VARIETIES OF IMPLICATIVE SEMILATTICES

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The main purpose of this paper is to investigate properties of the lattice of subvarieties of the variety of implicative semilattices. Also the distinct compositions of the operators of taking homomorphic images, subalgebras, and products of classes of implicative semilattices are determined.

A class K of similar universal algebras [2, pp. 33-34] is called a variety provided K consists of all the algebras which satisfy some set of identities. If K is a variety and $K' \subseteq K$, then K' is called a subvariety of K provided K' is itself a variety. The subvarieties of a given variety form a lattice when ordered by inclusion. A basic theorem of Birkhoff [1] states that a class of algebras K is a variety if and only if K is closed under the taking of homomorphic images, subalgebras, and direct products.

An implicative semilattice is an algebra $\langle L; \wedge, * \rangle$ where $\langle L; \wedge \rangle$ is a semilattice, and $*$ is a binary operation such that $x \wedge y \leq z$ if and only if $x \leq y * z$ (here $w \leq u$ means $w \wedge u = w$). Every implicative semilattice has a largest element which we denote by 1. Monteiro [4] has given a set of equational axioms for implicative semilattices thus showing that the class of implicative semilattices is a variety. In this paper we consider the lattice of subvarieties of this variety. We shall denote the variety of all implicative semilattices by I.

An ideal K of an implicative semilattice L is a subset of L such that $x \in K$ whenever $x \leq y \in K$. A filter J of L is a subset of L such that $x \wedge y, z \in J$ whenever $x, y \in J, z \in L$ and $x \leq z$. It is shown in [5] that filters are related to homomorphisms for implicative semilattices in the same way they are for boolean algebras. In particular, if by the kernel of a homomorphism we mean the pre-image of the greatest element of the range, then the kernel of any homomorphism is a filter, every filter is the kernel of a homomorphism, and the congruence relation and quotient algebra determined by the homomorphism are also determined by the kernel of the homomorphism. Also, in [6] it is shown that the lattice of filters and therefore the lattice of congruence relations of an implicative semi-lattice is distributive. Thus we are able to make use of the results and techniques of Jónsson [3].

In § 2 we determine several elementary properties of subdirectly irreducible implicative semilattices. Here it is also shown that the variety of implicative semilattices is generated by its finite members.

covers no subvariety, and is not the lattice join of any two proper subvarieties. In § 3 we determine the varieties generated by chains and give identities for these. In § 4 we introduce the notion of sublength for implicative semilattices and show that this is a variety property. The varieties of sublength three are determined by their generators.

For a class K of algebras $H(K)$, $S(K)$, $P(K)$ denote respectively the class of all homomorphic images of elements of K, the class of all subalgebras of elements of K, and the class of all direct products of elements of K. Pigozzi [8] has determined all direct compositions of the operations H, S, and P for arbitrary algebras. In the final section we consider this problem when restricted to classes of implicative semilattices.

For the basic arithmetic of implicative semilattices, the reader is referred to [5].

2. **Subdirectly irreducible implicative semilattices.** Since a subdirectly irreducible algebra is one with a smallest proper congruence relation, we see that a subdirectly irreducible implicative semilattice is one with a smallest proper filter, thus one with a unique dual atom which dominates all elements other than 1. If we let K^e denote the smallest variety containing the class of algebras K, then $K^e = HSP(K)$. Since a variety is determined by its subdirectly irreducible members, for a given class K of algebras, it is helpful to know the subdirectly irreducible members of K^e. It follows from [3, Cor. 3.4] that if K is a finite set of finite implicative semilattices, then every subdirectly irreducible member L of K^e is in $HS(K)$. The following lemma shows that L is actually in $S(K)$.

LEMMA 2.1. *If L is an implicative semilattice every filter of which is principal, then any homomorphic image of L is isomorphic to a subalgebra of L.*

Proof. Suppose $\varphi\colon L \to L'$ is a homomorphism. The kernel of φ is principal, say generated by a. It is easy to check that the mapping $\psi\colon L \to L$ which takes x to $a * x$ is a homomorphism with the same kernel as φ. Thus $\psi(L) \cong \varphi(L)$.

