $\{x_i\}$ if there exists a sequence of scalars (a_i) and $0 = p_1 < p_2 < \cdots$ such that $z_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i$. By (1), $\{z_n\}$ is a basic sequence.

If A and B are subspaces of X we will write $A \oplus B$ to denote the direct sum of A and B, when for each $x \in [A, B]$ (where $[A, B]$ denotes the closed span of $A \cup B$) there exists unique $\alpha \in A$, $\beta \in B$ such that $x = \alpha + \beta$. If $X = A \oplus B$ and dim $B = n$ (dim $B = +\infty$) we write $\mathrm{codim}_X A = n$ ($\mathrm{codim}_X A = +\infty$). We will also write $\mathrm{codim}_X A = +\infty$ if X has no subspace B such that $X = A \oplus B$.

LEMMA 1.1. *If $X = [A, B]$ where A and B are closed subspaces of X and if dim $B = n$ and $A \cap B = 0$ then codim$_X A = n$ and $X = A \oplus B$.*

I. Singer has shown [8]:

LEMMA 1.2. *Let A be a closed subspace of X.*

1° *The intersection of every $(n+1)$-dimentional subspace of X with A contains a nonzero element if and only if* $\operatorname{codim}_X A \leqq n$.

2° *There exists an n-dimensional subspace of X whose intersection with A contains only the zero element if and only if* $\operatorname{codim}_X A \geqq n$.

2. k-shrinking and k-boundedly complete basic sequences.

DEFINITION. A basic sequence $\{x_i, f_i\}$ is k-shrinking if $\operatorname{codim}_{[x_i]^*}[f_i] = k$ [8].

We note that a basic sequence is 0-shrinking if and only if it is shrinking [3].

LEMMA 2.1. *If $\{x_i, f_i\}$ is a basic sequence and $f \in [x_i]^*$, then $f \in [f_i]$ if and only if $\| f \mid [x_{n+1}, x_{n+2} \cdots] \| \to 0$ as $n \to \infty$* (where $f \mid [x_{n+1}, x_{n+2}, \cdots]$ denotes the functional f restricted to $[x_{n+1}, x_{n+2}, \cdots]$).

The proof is in [8].

LEMMA 2.2. *If $\{x_i, f_i\}$ is an n-shrinking basic sequence and $\{z_i\}$ is a block basic sequence of $\{x_i\}$ then $\{z_i\}$ is k-shrinking for some $k \leqq n$.*

Proof. Let $\{h_i\}$ be the functionals biorthogonal to $\{z_i\}$. Suppose $[z_i]^*$ contains an $(n+1)$-dimensional subspace, spanned by the linearly independent elements $g_1, g_2, \cdots g_{n+1}$, which intersects $[h_i]$ in only the zero element. Let $g'_i \in [x_i]^*$ be such that $g'_i \mid [z_i] = g_i$ $(i = 1, 2, \cdots n+1)$. Then by Lemma 2.1 the $(n+1)$-dimensional subspace of $[x_i]^*$ spanned by $\{g'_i : 1 \leqq i \leqq n+1\}$ intersects $[f_i]$ in only the zero element. This contradicts Lemma 1.2, 2°. Hence by Lemma 1.2, 1°, $\operatorname{codim}_{[z_i]^*}[h_i] \leqq n$. This completes the proof.

THEOREM 2.3. *If $\{x_i, f_i\}$ is an n-shrinking basic sequence and $0 \leqq k \leqq n$ then there is a k-shrinking block basic sequence of $\{x_i\}$.*

To prove this theorem we need two lemmas.

LEMMA 2.4. *If $\{x_i, f_i\}$ is a basic sequence and $\{g_i : 1 \leqq i \leqq n\}$ is a linearly independent set in $[x_i]^*$ such that $[g_i : 1 \leqq i \leqq n] \cap [f_i] = 0$ then there is a $\delta > 0$ such that*

$$\left\| g_j \mid [x_i]_{i=m}^{\infty} \cap \bigcap_{\substack{i=1 \\ i \neq j}}^{n} g_i^{-1}(0) \right\| > \delta \tag{2}$$

for $m = 1, 2, \cdots$ and $j = 1, 2, \cdots, n$.

Proof. Without loss of generality let $j = n$. Let

$$B_m = [f_1, \cdots, f_{m-1}, g_1, \cdots, g_{n-1}]^{\perp} .$$

From the isometry between $[x_i]^*/[g_1, g_2, \cdots, g_{n-1}, f_1, f_2, \cdots, f_{m-1}]$ and B_m^* [9] we have

$$\begin{aligned}||g_n | B_m|| &= \text{dist}\,(g_n, [g_1, g_2, \cdots, g_{n-1}, f_1, f_2, \cdots, f_{m-1}]) \\ &\geqq \text{dist}\,(g_n, [g_1, g_2, \cdots, g_{n-1}, f_1, f_2, \cdots]) > \delta > 0\end{aligned}$$

for $m = 1, 2, \cdots$ and for some δ since $g_n \notin [g_1, g_2, \cdots, g_{n-1}, f_1, f_2, \cdots]$.

