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A CONGRUENCE THEOREM FOR ASYMMETRIC TREES

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The question is studied how a given tree is determined by the collection of its asymmetric subtrees. The results are analogous other partial answers to the Ulam-Kelly conjecture.

In [1], [2], [4], [5] several theorems are proved concerning the following conjecture posed by P. J. Kelly [4]: If G and H are two graphs with p vertices v_i and u_i respectively ($p \geqq 3$) such that for all i: $G - v_i \cong H - u_i$ then G and H are themselves isomorphic. In [4] it is shown that this conjecture is true when G, H are trees. In [1], [2], [5] improvements of this result are obtained, namely, knowledge any of the following collections is sufficient to conclude $G \cong H$ providing G, H are trees:

(1) all maximal proper subtrees [2]
(2) subtrees $T - v_i$ where v_i is a peripheral vertex [1]
(3) non-isomorphic maximal subtrees [5].

Let $G(T)$ denote the automorphism group of a tree T. If $G(T) =$ {identity} then T is called an asymmetric tree. Let $\mathfrak{A}$ denote the class of all asymmetric trees.

For a tree T consider the set of all asymmetric proper subtrees of T. This set is naturally partially ordered by inclusion, denote by $A(T)$ the set of all maximal elements of this set, i.e. the set of all maximal asymmetric subtrees. (By subtree is meant proper subtree from now on.) Further denote by $\mathfrak{A}(T)$ the set of all isomorphism types of $A(T)$. (We denote by $[G]$ the isomorphism type of the graph G, hence $\mathfrak{A}(T) = \{[T']: T' \in A(T)\}$.) We write $A(T) \cong A(S)$ for trees T and S, if there is a one-to-one mapping $\varphi: A(T) \to A(S)$ such that $\varphi(T_i) \cong T_i$ for every $T_i \in A(T)$.

We write $\mathfrak{A}(T) = \mathfrak{A}(S)$ if the sets $\mathfrak{A}(T)$ and $\mathfrak{A}(S)$ are equal. We write $T_{i,j,k}$ for the tree consisting of three edge disjoint paths that start from a common point and have lengths i, j, k.

We will investigate the dependence of $[T]$ on $A(T)$ and $\mathfrak{A}(T)$.

It is obvious that not every tree T will be determined by $A(T)$, since there are nonisomorhic trees with $A(T) = \varnothing$ (we do not include the trivial tree in the collections $A(T)$ and $\mathfrak{A}(T)$). But such trees are characterized by the following known result:

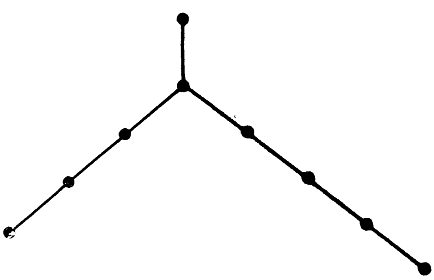
PROPOSITION 0.1. *We have $A(T) \neq \varnothing$ iff $T_7 < T$, where $T_7 = T_{1,2,3}$ with 7 vertices is the minimal asymmetric tree and $G < H$ means that G is a proper full subgraph of H.*

Moreover, assuming $A(T) \neq \emptyset$, the minimal asymmetric subtrees cover T, i.e., every edge of T belong to some $T' \in A(T)$, with the exception of the trees of one type. In view of this statement it would seem reasonable to conjecture that $\mathfrak{A}(T)$ and T are in one-to-one correspondence (up to isomorphism) providing $\mathfrak{A}(T) \neq \emptyset$. But this is not true, as is shown by the following class of examples:

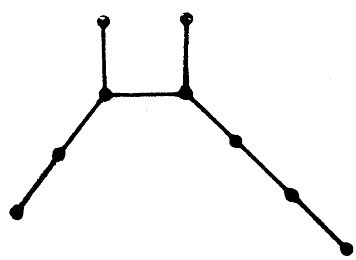
Let $i(1), \cdots, i(n)$ be n natural numbers. We denote by $T_{i(1),\cdots,i(n)}$ the subdivision of the n-star (*i.e.*, $K_1 + \bar{K}_n$, see [3]) obtained by inserting $i(k) - 1$ points in the kth edge. Obviously $\mathfrak{A}(T_{i(1)\cdots,i(n)}) = \{[T_{1,2,3}]\}$ for every n if $i(k) \leqq 3$ for $k = 1, \cdots, n$. The situation cannot be saved by considering $A(T)$ rather that $\mathfrak{A}(T)$ since $A(T_{3,2,2}) \cong A(T_{3\,2,1,1})$. The examples given here are not unique. We prove:

Main Theorem weaker form. Let T, S be asymmetric trees. Then $A(T) \cong A(S) \Leftrightarrow S \cong T$

Main Theorem stronger form. Let S, T be asymmetric trees. Then $\mathfrak{A}(T) = \mathfrak{A}(S) \Leftrightarrow T \cong S$, with the exception of the following two trees:



$T_{1,3,4}$



T_1

Since obviously $A(T_{1,3\,4}) \not\cong A(T_1)$ it is enough to prove the stronger form of the main theorem. In fact we prove this theorem in reformulation of the problem as a reconstruction of a tree (see Theorem 2.1).

The paper has two parts. In first of them we investigate the group of automorphisms of a tree in general and its connection to asymmetry (Corollary 1.2), in the second part we prove the main theorem (Theorem 2.1). The notions of the graph theory not defined here may be found in [3].

1. The automorphism group of a tree.

[1] I thank B. Manvel, who found independently the examples of exceptional trees $T_{1,3,4}$ and T_1 and called my attention to them.

THEOREM 1.1. *Let T be a tree. Let $C_2(T) = \{f \in G(T): f \circ f = id.\}$. Then $C_2(T)$ generates $G(T)$.*

Proof. Since every symmetric group is generated by transpositions and direct products and compositions see [3] preserve generators, the theorem follows.

THEOREM 1.1 has some interesting consequences:

COROLLARY 1.1. *Let T be a symmetric tree* (i.e., $G(T) \neq id.$). *Then there is $f \in G(T)$, $f \neq id.$ and $f \circ f = id.$*

This is clear by the above theorem. We remark that this is already false for unicyclic graphs, since there is a graph X with $G(X) \cong C_3$ (the cyclic group of order 3), see [3] p. 169.

COROLLARY 1.2. *Let T be an asymmetric tree, $d(x, T) = 1$. (By $d(x, T)$ we denote the degree of the point x in the tree T.) Then* $|G(T - x)| \leqq 2$ (i.e. *the removing of an endpoint of an asymmetric tree gives rise to at most one symmetry*).

Proof. Suppose for the contrary $|G(T - x)| > 2$ for some $d(x, T) = 1$. By Theorem 1.1 there are $f_1, f_2 \in G(T - x)$ and $f_1 \circ f_1 = f_2 \circ f_2 = id.$, $f_1 \neq f_2$. Let $[x, y] \in E(T)$, then necessarily $y \neq f_1(y) \neq f_2(y) \neq y$. Let us distinguish two cases: (i) $T - x$ is a central tree (see [1]), c is the only center of $T - x$. Let $W(c, y)$ be the path joining y and c. Put $n_i = \min\{\rho(c, z); z \in W(c, y), f_i(z) \neq z\}$, $i = 1, 2$. ($\rho(c, z)$ is the distance between c and z.) It can be proved easily that $n_1 = n_2$ and that $f_1(z) = f_2(z)$ where $z \in W(c, y)$, $\rho(z, c) = n_1$. But then $f_1 = f_2$, for otherwise the number n, defined for $f_1^{-1} \circ f_2$ as n_1 was defined for f_1, would be greater than n_1. But $f_1 \neq f_2$ by hypothesis. (ii) Let $T - x$ be bicentral. We can use the same argument as in (i) for the tree $(T - x)^{\vee}$, where for every T-bicentral tree the central tree $T^{\vee}$ is defined by: $V(T^{\vee}) = V(T) \cup \{c\}$, $c \notin V(T)$ and $E(T^{\vee}) = (E(T) - [c_1, c_2]) \cup [c, c_1] \cup [c, c_2]$, where c_1, c_2 are two centers of T.