If L is any implicative semilattice, we let $\hat{L}$ be the result of adjoining an element u to L in such a way that $x < u < 1$ for each $x \in L \sim \{1\}$. The operations of L extend those of $\hat{L}$ and satisfy the formulas

$$1 * u = u,\ u * 1 = 1\ ;$$

and

$$x * u = 1, u * x = x \quad \text{for all} \quad x \in L \sim \{1\} \, .$$

Clearly $\hat{L}$ is subdirectly irreducible and $L \leq_s \hat{L}$ (L is a subalgebra of $\hat{L}$). Also note that every subdirectly irreducible implicative semilattice can be so obtained. It is easy to check that the mapping φ defined by $\varphi(x) = u * x$ is a homomorphism of $\hat{L}$ onto L. Thus we get the following lemma.

LEMMA 2.2. *Every implicative semilattice is both a subalgebra and a homomorphic image of the same subdirectly irreducible implicative semilattice.*

By [3, Lemma 4.1] we see that if V_1 and V_2 are varieties of implicative semilattices and if L is a subdirectly irreducible member of $(V_1 \cup V_2)^e$, then $L \in V_1$ or $L \in V_2$. Using Lemma 2.2 and a proof as in [3, Thm. 5.4] we get the following corollary.

COROLLARY 2.3. *If V_1 and V_2 are proper subvarieties of I, then so is the lattice join of V_1 and V_2.*

The following lemma shows that I is generated by its finite members.

LEMMA 2.4. *Let L be an implicative semilattice, and let S be a finite, nonempty subset of L which is closed under $\wedge$. Then there is a finite implicative semilattice L' and a one-to-one, order preserving mapping φ of S into L' such that*
  (i) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ *if $x, y \in S$,*
  (ii) $\varphi(x * y) = \varphi(x) * \varphi(y)$. *If $x, y, x * y \in S$.*

Proof. Let L' be the set of all ideals of S ordered by inclusion. Clearly L' is closed under intersection. For $x \in S$, let $(x) = \{y \in S: y \leq x\}$ so $(x) \in L'$. For $A_1, A_2 \in L'$, let

$$A_1 \circ A_2 = \{x \in S: (x) \cap A_1 \subseteq A_2\} \, .$$

Obviously $A_1 \circ A_2 \in L'$. If $x \in A_1 \cap (A_1 \circ A_2)$, then $(x) = (x) \cap A_1 \subseteq A_2$ so $x \in A_2$. Thus $A_1 \cap (A_1 \circ A_2) \subseteq A_2$. If $A \in L'$, $A_1 \cap A \subseteq A_2$, and $x \in A$, then $(x) \cap A_1 \subseteq A_2$ so $x \in A_1 \circ A_2$. Hence $A \subseteq A_1 \circ A_2$. Therefore, $\langle L'; \cap, \circ \rangle$ is an implicative semilattice.

Now let $\varphi: S \to L'$ by $\varphi(x) = (x)$. This mapping is clearly one-to-one and order preserving. Also it is clear that $\varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$ if $x, y \in S$. Hence $\varphi(x) \cap \varphi(x * y) \subseteq \varphi(y)$, so $\varphi(x * y) \subseteq \varphi(x) \circ \varphi(y)$. If $w \in \varphi(x) \circ \varphi(y)$, then $(w) \cap (x) \subseteq (y)$ so $w \wedge x \leq y$. Thus $w \leq x * y$ and $w \in (x * y) = \varphi(x * y)$. Hence $\varphi(x * y) = \varphi(x) \circ \varphi(y)$ and the proof is complete.

THEOREM 2.5. *I is the smallest variety of implicative semilattices which contains all of the finite implicative semilattices.*

Proof. From Lemma 2.4 it follows that any identity which does not hold in every implicative semilattice fails in some finite implicative semilattice.

COROLLARY 2.6. *There is no variety V which I covers in the lattice of subvarieties of I.*

Proof. Same as [3, Cor. 5.5].