LEMMA 2.5. *Let $\{x_i, f_i\}$ be a basic sequence and $||x_i|| > \delta > 0$ $(i = 1, 2, \cdots)$ for some δ. If $f \in [x_i]^*$ and $\sum_{i=1}^{\infty} |f(x_i)| < +\infty$ then $||f \mid [x_{n+1}, x_{n+2}, \cdots]|| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof. Let K satisfy (1) for the sequence $\{x_i\}$. Thus, since $|f_i(x)| < 2K\delta^{-1}$ where $||x|| \leqq 1$,

$$\begin{aligned}&\sup\left\{\left| f\left(\sum_{i=m+1}^{\infty} f_i(x)x_i\right|: x \in [x_{n+1}, x_{n+2}, \cdots], ||x|| \leqq 1\right\}\right. \\ &\leqq 2K\delta^{-1} \sum_{i=m+1}^{\infty} |f(x_i)| .\end{aligned}$$

Proof of theorem. Since the basic sequence $\{x_i, f_i\}$ is n-shrinking there exists a linearly independet set $\{g_i: 1 \leqq i \leqq n\} \subseteqq [x_i]^*$ such that

(3) $$[x_i]^* = [f_i] \oplus [g_i: 1 \leqq i \leqq n] .$$

By (2) in Lemma 2.4 we can construct a block basic sequence $\{y_i\}$ of $\{x_i\}$ with the following properties:

(4) $$\tfrac{1}{2} < ||y_i|| < \tfrac{3}{2}, i = 1, 2, \cdots ,$$

(5) $$|g_i(y_{nq+i})| > \delta > 0 \quad \text{for some } \delta, \text{ for } \quad i = 1, 2, \cdots, n$$

and $q = 1, 2, \cdots$, and

(6) $$|g_i(y_{nq+j})| < 1/2^q \text{ for } i \neq j .$$

Let $1 \leqq k \leqq n$ and let $\{z_i\}$ be a subsequence of $\{y_i\}$ consisting of the elements of the form y_{nq+i} where $i = 1, 2 \cdots k$ and $q = 1, 2, \cdots$. Let $\{h_i\}$ be the sequence of functionals biorthogonal to $\{z_i\}$. If $f \in [f_i]$ then, by Lemma 2.1, $f | [z_i] \in [h_i]$. Let $g'_j = g_j | [z_i]$ $(j = 1, 2, \cdots n)$. Since every functional in $[z_i]^*$ is the restriction of some functional in $[x_i]^*$ we conclude by (3) that

(7) $$[z_i]^* = [g'_1, g'_2, \cdots, g'_n, h_1, h_2, \cdots] .$$

From Lemmas 2.1 and 2.5 and (4), (6) above it follows that $g'_i \in [h_i]$,

$k+1 \leqq i \leqq n$. Assume there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_k$ and $h \in [h_i]$ such that $\sum_{i=1}^k \alpha_i g'_i = h$. Hence

$$\alpha_1 g'_1 = h - \sum_{i=2}^k \alpha_i g'_i .$$

But by (5) and (4), $\|g'_1 | [y_{np+1}: p \geqq m]\| > \frac{2}{3}\delta$ for $m = 1, 2, \cdots$. Also by (4), (6) and Lemma 2.1, $\|(h - \sum_{i=2}^k \alpha_i g'_i) | [y_{np+1}: p \geqq m]\| \to 0$ as $m \to \infty$. Therefore $\alpha_1 = 0$. Similarly $\alpha_i = 0$ for $i = 2, 3, \cdots, k$. Thus we have shown that the set $\{g'_i: 1 \leqq i \leqq k\}$ is linearly independent and $[g'_i: 1 \leqq i \leqq k] \cap [h_i] = 0$. Thus by (7) and Lemma 1.1 we have $\operatorname{codim}_{[z_i]^*}[h_i] = k$ and hence $\{z_i\}$ is k-shrinking.

The case $k = 0$ follows from [1, Thm. 3, p. 154] and the fact that a quasi-reflexive space contains an infinite dimensional reflexive subspace [5].