REMARK. Corollary 1.2 gives a necessary condition for a tree T to have an asymmetric extension to $|T| + 1$ vertices, which is itself a tree. This condition is not sufficient.

2. Asymmetric congruence of trees. We are going to prove the main theorem. This will be done in Propositions 2.1 – 2.8. A difference between the proof presented here and the proofs used in [1], [2], [4] is that we know less about the structure of $\mathfrak{A}(T)$. Thus to prove that some basic parameters of T are determined by $\mathfrak{A}(T)$ we need existence theorems.

Let $T \in \mathfrak{A}$ be fixed from now on.

PROPOSITION 2.1. $\mathfrak{A}(T) = \varnothing$ *iff* $T \cong T_7$.

This follows from Proposition 0.1. Thus let $\mathfrak{A}(T) \neq \varnothing$ from now on.

Let (T, x) be a rooted tree, by $G(T, x)$ we denote the group of all root-automorphisms, i.e., all automorphisms of T which leave x fixed. In an obvious sense we will speak about root-asymmetric tree, root-isomorphic trees $(T, x) \cong (S, y)$ and so on.

Let T be a tree, the branch S of T at a point x is every maximal subtree of T which contains x as an endpoint. Every branch at a center of T is called limb.

To determine $|T|$ we prove the existence of $[T_0] \in \mathfrak{A}(T)$, $|T_0| = |T| - 1$. We prove first:

LEMMA 2.1. *Let (T, x_0) be a root-asymmetric tree. Then there is a vertex x, $x \neq x_0$, $d(x, T) = 1$, such that $(T - x, x_0)$ is root-asymmetric.*

Proof. For $|T| = 2$ the statement obviously holds. Let the lemma hold for every (S, y), $|S| < n$. Let (T, x_0) be a root-asymmetric tree, $|T| = n$. Define the relation $\prec$ on $V(x_0, T) = \{x; [x, x_0] \in E(T)\}$ by: $x \prec y \Leftrightarrow$ there is an endpoint of T_y and $(T_y - z, x_0) \cong (T_x, x_0)$. (Equivalently by Corollary 1.2: there is an endpoint $z \in V(T_y)$ and $f \in G(T - z)$ such that $f(y) = x$, $f(x) = y$.) Here T_y denotes the branch of T at x_0 containing y.

Let x_1 be a minimal vertex for the relation $\prec$. Then by the induction hypothesis there is $x \in T_{x_1}$ such that $(T_{x_1} - x, x_0)$ is root-asymmetric and by the definition of $\prec$ $(T - x, x_0)$ is a root-asymmetric tree.

According to [1], a vertex x of a tree T is called peripheral if there is $y \in V(T)$ and $\rho(x, y) = \operatorname{diam} T$. The couple x, y we call a peripheral couple.

PROPOSITION 2.2. (I) *Let T be a central tree, then either* (i) $T \cong T_{1,k,m}$, $k + m$ *odd or* (ii) *there is* $[T_i] \in \mathfrak{A}(T)$, T_i *central*, $|T_i| = |T| - 1$.

(II) *Let T be bicentral, then either* (i) $T \cong T_{1,k,m}$, $k + m$ *even or* (ii) *there is* $[T_i] \in \mathfrak{A}(T)$, T_i *bicentral*, $|T_i| = |T| - 1$.

(III) *For every T there is $T_i \in A(T)$, T_i is a maximal subtree.*

Proof I. Let T be a central asymmetric tree with the center c. The following cases are possible:

(a) for every peripheral vertex of T, there is a peripheral couple disjoint with it

(b) T contains more than one peripheral couple and (a) does not hold

(c) T contains exactly two peripheral vertices.