LEMMA 2.6. *If K is a class of similar algebras and if A is a subdirectly irreducible member of $SP(K)$, then $A \in S(K)$.*

Proof. Suppose $A \leq_s \prod_{i \in I} A_i$ where each $A_i \in K$. Let φ_i be the projection of A into A_i. Then A is a subdirect product of $\{\varphi_i(A) : i \in I\}$. Thus one of the projections must be a monomorphism.

3. Varieties generated by chains.[1] From the remarks of the preceeding section it is clear that any chain with a largest element and a dual atom is a subdirectly irreducible implicative semilattice. For $n \in \omega \sim \{0\}$, let $\mathscr{C}_n$ denote the variety generated by an n-element chain. Let $\mathscr{C}$ denote the variety generated by all finite chains; $\mathscr{C} = \bigvee \mathscr{C}_n$. From Lemma 2.1 of the preceeding section it is clear that for $m < n$ we have $\mathscr{C}_m \subset \mathscr{C}_n$. In fact $\mathscr{C}_{n+1}$ covers $\mathscr{C}_n$ in the lattice of subvarieties.

THEOREM 3.1. *If C is any infinite chain with 1, then $\mathscr{C}$ is the variety generated by C. Hence $\mathscr{C}$ is the variety generated by the class of all chains.*

Proof. Since C contains each finite chain as a subalgebra we clearly have $\mathscr{C} \subseteq \{C\}^e$. The reverse inclusion is obtained by a standard argument. If an identity fails in C it fails in a finite subchain of C. This subchain together with 1 is a subalgebra of C in which the identity fails. Thus C satisfies every identity which holds in each finite chain, so $C \in \mathscr{C}$.

FIGURE 1

NOTATION. Let D' denote the implicative semilattice of Figure 1.

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[1] T. Katrinak has obtained results similar to some of those in this and the next sections.

LEMMA 3.2. *If L is a subdirectly irreducible implicative semilattice and L is not a chain, then $D^1 \leqq_s L$.*

Proof. Let u be the dual atom of L and let a, b be elements of L which are not related. It is straightforward to check that

$$\{1, u, a * b, b * a, (a * b) \wedge (b * a)\}$$

is a subalgebra of L. To see that these elements are distinct it is enough to show that $a * b$ and $b * a$ are not related. But if $a * b \leqq b * a$, then

$$b = b \wedge b \leqq b \wedge (a * b) \leqq b \wedge (b * a) \leqq a$$

a contradiction.

In [7] the operation of pseudo-join for implicative semilattices was introduced. This operation is given by

$$a + b = ((a * b) * b) \wedge ((b * a) * a) .$$

It was shown that this operation is a lattice join precisely when the operation is associative. Semi-Boolean lattices are those implicative semilattices in which the pseudo-join is associative.

THEOREM 3.3. *$\mathscr{C}$ is the class of semi-Boolean lattices.*

Proof. It is clear that in any chain $a + b$ is the larger of a and b. Thus any chain is a semi-Boolean lattice. It then suffices to show that any subdirectly irreducible implicative semilattice which is not a chain is not semi-Boolean. For this it is enough to show that D^1 is not semi-Boolean. If a and b are the two unrelated elements of D^1 then $a + b = 1$ which is not the least upper bound of a and b.

THEOREM 3.4. *For $n = 1, 2, \cdots$, the identity*

$$I_n: (x_1 * x_2) + (x_2 * x_3) + \cdots + (x_n * x_{n+1}) = 1$$

holds in C_n but fails in C_{n+1}.

Proof. In the n-element chain, for any elements $a_1, a_2, \cdots, a_{n+1}$, we must have $a_i \leqq a_{i+1}$ for some i so $a_i * a_{i+1} = 1$.

