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# MUTUALLY APOSYNDETIC PRODUCTS OF CHAINABLE CONTINUA

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**In this paper it is proved that the Cartesian product of two compact metric chainable continua is mutually aposyndetic if and only if each of the two factors is an arc. Also some relationships are shown between indecomposability and a strong form of non-mutual aposyndesis.**

1. In [5], C. L. Hagopian developed the notion of mutual aposyndesis, a "Hausdorff" version of F. B. Jones' aposyndesis [6]. Mutual aposyndesis is stronger than aposyndesis but in general weaker than local connectedness. However, Theorem 1 of this paper shows that mutual aposyndesis and local connectedness are equivalent in a certain case.

Jones showed [7] that if a continuum is not aposyndetic at any point with respect to any other point, then it is indecomposable. A similar notion for mutual aposyndesis, called strict nonmutual aposyndesis by Hagopian, is closely related to indecomposability [5]. The author extends mutual aposyndesis to the notion of $n$-mutual aposyndesis and shows a relationship between strict non-$n$-mutual aposyndesis and $n$-indecomposability.

2. Definitions and notation. All spaces considered in this paper are compact and metric. A *continuum* is a nondegenerate closed connected set. The continuum $M$ is *aposyndetic* at a point $x$ with respect to a point $y$ if there is a subcontinuum in $M - y$ containing $x$ in its interior [6]. We shall say that $M$ is *semi-aposyndetic* at $\{x, y\}$ if $M$ is aposyndetic either at $x$ with respect to $y$ or at $y$ with respect to $x$. If $n \geqq 2$ and $A$ is an $n$-point set, we say that $M$ is *$n$-mutually aposyndetic at $A$* if there are $n$ disjoint subcontinua of $M$, each containing a point of $A$ in its interior. If $M$ is $n$-mutually aposyndetic at each $n$-point set, then $M$ is said to be *$n$-mutually aposyndetic.* If $M$ is $n$-mutually aposyndetic at no $n$-point set, then $M$ is *strictly non-$n$-mutually aposyndetic.* For $n = 2$ we obtain the notions of mutual aposyndesis and strict nonmutual aposyndesis [5]. For each point $x$ in $M$, $L_x$ denotes the set of all points $y$ such that $M$ is not aposyndetic at $y$ with respect to $x$, and $K_x$ denotes the set of all points $y$ such that $M$ is not aposyndetic at $x$ with respect to $y$. If $p$, $q$, and $r$ are distinct points of $M$, *$p$ cuts $q$ from $r$* if each continuum in $M$ containing both $q$ and $r$ also contains $p$. ("Cut weakly" is sometimes used; this is not the same as to "separate".)

A *chain* is a finite collection $\{E_1, \cdots, E_m\}$ of open sets such that $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| \leqq 1$. The elements of a chain are called *links*. For $\varepsilon > 0$ an *$\varepsilon$-chain* is a chain in which each link has diameter less than $\varepsilon$. A continuum is *chainable* if for each $\varepsilon > 0$, it can be covered by an $\varepsilon$-chain. An *$\varepsilon$-map* on a continuum $M$ will denote a continuous function $f$ from $M$ onto $[0, 1]$ such that for each $r \in [0, 1]$, diam $f^{-1}(r) < \varepsilon$. A chainable continuum $M$ is also characterized by the property that for each $\varepsilon > 0$, there is an $\varepsilon$-map on $M$. An *endpoint* of a chainable continuum is a point $p$ such that for each $\varepsilon > 0$, $p$ is in the first link of some $\varepsilon$-chain covering $M$.

A continuum irreducible between two points is of *type A* [10] if there is a monotone upper semi-continuous decomposition of $M$ onto an arc. A continuum $M$ is of *type A'* [10] if $M$ is of type $A$ and has a decomposition in which no element has interior.

A subcontinuum $T$ of the continuum $M$ is *terminal* [4] if for each pair of subcontinua $A, B$ which intersect $T$, either $A \subset B \cup T$ or $B \subset A \cup T$. If $p$ is a point of an indecomposable subcontinuum $K$ of $M$, $p$ is an *inaccessible point* of $K$ [4] if for each subcontinuum $R$ of $M$ which contains $p$, either $R \subset K$ or $K \subset R$.