DEFINITION. Let $\{x_i\}$ be a basic sequence. We define two spaces of sequences $B(x_i)$ and $C(x_i)$ by

$$B(x_i) = \left\{(a_i x_i): \sup_n \left\| \sum_{i=1}^n a_i x_i \right\| < +\infty \right\}$$

and

$$C(x_i) = \left\{(a_i x_i): \sum_{i=1}^\infty a_i x_i \text{ exists} \right\} .$$

Define a norm on $B(x_i)$ and $C(x_i)$ by $|||(a_i x_i)||| = \sup_n \|\sum_{i=1}^n a_i x_i\|$. With this norm $B(x_i)$ and $C(x_i)$ are Banach spaces and $B(x_i) \supseteqq C(x_i)$. We say $\{x_i\}$ is k-boundedly complete if $\operatorname{codim}_{B(x_i)} C(x_i) = k$ [8].

We note that a basic sequence $\{x_i\}$ is 0-boundedly complete if and only if $\{x_i\}$ is boundedly complete [3].

LEMMA 2.6. *If $\{x_i\}$ is an n-boundedly complete basic sequence and $\{z_i\}$ is a block basic sequence of $\{x_i\}$ then $\{z_i\}$ is k-boundedly complete for some $k \leqq n$.*

Proof. Assume $B(z_i)$ has an $(n+1)$-dimensional subspace W which intersects $C(z_i)$ in only the zero element. But then $\phi(W)$ would be an $(n+1)$-dimensional subspace of $B(x_i)$ which intersects $C(x_i)$ in only the zero element, where ϕ denotes the natural embedding of $B(z_i)$ into $B(x_i)$ (i.e., $\phi(a_i z_i) = (b_i x_i)$ if for each n there is a $m \geqq n$ such that $\sum_{i=1}^n a_i z_i = \sum_{i=1}^m b_i x_i$). This contradicts Lemma 1.2.1°. By Lemma 1.2.1°, $\operatorname{codim}_{B(z_i)} C(z_i) \leqq n$.

THEOREM 2.7. *Let $\{x_i\}$ be an n-boundedly complete basic sequence for $n \geqq 1$. Then for $k \in \{0, 1\}$ there is a block basic sequence $\{z_i\}$ of $\{x_i\}$ which is k-boundedly complete.*

Proof. For the case $k = 1$, it is clearly sufficient to show that $\{x_i\}$ admits a m-boundedly complete block basic sequence for some m, $1 \leqq m < n$ whenever $n > 1$. Since $\{x_i\}$ is not 0-boundedly complete there is an element $(a_i x_i) \in B(x_i) - C(x_i)$. Hence there exists $0 = p_1 < p_2 < \cdots$ and $\delta > 0$ such that if

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i,\ \|y_n\| > \delta \text{ for } n = 1, 2, \cdots .$$

By Lemma 2.6 $\{y_i\}$ is m-boundedly complete for some $m \leqq n$.

Assume $m = n$. Then there exists

$$\{(b_{ki}y_i): 1 \leqq k \leqq n-1\} \subseteqq B(y_i) - C(y_i)$$

such that $B(y_i) = C(y_i) \oplus [(b_{ki}y_i): 1 \leqq k \leqq n-1] \oplus [(y_i)]$. By (1) there exists $M > 0$ such that $\|b_{ki}y_i\| < M$ and thus $|b_{ki}| \leqq M\delta^{-1}$ ($i = 1, 2, \cdots, 1 \leqq k \leqq n-1$). Hence there is an increasing sequence of positive integers (n_i) and $b_1, \cdots, b_{n-1}$ such that $\lim_{i\to\infty} b_{kn_i} = b_k$ and $|b_k - b_{kn_i}| < 1/2^i$ ($i = 1, 2, \cdots, 1 \leqq k \leqq n-1$). Let $c_{ki} = b_{ki} - b_i$ and $d_{ki} = c_{ki} - c'_{ki}$ where $c'_{kj} = c_{kj}$ for $j \in \{n_i\}$ and $c'_{kj} = 0$ for $j \notin \{n_i\}$. Then $(c'_{ki}y_i) \in C(y_i)$ and

$$B(y_i) = C(y_i) \oplus [(d_{ki}y_i): 1 \leqq k \leqq n-1] \oplus [(y_i)] \tag{8}$$

and $d_{kj} = 0$ for $j \in \{n_i\}$. Let $\{m_i\}$ be the sequence of positive integers complementary to $\{n_i\}$.

We will show that $\{y_{m_i}\}$ is $(n-1)$-boundedly complete. Let $(e_{m_i}y_{m_i}) \in B(y_{m_i})$. Therefore $(e_i y_i) \in B(y_i)$ where $e_j = 0$ if $j \notin \{m_i\}$. Thus by (8) there exist scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $(u_i y_i) \in C(y_i)$ such that

$$(e_i y_i) = (u_i y_i) + \sum_{k=1}^{n-1} \alpha_k (d_{ki} y_i) + \alpha_n (y_i) .$$

Thus we obtain $\alpha_n = 0$ and $u_j = 0$ for $j \notin m_i$. Hence

$$(e_{m_i} y_{m_i}) = (u_{m_i} y_{m_i}) + \sum_{k=1}^{n-1} \alpha_k (d_{km_i} y_{m_i}) .$$

Thus by Lemma 1.1, $\{y_{m_i}\}$ is $(n-1)$-boundedly complete.