In the $n + 1$ element chain, we have $a_1 > a_2 > \cdots > a_{n+1}$ so

$$(a_1 * a_2) + (a_2 * a_3) + \cdots + (a_n * a_{n+1}) = a_2 + a_3 + \cdots + a_{n+1}$$
$$= a_2 .$$

COROLLARY 3.5. *$\mathscr{C}_n$ is characterized by I_n together with the associative law for $+$.*

Proof. Let $\dot{V}$ be the variety defined by I_n and the associative law for $+$. By Theorems 3.3 and 3.5 we see that V is a proper subvariety of $\mathscr{C}$ which contains $\mathscr{C}_n$. Since D^1 is not semi-Boolean, we see by Lemma 3.2 that the only subdirectly irreducible members of $\mathscr{C}$ are chains. Thus any subvariety of $\mathscr{C}$ which properly contains $\mathscr{C}_n$ must contain C_{n+1}. Since V does not contain C_{n+1} we have $V = \mathscr{C}_n$.

4. Semilattices of finite sublength.

DEFINITION 4.1. For $n = 1, 2, \cdots$ an implicative semilattice L is said to have sublength n provided C_n is a subalgebra of L but C_{n+1} is not. We let $\overline{\mathscr{C}_n}$ denote the class of implicative semilattice having sublength less than or equal to n.

LEMMA 4.2. *If* $f: L \to C_n$ *is an epimorphism, then* L *has a subalgebra isomorphic to* C_n *which is mapped onto* C_n *by* f.

Proof. The proof is by induction on n. The case $n = 1$ is clear. Suppose the result is true for $n = k$ and that $f: L \to C_{k+1}$. Let $C_{k+1} = \{1 = a_1, a_2, \cdots, a_k, a_{k+1}\}$ where $a_1 > a_2 > \cdots > a_k > a_{k+1}$. Let

$$\{1 = x_1, x_2, \cdots, x_k, x_{k+1}\} \subseteq L$$

such that $f(x_i) = a_i$ for $i = 1, \cdots, k + 1$. We may assume that $x_1 > x_2 > \cdots > x_k > x_{k+1}$ since $x_1 > x_1 \wedge x_2 > \cdots > x_1 \wedge \cdots \wedge x_{k+1}$ and

$$f(x_1 \wedge \cdots \wedge x_j) = a_j .$$

If we restrict f to the principal filter generated by x_k and apply the induction hypothesis, we may assume that $x_i * x_j = x_j$ if $i < j < k$. We focus our attention on the subalgebra of L generated by

$$\{x_1, \cdots, x_{k+1}\} .$$

Clearly x_{k+1} is the least element of this subalgebra so we denote it by 0. Also we denote a_{k+1} by 0. For $2 \leq j \leq k$ let $y_j = x_k^{**} * x_j$ where $x^* = x * 0$. We now claim that $\{1, y_2, \cdots, y_k, 0\}$ is the subalgebra of L which we desire.

It is clear that $1 \geq y_2 \geq \cdots \geq y_k \geq 0$. However,

$$\begin{aligned} f(y_i) &= f(x_k^{**} * x_i) \\ &= f(x_k^{**}) * a_i \\ &= [(a_k * 0) * 0] * a_i \\ &= (0 * 0) * a_i \\ &= a_i . \end{aligned}$$

Thus we have $1 > y_2 > \cdots > y_k > 0$ and $f(y_i) = a_i$.

To check that $\{1, y_2, \cdots, y_k, 0\}$ is a subalgebra of L, we only need to show that $y_j * 0 = 0$ and that $y_i * y_j = y_j$ if $1 < i < j \leq k$. Since $y_j = x_k^{**} * x_j \geqq x_j \geqq x_k$, we have $y_j * 0 \leqq x_k * 0 = x_k^*$. Also,

$$ y_i = x_k^{**} * x_i \geqq x_k^{**} * 0 = x_k^{***} = x_k^* \ . $$

Thus $y_i * 0 \leqq x_k^* * 0 = x_k^{**}$. Hence $y_i * 0 \leqq x_k^* \wedge x_k^{**} = 0$ so $y_i * 0 = 0$. Furthermore,

$$ \begin{aligned} y_i * y_j &= (x_k^{**} * x_i) * (x_k^{**} * x_j) \\ &= x_k^{**} * (x_i * x_j) \\ &= x_k^{**} * x_j \\ &= y_j \end{aligned} $$

and the proof is complete.