REMARK. If $\varepsilon > 0$ and $T$ is a terminal subcontinuum of a chainable continuum $M$, then there is an $\varepsilon$-map $f$ on $M$ such that $f(T)$ is an initial segment of $[0, 1]$. (This can be shown using Lemma 1 of [4].)

A continuum $M$ is the *finished sum* [9] of subcontinua $A_1, \cdots, A_k$ if $M = \bigcup A_i$ and for each $j$, $A_j \not\subset \bigcup_{i \neq j} A_i$. The continuum $M$ is *$n$-indecomposable* [9; 2] if $M$ is the finished sum of $n$, but not of $n + 1$, subcontinua.

It is well-known [1] that chainable continua are atriodic, hereditarily unicoherent, irreducible between two points, and that each subcontinuum is chainable also. For definitions of other terms see [7] and [8].

## 3. Mutually aposyndetic products.

LEMMA 1. *Suppose the semi-aposyndetic continuum $M$ is irreducible between two points. Then $M$ is an arc.*

*Proof.* By [3, p. 116], $M$ is aposyndetic. But every aposyndetic irreducible continuum is an arc [11, p. 738].

LEMMA 2. *Suppose that*

(1) *$M$ is a chainable continuum of type $A'$,*

(2) *$M$ is not semi-aposyndetic at $\{x, y\}$,*

(3) $T = K_x \cap K_y$,

(4) *q is a point of a continuum N,*

(5) *H is a continuum in* $M \times N$ *containing the point* $(x, q)$ *in its interior, and*

(6) *D denotes the* $(x, q)$*-component of* $H \cap (T \times N)$.

*Then* $\pi_1(D) = T$. ($\pi_j$ *is the projection map onto the* $j$th *factor space.*)

*Proof.* By [10, p. 8], there is a minimal (with respect to refinement) monotone upper semi-continuous decomposition $\mathscr{D}$ of $M$ onto $[0, 1]$. Let $f$ be the associated quotient map.

For each $z \in M$, $L_z$ is a continuum in $M$ [7, p. 405]. Since $M$ is irreducible, each $K_z = L_z$ [3, p. 116]. Hence $T = L_x \cap L_y$, a continuum (by unicoherence). And by the definition of $K_z$ we have

(*) For each continuum $R$ containing either $x$ or $y$ in its interior, $T \subset R$.

Suppose the lemma fails. Let $s \in T - \pi_1(D)$. By [10, p. 25] there is a point $r \in [0, 1]$ such that $T \subset f^{-1}(r)$. In order to prove (**) below, we temporarily assume that $0 < r < 1$. Let $A$, $B$, and $C$ denote the sets $f^{-1}([0, r))$, $f^{-1}((r, 1])$, and $f^{-1}(r)$ respectively. Since $C$ cannot have interior, $M = \mathrm{Cl}\, A \cup \mathrm{Cl}\, B$ (Cl denotes closure). Using this fact and (*), it can be shown that either $\mathrm{Cl}\, A$ or $\mathrm{Cl}\, B$ must contain all three of the points $x$, $y$, and $s$. We shall assume that $\{x, y, s\} \subset \mathrm{Cl}\, A$. By [10, p. 10] $\mathscr{D} \cap \mathrm{Cl}\, A$ is a monotone upper semi-continuous descomposition of $\mathrm{Cl}\, A$ onto $[0, 1]$, and it is easily seen to be minimal. By [10, p. 30] we have

(**) If $p \in A$ and $q, t \in C \cap \mathrm{Cl}\, A$, then $t$ cuts $p$ from $q$ (in the continuum $\mathrm{Cl}\, A$).

Note that (**) holds also in the case that $r$ is an end point of $[0, 1]$, so that (**) holds for each $r \in [0, 1]$.

