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ON THE OTHER SET OF THE BIORTHOGONAL POLYNOMIALS SUGGESTED BY THE LAGUERRE POLYNOMIALS

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Recently Konhauser considered the biorthogonal pair of polynomial sets $\{Z_n^\alpha(x; k)\}$ and $\{Y_n^\alpha(x; k)\}$ over $(0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$ and the basic polynomials x^k and x. For the polynomials $Y_n^\alpha(x; k)$, a generating function, some integral representations, two finite sum formulae, an infinite series and a generalized Rodrigues formula are obtained in this paper.

Biorthogonality and some other properties of $Z_n^\alpha(x; k)$ and $Y_n^\alpha(x; k)$ for any positive integer k were discussed by Konhauser ([1], [2]). For $k = 2$, the polynomials were discussed earlier by Preiser [4]. For $k = 1$, the polynomials $Y_n^\alpha(x; k)$, as also $Z_n^\alpha(x; k)$, reduce to the generalized Laguerre polynomials $L_n^\alpha(x)$.

In a recent paper [3], we obtained generating functions and other results for the polynomials $Z_n^\alpha(x; k)$ in x^k. The present paper is concerned only with the polynomials $Y_n^\alpha(x; k)$ in x which form the other set of the biorthogonal pair. The results of the paper reduce, when $k = 1$, to some standard properties of $L_n^\alpha(x)$. Simplicity of the procedure for deriving the generating relation (2.1) which may be regarded as our principal result, seems to be of some passing interest.

2. A generating function for $Y_n^\alpha(x; k)$. We begin with the contour integral representation [2, (26)]

$$(2.1) \qquad Y_n^\alpha(x; k) = (k/2\pi i)\int_C e^{-xt}(t+1)^{\alpha+kn}[(t+1)^k - 1]^{-(n+1)}\, dt$$

where we take C as a closed contour enclosing $t = 0$ and lying within $|t| < 1$. If we make the substitution $u = 1 - (t+1)^{-k}$, we get another integral representation for $Y_n^\alpha(x; k)$, viz.

$$(2.2) \qquad Y_n^\alpha(x; k) = (2\pi i)^{-1}\int_{C'} (1-u)^{-(\alpha+1)/k} \exp[x\{1 - (1-u)^{-1/k}\}]u^{-n-1}\, du$$

C' being a circle with centre $u = 0$ and a small radius. By standard arguments of complex analysis we obtain the generating relation

$$(2.3) \qquad \sum_{n=0}^{\infty} Y_n^\alpha(x; k)u^n = (1-u)^{-(\alpha+1)/k} \exp[x\{1 - (1-u)^{-1/k}\}]$$

for $\operatorname{Re}(\alpha + 1) > 0$, $|u| < 1$ and positive integers k.

Since the generating relation (2.3) is of the form

$$A(u)\exp[xH(u)] = \sum_{n=0}^{\infty} Y_n^{\alpha}(x;k)u^n ,$$

it at once follows ([6], [5]) that the set $\{Y_n^{\alpha}(x;k)\}$ is of Sheffer A-type zero. One of the several immediate consequences of this fact [5, Theorems 73-76] is that there exists a sequence $\{h_i\}$ independent of x and n such that

$$DY_n^{\alpha}(x;k) = \sum_{m=0}^{n-1} h_m Y_{n-m-1}^{\alpha}(x;k) . \tag{2.4}$$

In (2.2) putting $s = x^k(1-u)^{-1}$, we are led to still another integral representation

$$Y_n^{\alpha}(x;k) = (2\pi i)^{-1}e^x x^{k-\alpha-1}\int_{\sigma} s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})(s-x^k)^{-n-1}\,ds \tag{2.5}$$

where σ denotes the circle $|s-x^k| = r$ with small r. Evidently σ may be any small closed contour encircling $s = x^k$.