In the case (a) we can use Lemma 2.1 since each $T - x$ is a central tree and hence $(T - x, c)$ is a root-asymmetric tree if $T - x$ is an asymmetric tree. Suppose that (c) holds: Let a, b be the only peripheral vertices of T, $a \in R_1$, $b \in R_2$ the only radial limbs of T (i.e., branches at the center with a peripheral point). Let $|R_1| \leqq |R_2|$. If there are other limbs of T, then we can apply Lemma 2.1 to their union and find x such that $[T - x] \in \mathfrak{A}(T)$. Thus let R_1, R_2 be the only branches of T at c. The proof can be finished by choosing a convenient point $z \in W(a, b)$ (the unique path connecting a and b), $d(z, T) \geqq 3$ and considering the union of all branches of T at z which contain neither a nor b. Using Lemma 2.1 we get an asymmetric tree with the only exception $T \cong T_{1,k,m}$, $k + m$ odd.

The case (b) can be handled similarly.

(II) can be proved by use of the graph $T^{\vee}$(see the proof of the Corollary 1.2).

(III) is obvious by (I) and (II), since $T_{1,2,3} \neq T_{1,k,m}$ implies that $T_{1,k\,m}$ contains a maximal subtree which is asymmetric.

REMARK. The Proposition 2.1. (III) was recently proved in a different context by J. Sheehan and J. A. Zimmer Jr. from the University of Waterloo.

PROPOSITION 2.3. *Let* $\{k, m\} \neq \{3, 4\}$. *Then* $T_{1,k,m}$ *is reconstructible from* $\mathfrak{A}(T_{1,k,m})$. *There is* $\mathfrak{A}(T_{1,3,4}) = \mathfrak{A}(T_1)$ (see the Introduction) *and there are no other such graphs.*

Outline of proof. Obviously $[T'] \in \mathfrak{A}(T_{1,k,m})$ implies $T_{2,2,2} \nleqq T'$, T' has only one vertex of degree $\geqq 3$, and further $|\mathfrak{A}(T_{1,k,m})| \leqq 2$. From these facts one can verify the statement by exhaustion of cases.

PROPOSITION 2.4. (i) *T is central if diam $T' <$ diam T for every bicentral subtree T' of T* (diam T is the diameter of the tree T) (ii) *T is bicentral if diam $T' <$ diam T for every central subtree T' of T.*

The proof is clear and is omitted.

By Propositions 2.2, 2.3, 2.4 we can determine from $\mathfrak{A}(T)$ whether

T is central or bicentral. Thus from now on let T be central, $T \not\cong T_{1,k,m}$, $T \not\cong T_1$. Let c be the center of T, r the radius of T, (see [3]). The case T bicentral will be investigated later. The following lemma deals with a special kind of trees, one that has a radial limb which is a path (called a radial path).

LEMMA 2.2. *The following two statements are equivalent:*

(i) *T satisfies one of the following properties:* (a) *T contains a radial path* (which is necessarily unique). (b) *$T \cong T_n$ for some $n > 1$, where T_n* (n natural number) *is the tree defined by* $V(T_n) = \{1, \cdots, n+8\}$, $E(T_n) = \{[i, i+1]; i = 1, \cdots, n+5\} \cup [4, n+7] \cup [n+4, n+8]$, (the tree T_1 is defined in the Introduction)

(ii) *$\mathfrak{A}(T)$ satisfies one of the following properties:* (a) *For every $[T_i] \in \mathfrak{A}(T)$, T_i contains a branch which is the path of length r.* (b) *There is $[T_0] \in \mathfrak{A}(T)$ such that every tree T_i, $[T_0] \neq [T_i] \in \mathfrak{A}(T)$ contains a branch which is the path of length r, the tree T_0 itself contains a branch at one of its centers which is the path of length $r-1$.*

Outline of proof. Let T contain a radial path. Since T is central, we can assume that $W(c, y)$ is a radial path ($d(y, T) = 1$). By asymmetry this is the only radial path in T. Furthermore: If $x \in V(T) - W(c, y)$, $x \notin T' \in A(T)$ then T' has a radial path. From this it follows, by the maximality of the elements of $A(T)$, that there is at most one $[T_0] \in \mathfrak{A}(T)$, such that T_0 has no branch which is the path of length r. It is now easy to conclude that either (iia) or (iib) holds. Conversely let T have no radial path and suppose that $\mathfrak{A}(T)$ satisfies (iia) or (iib). We can conclude that $T \cong T_n$ for some $n > 1$. We can prove first that every limb of T is a radial limb and by a similar method to that in the proof of Proposition 2.2 we can prove $T \cong T_n$. The details are omitted.