If $C \cap \mathrm{Cl}\, A \neq T$, then there is a point $c \in C \cap \mathrm{Cl}\, A - T$, hence (by definition of $K_x$ and $K_y$) a subcontinuum $L \subset M - c$ containing $x$, say, in its interior. But then $L \cap \mathrm{Cl}\, A$ is a subcontinuum (by unicoherence) of $\mathrm{Cl}\, A$ which contains $x$ and $L^0 \cap A$ but not the point $c$, contrary to (**). Thus $C \cap \mathrm{Cl}\, A = T$.

For each $\varepsilon > 0$ define $H_\varepsilon = H \cap [\mathrm{Cl}\, f^{-1}((r - \varepsilon, r)) \times N]$. Suppose that for each $\varepsilon > 0$ there is a continuum in $H_\varepsilon$ intersecting both $s \times N$ and $D$. The lim sup of such continua would then intersect both $s \times N$ and $D$, and would be contained in $T \times N$, hence in $D$ by the definition of $D$. Since this contradicts the choice of $s$, there must exist an $\varepsilon > 0$ such that no continuum in $H_\varepsilon$ intersects both $s \times N$ and $D$. By [8, p. 15] there are closed disjoint sets $E_s$ and $E_D$ such that $H_\varepsilon = E_s \cup E_D$, $(s \times N) \cap H_\varepsilon \subset E_s$, and $D \subset E_D$. Let $z_1, z_2, \cdots$ be a sequence of points in $E_D \cap H^0 - T \times N$ which converges to the point $(x, q)$. For each $i$, let $F_i = z_i$-component of $H \cap [f^{-1}((r - \varepsilon, r)) \times N]$. By [8, p. 18] each

$F_i$ has a limit point (relative to $H$) in either $T \times N$ or in $f^{-1}(r - \varepsilon) \times N$. If some $F_j$ has a limit point in $T \times N$, then $\mathrm{Cl}\, F_j$ is a continuum in $E_D$ from $z_j$ to $T \times N$, whereupon its projection onto $M$ would contradict (**). Hence each $F_i$ has a limit point in $f^{-1}(r - \varepsilon) \times N$. Then $\limsup F_i$ is a continuum in $E_D$ from $f^{-1}(r - \varepsilon) \times N$ to $(x, q)$, whereupon its projection is a continuum in $\mathrm{Cl}\, A$ containing $x$ and a point of $A$, but not containing $s$, contrary to (**).

LEMMA 3. *Suppose that*

(1) *$M$ is a chainable continuum containing an indecomposable subcontinuum $T$,*

(2) *$q$ is a point of a continuum $N$,*

(3) *$x$ is an inaccessible point of $T$,*

(4) *$H$ is a continuum in $M \times N$ containing $(x, q)$ in its interior, and*

(5) *$D$ denotes the $(x, q)$-component of $H \cap (T \times N)$.*
*Then $\pi_1(D) = T$.*

*Proof.* Assume that $T \neq M$; otherwise $\pi_1(D) = T$ clearly. Suppose $s \in T - \pi_1(D)$. For each $\varepsilon > 0$ define $H_\varepsilon = H \cap [\mathrm{Cl}\, N_\varepsilon(T) \times N]$ where $N_\varepsilon(T)$ denotes the $\varepsilon$-neighborhood of $T$. As in the proof of Lemma 2, there exists an $\varepsilon > 0$ and disjoint closed sets $E_s$ and $E_D$ such that $H_\varepsilon = E_s \cup E_D$, $(s \times N) \cap H_\varepsilon \subset E_s$, and $D \subset E_D$. The closure of the $(x, q)$-component of $H \cap [N_\varepsilon(T) \times N]$ is then a continuum in $E_D$ from $(x, q)$ to the boundary of $N_\varepsilon(T) \times N$, whereupon its projection (onto $M$) is a subcontinuum of $M$ containing both $x$ and a point of $M - T$, but not $s$, contrary to the fact that $x$ is an inaccessible point of $T$.

THEOREM 1. *Let $M$ and $N$ be chainable continua. Then $M \times N$ is mutually aposyndetic if and only if $M = N = [0, 1]$.*

*Proof.* Clearly $[0, 1]^2$ is mutually aposyndetic. To prove the other implication, we consider two cases.