The existence of a 0-boundedly complete block basic sequence again follows from [1, Thm. 3, p. 54] and [5].

LEMMA 2.8. *Let the basic sequence $\{x_i\}$ be 1-shrinking and 1-boundedly complete. Then there is a block basic sequence $\{z_i\}$ of $\{x_i\}$ which is either 1-shrinking and 0-boundedly complete or 0-shrinking and 1-boundedly complete.*

Proof. Let $\{y_i\}$ be the block basic sequence constructed as in Theorem 2.7. Then $\{y_i\}$ is 1-boundedly complete. If $\{y_i\}$ is 0-shrinking we are done. If not, then by Lemma 2.2, $\{y_i\}$ is 1-shrinking. Thus by Lemma 2.1 there exists $f \in [y_i]^*$ and $0 = p_1 < q_1 < p_2 < q_2 < \cdots$ such that

$$(9) \quad \| f \mid [y_i\colon p_n \leqq i \leqq q_n] \| > \delta > 0, \text{ for some } \delta \text{ and } n = 1, 2, \cdots .$$

As in the proof of Theorem 2.7, the subsequence $\{z_i\}$ of $\{y_i\}$, formed by those elements in $[y_i\colon p_n \leqq i \leqq q_n]$ $(n = 1, 2, \cdots)$ is 0-boundedly complete. But by (9) $\{z_i\}$ is 1-shrinking.

For other results on k-shrinking and k-boundedly complete basic sequences see [4].

3. Quasi-reflexive spaces. We will write Ord $(X) = n$ to mean X is quasi-reflexive of order n.

Civin and Yood have shown [2]:

THEOREM 3.1. *If* Ord $(X) = n$ *and* Y *is a closed subspace of* X *then* Y *and the quotient space* X/Y *are quasi-reflexive and* Ord $(X) =$ Ord $(Y) +$ Ord (X/Y)

I. Singer has shown [8]:

THEOREM 3.2. *If* $\{x_i\}$ *is a basic sequence then* Ord $([x_i]) = n$ *if and only if there exist natural numbers* k_1 *and* k_2 *such that* $\{x_i\}$ *is* k_1*-shrinking and* k_2*-boundedly complete and* $n = k_1 + k_2$.

THEOREM 3.3. *If* $\{x_i\}$ *is a basic sequence and* Ord $([x_i]) = n > 0$ *then there exist block basic sequences* $\{y_i\}$ *and* $\{z_i\}$ *of* $\{x_i\}$ *such that* Ord $([y_i]) = 1$ *and* Ord $([z_i]) = 0$.

Proof. The existence of $\{z_i\}$ such that Ord $([z_i]) = 0$ again follows from [1] and [5].

By Theorem 2.3 and Lemma 2.6 there exists a block basic sequence $\{y_i\}$ of $\{x_i\}$ which is 1-shrinking and k-boundedly complete for some $k \leqq n$. If $k = 0$ then Ord $([y_i]) = 1$ by Theorem 3.2. If $k > 0$ there exists, by Lemma 2.6, a block basic sequence $\{y'_i\}$ of $\{y_i\}$ which is 1-boundedly complete. If $\{y'_i\}$ is 0-shrinking we are done. If not then $\{y'_i\}$ is 1-shrinking and we now apply Lemma 2.8 to complete the proof.

THEOREM 3.4. *Let* Ord $(X) = n > 0$. *There exists separable subspaces* $Y_0, Y_1, \cdots, Y_n$ *of* X *such that* Ord $(Y_k) = k$ *and* $Y_k \subseteqq Y_{k+1}$ *for* $k = 0, 1, \cdots, n - 1$.

Proof. By [6, p. 546] a quasi-reflexive space of order n contains a basic sequence $\{x_i\}$ which is n-shrinking. Thus $\{x_i\}$ is 0-boundedly complete. Let $Y_n = [x_i]$. Thus by Theorem 2.2, there is a block basic sequence of $\{x_i\}$ which is $(n-1)$-shrinking and 0-boundedly complete. Hence there exists Y_{n-1} such that Ord $(Y_{n-1}) = n - 1$ and $Y_n \supseteqq Y_{n-1}$. We construct $Y_{n-2}, Y_{n-3}, \cdots, Y_0$ similarly.

We note that we have also shown that each Y_k has a basis.

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