PROPOSITION 2.5. *Let $n > 1$. The tree T_n see Lemma 2.2 is reconstructible from $\mathfrak{A}(T_n)$.*

The proof is simple (using Proposition 2.3).

PROPOSITION 2.6. *If T contains a radial path, then T is reconstructible from $\mathfrak{A}(T)$.*

Proof. Let T' contain the radial path (which is unique), and let $\mathfrak{A}(T') = \mathfrak{A}(T)$. Then by Lemma 2.2 and Proposition 2.5 we know that T has the radial path. Assume that (iib) of the Lemma 2.2 holds; let $|T_0| = |T| - k$, then either diam T_0 = diam T and $T \cong T'$ follows easily, or diam $T_0 <$ diam T. This case can occur if T has only two radial branches at c. Since (iib) holds, we know that $R \neq \emptyset$, where R is the union of all non-radial limbs at c. Applying Lemma 2.2 to (R, c) we can determine both radial branches of T. It is easy to see also that R is determined uniquely.

Case Lemma 2.2 (iia) can be handled similarly.

Now we can prove the main theorem for central trees:

PROPOSITION 2.7. *Let T be central, then T is reconstructible from $\mathfrak{A}(T)$.*

Proof. By Propositions 2.1, 2.2, 2.5, 2.6 we can assume $T \not\cong T_{1,k,m}$, $T \not\cong T_n$, T does not contain a radial path. Consider $\mathfrak{A}_1 = \{[T_i] \in \mathfrak{A}(T); |T_i| + 1 = |T|, T_i \text{ is central}\}$.

We know (by the assumptions on T), that there is $x \in R$, where R is a limb of the minimal cardinality, such that $T - x \in \mathfrak{A}_1$. In this tree $T - x$ we know all limbs except R. Let R_1 be a limb of the minimal cardinality among all limbs different from R. Let (R_1', c) be a maximal root-asymmetric subtree of the limb (R_1, c), which contains a peripheral point. If there is $T' \in \mathfrak{A}_1$, such that T' does not contain the limb (R_1, c) and T' contains the limb (R_1', c), then the branch (R, c) is the only branch in T' which was still unknown. If there is no such tree T' then $(R, c) \cong (R_1', c)$.

To prove the main theorem for bicentral trees we could modify the proofs of the previous propositions. We use a different proof.

PROPOSITION 2.8. *Let T be a bicentral tree. Then T is reconstructible from $\mathfrak{A}(T)$.*

Proof. Let T be a bicentral tree of the diameter $2r + 1$; by the Proposition 2.2 we can assume $T \not\cong T_{1,k,m}$. If T contains a branch which is a path of length $r + 1$, then for such a tree a statement similar to Lemma 2.2 holds and T can be reconstructed from $\mathfrak{A}(T)$ in a similar manner to that used in Proposition 2.6. Assume that T does not contain a branch of length $r + 1$. Let us form the tree $T^{\vee}$ by the definition given in the proof of Corollary 1.2. As seen from the proof of the Proposition 2.7, $T^{\vee}$ is determined by all the trees in $\mathfrak{A}(T^{\vee})$ which have the same diameter as $T^{\vee}$. Since for such trees the operation "$\vee$" preserves isomorphism the proposition follows.

Thus we finally have:

THEOREM 2.1. *Every asymmetric tree is reconstructible from* $A(T)$. *Every asymmetric tree is reconstructible from* $\mathfrak{A}(T)$, *with the exception of* T_1 *and* $T_{1,3,4}$.

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