*Case I.* At least one of $M$ and $N$ has an end point.

Suppose $q$ is an end point of $N$, and $M$ is not semi-aposyndetic. In order to define sets $D_x$ and $D_y$, we consider the following two cases:

*Case* 1. The continuum $M$ is of type $A'$.

Let $x$ and $y$ be points of $M$ such that $M$ is not semi-aposyndetic at $\{x, y\}$, and let $T = K_x \cap K_y$. By mutual aposyndesis of $M \times N$, there are disjoint subcontinua $H_x$ and $H_y$ such that $(x, q) \in H_x^0$ and $(y, q) \in H_y^0$. Then for $z \in \{x, y\}$, let $D_z$ be the $(z, q)$-component of $H_z \cap (T \times N)$, whereupon $\pi_1(D_z) = T$ by Lemma 2.

*Case* 2. The continuum $M$ is not of type $A'$.

By [10, p. 15], $M$ contains an indecomposable subcontinuum $T$ with interior. Suppose that $A$, $B$, and $C$ are disjoint subcontinua of $M$, each of which intersects $T$ but is not contained in $T$. Then $A \cup B \cup C \cup T$ is a triod, contrary to the fact that $M$ is chainable. Hence there are at most two composants of $T$ which intersect subcontinua like $A$, $B$, and $C$ above. Consequently, all the other composants of $T$ contain inaccessible points of $T$. Let $x$ and $y$ be distinct inaccessible points of $T$. By mutual aposyndesis, there are disjoint subcontinua $H_x$ and $H_y$ such that $(x, q) \in H_x^0$ and $(y, q) \in H_y^0$. Defining $D_x$ and $D_y$ as in Case 1, it follows from Lemma 3 that both $D_x$ and $D_y$ project onto $T$.

Choose $\varepsilon > 0$ such that $D_x$ and $D_y$ are at least $2\varepsilon$ apart. Let $f$ be an $\varepsilon$-map on $T$ and let $g$ be an $\varepsilon$-map on $N$ such that $g(q) = 0$. Define the continuous function $h$ from $T \times N$ to $[0, 1]^2$ by $h(a, b) = ((f(a), g(b))$. Both $h(x, q)$ and $h(y, q)$ meet $[0, 1] \times \{0\}$. Since both $D_x$ and $D_y$ project onto $T$, both continua $h(D_x)$ and $h(D_y)$ must intersect both $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. But by [8, p. 158], $h(D_x)$ and $h(D_y)$ must intersect, contradicting the choice of $\varepsilon$. Consequently, our assumption that $M$ was not semi-aposyndetic must be false. Then by Lemma 1, $M$ is an arc, and hence has an end point. Now assume that $N$ is not semi-aposyndetic, and use the same argument (interchanging the roles of $M$ and $N$) to establish that $N$ also must be semi-aposyndetic, hence an arc.

*Case II*. Neither $M$ nor $N$ has an end point.

By [4, p. 385], there are indecomposable terminal subcontinua $L_M$ and $L_N$ of $M$ and $N$ respectively. Let $q$ be an inaccessible point of $L_N$, and let $x$ and $y$ be distinct inaccessible points of $L_M$. By mutual aposyndesis, there are disjoint subcontinua $H_x$ and $H_y$ of $M \times N$ such that $(x, q) \in H_x^0$ and $(y, q) \in H_y^0$. Let $\varepsilon > 0$ such that $H_x$ and $H_y$ are at least $2\varepsilon$ apart. Let $f$ be an $\varepsilon$-map on $L_M$ and let $g$ be an $\varepsilon$-map on $N$ such that $g(L_N) = [0, c]$ for some $c \leqq 1$. Define $h: L_M \times N \rightarrow [0, 1]^2$ by $h(a, b) = (f(a), g(b))$. For $z \in \{x, y\}$, let $D_z$ and $D'_z$ denote the $(z, q)$-components of $H_z \cap (L_M \times N)$ and $H_z \cap (M \times L_N)$ respectively. By Lemma 3, $\pi_1(D_x) = \pi_1(D_y) = L_M$ and $\pi_2(D'_x) = \pi_2(D'_y) = L_N$. By the choice of $\varepsilon$, $h(D_x) \cap h(D_y) = \varnothing$. Since $q$ is an inaccessible point of $L_N$, for each $z \in \{x, y\}$ either $\pi_2(D_z) \subset L_N$ or $L_N \subset \pi_2(D_z)$.

