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AN ESTIMATE FOR WIENER INTEGRALS CONNECTED WITH SQUARED ERROR IN A FOURIER SERIES APPROXIMATION

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If a function $x(\sigma)$, $0 \leqq \sigma \leqq t$, is in Lip-α, $0 < \alpha < 1$, $x(0) = 0$ and if c_k $(k = 0, 1, 2, \cdots)$ are its Fourier coefficients with respect to the functions $\sqrt{2/t}\sin[\pi(k+\frac{1}{2})\sigma/t]$, then it is known [1, pp. 171–172] that

$$\sum_{k \geqq n} c_k^2 \leqq \frac{A}{(n+\frac{1}{2})^{2\alpha}}\,, \qquad n \geqq 0 \tag{1}$$

where A is a positive number not depending on n. We will show a connection between this estimate and an estimate for Wiener integrals. Let $E_w\{\ \}$ denote expectation on a Wiener process, that is, a Gaussian process with mean function zero, covariance function $\min(\sigma, \tau)$, $0 \leqq \sigma, \tau \leqq t$ and sample functions $z(\sigma)$ with $z(0) = 0$.

THEOREM: Let $x(\sigma)$ be in $C[0, t]$ and let c_k be the Fourier coefficients of $x(\sigma)$ with respect to the normalized eigenfunctions associated with $\min(\sigma, \tau)$. That is

$$c_k = \sqrt{\frac{2}{t}}\int_0^t x(\sigma)\sin[\pi(k+\tfrac{1}{2})\sigma/t]d\sigma\,.$$

Let $0 < \alpha < 1$. Then estimate (1) is a necessary and sufficient condition for the estimate

$$e^{-(B/2)\nu^{1-\alpha}} \leqq \frac{E_W\left\{e^{-(\nu/2)\int_0^t [z(\sigma)-x(\sigma)]^2 d\sigma}\right\}}{E_W\left\{e^{-(\nu/2)\int_0^t z^2(\sigma)d\sigma}\right\}} \tag{2}$$

for all positive ν, where B is a positive number not depending on ν.

Proof. From Cameron and Donsker's proof of a lemma [2, p. 27–28], we have that, for the case $\rho_k = [\pi(k+\frac{1}{2})/t]^2$, the right side of (2) equals

$$e^{-\nu/2\sum_{k=0}^{\infty}\frac{c_k^2\rho_k}{\rho_k+\nu}}\,.$$

Hence estimate (2) holds if and only if

$$\sum_{k=0}^{\infty}\frac{c_k^2\rho_k}{\rho_k+\nu} \leqq \frac{B}{\nu^{\alpha}} \tag{3}$$

for all positive ν. To prove that (2) implies (1) note that for each fixed value of ν, as $k \to \infty$, $[\rho_k \mid (\rho_k + \nu)] \uparrow 1$. Therefore for each n, by the remark and (3),

$$\frac{\rho_n}{\rho_n + \nu} \sum_{k \geqq n} c_k^2 \leqq \sum_{k \geqq n} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leqq \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{\rho_k + \nu} \leqq \frac{B}{\nu^\alpha}$$

for all positive ν. Letting $\nu = \rho_n$ we have

$$\sum_{k \geqq n} c_k^2 \leqq \frac{2B}{[\pi(n + \frac{1}{2})/t]^{2\alpha}} = \frac{A}{(n + \frac{1}{2})^{2\alpha}}$$

which is estimate (1).

We now show that the latter estimate implies (3). Since the left side of (3) is bounded by $\sum_{k=0}^{\infty} c_k^2$, estimate (3) holds for $0 < \nu \leqq 1$. Hence it suffices to prove (3) for $\nu > 1$. To simplify notation set

$$S(n) = \sum_{k \geqq n} c_k^2 \leqq \frac{A}{(n + \frac{1}{2})^{2\alpha}} \tag{4}$$

by hypothesis. For any $n \geqq 1$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} &= \sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} \\ &\quad + \sum_{k=n+1}^{\infty} S(k) \left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right]. \end{aligned} \tag{5}$$

For the first two terms on the right side of (5) we have

$$\sum_{k=0}^{n-1} \frac{c_k^2 \rho_k}{(\rho_k + \nu)} + S(n) \frac{\rho_n}{\rho_n + \nu} \leqq \frac{2S(0)\rho_n}{\nu} < \frac{2S(0)\rho_n}{\nu^\alpha}. \tag{6}$$

To estimate the third term consider first

$$\left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] = \frac{\nu(\rho_k - \rho_{k-1})}{(\rho_k + \nu)(\rho_{k-1} + \nu)}. \tag{7}$$

Since $\rho_k - \rho_{k-1} = 2(\pi/t)^2 k$ and $(\rho_k + \nu)(\rho_{k-1} + \nu) \geqq [(\pi k/2t)^2 + \nu]^2$, the right side of (7) is dominated by $2(\pi/t)^2 \nu k [(\pi k/2t)^2 + \nu]^{-2}$. Applying (4) and the above, we have

$$\sum_{k=n+1}^{\infty} S(k) \left[\frac{\rho_k}{\rho_k + \nu} - \frac{\rho_{k-1}}{\rho_{k-1} + \nu} \right] \leqq 2(\pi/t)^2 A \nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2 + \nu]^2}. \tag{8}$$

To get the desired estimate we will use standard integral estimates. For $\alpha \geqq \frac{1}{2}$, the summands in the right side of (8) decrease monotonically with k for fixed ν. If $\alpha < \frac{1}{2}$, the function

$$g(\xi) = \xi^{1-2\alpha} \left[\left(\frac{\pi \xi}{2t} \right)^2 + \nu \right]^{-2}. \qquad \xi \geqq 0$$

has a unique local and absolute maximum at

$$\xi^* = \frac{2t}{\pi}\left(\frac{1-2\alpha}{3+2\alpha}\nu\right)^{1/2} .$$

In this case if $n \geqq \xi^*$, the summands in the right side of (8) decrease monotonically as k increases and

$$\begin{aligned} 2\left(\frac{\pi}{t}\right)^2 A\nu \sum_{k=n+1}^{\infty} \frac{k^{1-2\alpha}}{[(\pi k/2t)^2+\nu]^2} &\leqq \frac{2(\pi/t)^2 A}{\nu}\int_n^{\infty}\frac{\xi^{1-2\alpha}}{[(\pi\xi/2t\sqrt{\nu})^2+1]^2}d\xi \\ &= \frac{8(\pi/2t)^{2\alpha}A}{\nu^{\alpha}}\int_{\pi n/2t\sqrt{\nu}}^{\infty}\frac{\eta^{1-2\alpha}}{(\eta^2+1)^2}d\eta \\ &< 8\left(\frac{\pi}{2t}\right)^{2\alpha} A\int_0^{\infty}\frac{\eta^{1-2\alpha}}{(\eta^2+1)^2}d\eta\,\frac{1}{\nu^{\alpha}} . \end{aligned} \tag{9}$$

In the case $\alpha \geqq \frac{1}{2}$, (9) holds for any n and in both cases the last integral converges since $0 < \alpha < 1$. To complete the proof we fix in (5) $n = n^* \geqq \xi^*$ in the case $\alpha < \frac{1}{2}$ or $n = n^* \geqq 1$ if $\alpha \geqq \frac{1}{2}$. Estimates (6), (8), and (9) complete the proof.

REFERENCES

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