Evaluating the integral in (2.5) by the residue theorem, we obtain a generalized Rodrigues formula:

$$Y_n^{\alpha}(x;k) = (n!)^{-1}e^x x^{k-\alpha-1}[D^n s^{n-1+(\alpha+1)/k}\exp(-s^{1/k})]_{s=x^k} . \tag{2.6}$$

For $k = 1$, it reduces to the Rodrigues formula for $L_n^{\alpha}(x)$.

3. Applications. In this section we apply the generating relation of the previous section to obtain two finite sum formulae for $Y_n^{\alpha}(x;k)$ and also to prove a result involving an infinite series of these polynomials.

a. Two finite sums involving $Y_n^{\alpha}(x;k)$. From the generating relation (2.3) and the simple relation

$$(1-u)^{-(\alpha+1)/k} = (1-u)^{-(\beta+1)/k}\sum_{m=0}^{\infty}(m!)^{-1}\left(\frac{\alpha-\beta}{k}\right)_m u^m ,$$

if follows that

$$Y_n^{\alpha}(x;k) = \sum_{m=0}^{n}(m!)^{-1}\left(\frac{\alpha-\beta}{k}\right)_m Y_{n-m}^{\beta}(x;k) \tag{3.1}$$

where α and β are arbitrary.

Also from (2.3), on using

$$\begin{aligned}(1-u)^{-\{(\alpha+\beta+1)+1\}/k}&\exp[(x+y)\{1-(1-u)^{-1/k}\}]\\ &= (1-u)^{-(\alpha+1)/k}\exp[x\{1-(1-u)^{-1/k}\}]\cdot(1-u)^{-(\beta+1)/k}\\ &\quad\times\exp[y\{1-(1-u)^{-1/k}\}]\end{aligned}$$

we get that

$$Y_n^{\alpha+\beta+1}(x+y;k)=\sum_{m=0}^{n} Y_m^{\alpha}(x;k)\,Y_{n-m}^{\beta}(y;k) \tag{3.2}$$

for arbitrary α and β.

b. A series of polynomials $Y_n^\alpha(x;k)$. We show that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(n+m)!}{n!\,m!}Y_{n+m}^{\alpha}(x;k)u^n\\ &\quad=(1-u)^{-(\alpha+mk+1)/k}\exp\left[x\{1-(1-u)^{-1/k}\}\right]Y_m^{\alpha}(x(1-u)^{-1/k};k)\,.\end{aligned} \tag{3.3}$$

Using the obvious result

$$1-u-v=(1-u)\{1-v(1-u)^{-1}\}$$

we have that

$$\begin{aligned}F(u,v)&\equiv(1-u-v)^{-(\alpha+1)/k}\exp\left[x\{1-(1-u-v)^{-1/k}\}\right]\\ &=(1-u)^{-(\alpha+1)/k}\exp\left[x\{1-(1-u)^{-1/k}\}\right]\cdot(1-v(1-u)^{-1})^{-(\alpha+1)/k}\\ &\quad\cdot\exp\left[x(1-u)^{-1/k}\{1-(1-v(1-u)^{-1})^{-1/k}\}\right]\\ &=(1-u)^{-(\alpha+1)/k}\exp\left[x\{1-(1-u)^{-1/k}\}\right]\\ &\quad\cdot\sum_{m=0}^{\infty}Y_m^{\alpha}(x(1-u)^{-1/k};k)\left[v(1-u)^{-1}\right]^m\,,\end{aligned}$$

applying (2.3). But using (2.3), we also find that

$$\begin{aligned}F(u,v)&=\sum_{n=0}^{\infty}Y_n^{\alpha}(x;k)(u+v)^n\\ &=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}u^{n-m}v^m Y_n^{\alpha}(x;k)\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(m+n)!}{m!\,n!}Y_{n+m}^{\alpha}(x;k)u^n v^m\,.\end{aligned}$$

Comparing the coefficients of v^m in the two expansions obtained for $F(u,v)$, we obtain (3.3).

This result is analogous to a property possessed by almost all the classical orthogonal polynomials [5; 95(7), 111(1), 120(9), 144(23)] except possibly by the Jacobi polynomials.

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