Suppose that $L_N \subset \pi_2(D_x)$. Then $h(D_x)$ intersects both $[0, 1] \times \{0\}$ and $[0, 1] \times \{c\}$.

*Case* 1. $\pi_2(D_y) \subset L_N$.

Then $D_y \subset L_M \times L_N$. Let $B$ be a subcontinuum of $h(D_x)$ irreducible from $[0, 1] \times \{0\}$ to $[0, 1] \times \{c\}$. Since $\pi_1(D_y) = L_M$, $h(D_y)$ intersects both $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. By [8, p. 158], the continua $B$ and $h(D_y)$ must intersect, contrary to the fact that $h(D_x) \cap h(D_y) = \emptyset$.

*Case* 2. $L_N \subset \pi_2(D_y)$.

For $z \in \{x, y\}$, let $d_z$ denote the maximum of the numbers $b \in [0, 1]$ such that the point $(0, b) \in h(D_z)$. If $d_x > d_y$, then $h(D_x)$ intersects both $[0, 1] \times \{0\}$ and the point $(0, d_x)$, and $h(D_y)$ intersects both $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$. Hence $h(D_x)$ and $h(D_y)$ must intersect [8, p. 158]. A similar contradiction is reached in case $d_y > d_x$.

Since the supposition that $L_N \subset \pi_2(D_x)$ results in a contradiction, we have that $\pi_2(D_x) \subset L_N$.

In a similar manner (by interchanging the roles of $L_M$ and $L_N$, and of $D_x$ and $D'_y$, and making the other obvious modifications) it can be shown that $\pi_1(D'_y) \subset L_M$. Hence both $D_x$ and $D'_y$ are contained in $L_M \times L_N$. Let $g'$ be an $\varepsilon$-map on $L_N$, and define $h' \colon L_M \times L_N \to [0, 1]^2$ by $h'(a, b) = (f(a), g'(b))$. By the choice of $\varepsilon$, $h'(D_x) \cap h'(D'_y) = \emptyset$. But since $h'(D_x)$ intersects both $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, and since $h'(D'_y)$ intersects both $[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$, the continua $h'(D_x)$ and $h'(D'_y)$ must intersect [8, p. 158]. This contradiction concludes Case II, and hence the proof of the theorem.

The chainability requirement in the hypothesis of Theorem 1 cannot be replaced by the the requirement that the continua be of type $A'$:

EXAMPLE. *A nonchainable planar continuum $M$ of type $A'$ such that $M^2$ is mutually aposyndetic.* Let $M$ be the union of two disjoint circles plus an open ray (copy of $(0, 1)$) which spirals down on one circle at one end and on the other circle at the other end. The minimal decomposition of $M$ would have only the two circles as nondegenerate elements. Since $M$ contains a circle, it is clearly not chainable. However, it can be shown that $M^2$ is mutually aposyndetic.

**4. Strict non-$n$-mutual aposyndesis.** Hagopian has shown [5, p. 621] that the product of two chainable continua is strictly non-mutually aposyndetic if and only if each of the two continua is indecomposable. [Hagopian actually showed it for the case when the two factors are the same continuum; however it is clear that with slight modifications his proof will prove this more general result.] One direction of implication generalizes easily to $n$-mutual aposyndesis:

THEOREM 2. *Let $n \geqq 2$. Suppose $M_1$ and $M_2$ are continua, and $M_1 \times M_2$ is strictly non-n-mutually aposyndetic. Then for each $i$ $(i = 1, 2)$, $M_i$ is $r_i$-indecomposable for some integer $r_i < n$.*

*Proof.* Suppose $M_1$ is the finished sum of $n$ subcontinua $A_1, \cdots, A_n$. Then for each $j \leqq n$ there is a point $p_j \in A_j - \bigcup_{i \neq j} A_i$. In $M_2$ let $U_1, \cdots, U_n$ be open sets with disjoint closures. Then for each $j \leqq n$, let $H_j = (A_j \times \mathrm{Cl}\ U_j) \cup (p_j \times M_2)$, clearly a continuum with interior. Since the $H_j$'s are disjoint, $M_1 \times M_2$ is not strictly non-$n$-mutually aposyndetic. This contradiction implies that $M_1$ is the finished sum of at most $n - 1$ subcontinua, and the proof is complete.

The other direction of implication in Hagopian's result is represented by

(***) Suppose $M$ is an $m$-indecomposable chainable continuum and $N$ is an $n$-indecomposable chainable continuum. Then $M \times N$ is strictly non-$(mn + 1)$-mutually aposyndetic.

*Question.* Is (***) true for all values of $m$ and $n$?

By the above remarks, (***) holds for $m = n = 1$. The next theorem shows that $m = 2$ and $n = 1$ are also values for which (***) is true.

THEOREM 3. *Suppose that $M_1$ and $M_2$ are chainable continua, and $M_2$ is indecomposable. Then $M_1 \times M_2$ is strictly non-3-mutually aposyndetic if and only if $M_1$ is either indecomposable or 2-indecomposable.*

*Proof.* If $M_1 \times M_2$ is strictly non-3-mutually aposyndetic, then the conclusion follows from Theorem 2.

Conversely, suppose that $M_1$ is either indecomposable or 2-indecomposable. In case $M_1$ is indecomposable, then $M_1 \times M_2$ is strictly nonmutually aposyndetic, hence strictly non-3-mutually aposyndetic. So we assume that $M_1$ is 2-indecomposable.

Suppose that there are three disjoint continua $H_1$, $H_2$, and $H_3$ with interior in $M_1 \times M_2$. By [9, p. 649], $M_1 = A \cup B$ where $A$ and $B$ are proper indecomposable subcontinua. One of $A \times M_2$ and $B \times M_2$ (say $A \times M_2$) must contain interior points of at least two of the three $H_i$'s (say $H_1$ and $H_2$). Since $M_2$ is indecomposable, $\pi_2(H_1) = \pi_2(H_2) = M_2$. Similarly for $i = 1, 2$, $\pi_1(H_i) \supset A$; otherwise $\pi_1(H_i) \cap A$ would be a proper subcontinuum of $A$ with interior, contrary to the fact that $A$ is indecomposable.

Let $\varepsilon > 0$ such that $H_1$ is of distance at least $2\varepsilon$ from $H_2$. Let $g$ be an $\varepsilon$-map on $M_2$ and let $f$ be an $\varepsilon$-map on $M_1$ such that $f(A)$ is an initial segment of $[0, 1]$. Define the continuous function $h$ from $M_1 \times M_2$ to $[0, 1]^2$ by $h(x, y) = (f(x), g(y))$. By the choice of $\varepsilon$, the continua $h(H_1)$ and $h(H_2)$ are disjoint. For $i = 1, 2$, $h(H_i)$ meets both $y = 0$ and $y = 1$ since $\pi_2(H_i) = M_2$. And for $i = 1, 2$, since $\pi_1(H_i) \supset A$, $h(H_i)$ projects onto $f(A)$. Let $a_1$ be the left-most point (i.e., smallest first coordinate) of $h(H_1)$ on the top edge ($y = 1$), and let $a_2$ be the corresponding point for $H_2$. We shall assume, without loss of generality, that $a_1$ lies to the left of $a_2$. Since $h(H_1)$ intersects $y = 0$ and $h(H_2)$ intersects $x = 0$, the continua $h(H_1)$ and $h(H_2)$ must intersect [8, p. 158]. This contradiction concludes the proof.

*Question.* For what values of $m$ and $n$ does (***) hold without the requirement that $M$ and $N$ be chainable [cf. 5, p. 622]?

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