THEOREM 4.3.   *For* $n = 1, 2, \cdots, \overline{\mathscr{C}}_n$ *is a variety.*

Proof. Clearly $\overline{\mathscr{C}}_n$ is closed under the taking of subalgebras, by Lemma 2.6 it is closed under products, and by Lemma 4.2 it is closed under homomorphisms.

COROLLARY 4.4.   *Let* I *be the variety of all implicative semi-lattices. Then* $\bigvee_{n \in \omega} \overline{\mathscr{C}}_n = I$.

Proof. This is clear since $\bigvee_{n \in \omega} \overline{\mathscr{C}}_n$ contains all of the finite implicative semilattices.

THEOREM 4.5.   *For* $n = 1, 2, \cdots \overline{\mathscr{C}}_n$ *is characterized by* I_n *(with all association in* I_n *being to the right).*

Proof. Let V_n be the variety determined by I_n. We show by induction on n that $V_n = \overline{\mathscr{C}}_n$. If $L \in \overline{\mathscr{C}}_n$, then L contains C_{n+1} as a subalgebra; so by Theorem 3.4 I_n fails in L. Thus we see that $V_n \subseteqq \overline{\mathscr{C}}_n$. Hence we only need to show that the subdirectly irreducible members of $\overline{\mathscr{C}}_n$ are in V_n. The case $n = 1$ is clear since $\overline{\mathscr{C}}_1$ consists only of one element algebras.

Now assume that I_k characterizes $\overline{\mathscr{C}}_k$ and that $\hat{L}$ is a subdirectly irreducible member of $\overline{\mathscr{C}}_{k+1}$. Suppose $a_1, a_2, \cdots, a_{k+2} \in \hat{L}$ and that

$$ (a_1 * a_2) + ((a_2 * a_3) + \cdots + ((a_k * a_{k+1}) + (a_{k+1} * a_{k+2})) \cdots) < 1 \ . $$

Then we must have $a_1 * a_2, a_2 * a_3, \cdots, a_k * a_{k+2} < 1$. From this we get $a_2, a_3, \cdots, a_{k+2} < 1$. Also we have

$$(a_2 * a_3) + ((a_3 * a_4) + \cdots + ((a_k * a_{k+1}) + (a_{k+1} * a_{k+2})) < 1 \,.$$

By the induction hypothesis we have I_k holding in L since $L \in \overline{\mathscr{C}}_k$. Thus one of $a_2, a_3, \cdots, a_{k+2}$ must be the dual atom of $\hat{L}$, call it u. Now since $a_2 \leqq u$, $a_2 * a_3 \leqq u$ we must have $a_3 < u$. Similarly $a_i < u$ for $i = 3, 4, \cdots, k + 2$. Hence $a_2 = u$. Then we get

$$1 > (a_2 * a_3) + \cdots + ((a_k * a_{k+1}) + (a_{k+1} * a_{k+2})) \cdots) \,.$$
$$= (u * a_3) + \cdots + ((a_k * a_{k+1}) + (a_{k+1} * a_{k+2})) \cdots) \,.$$
$$= (1 * a_3) + \cdots + ((a_k * a_{k+1}) + (a_{k+1} * a_{k+2})) \cdots) \,.$$

But this contradicts the assumption that I_k holds in L, and the proof is complete.

LEMMA 4.6.   *Let $\hat{L}$ be any subdirectly irreducible member of $\overline{\mathscr{C}}_3$ which has a least element 0.   Then L is a Boolean algebra.*

Proof.   If $x \in L \sim \{1\}$ we can not have $x * 0 = 0$ or else $\{1, u, x, 0\}$ is a four element subchain of L. Using Theorems 4.3 and 4.4 of [5] we see that L is a Boolean algebra.   (Note that $\hat{L} \in \overline{\mathscr{C}}_{n+1}$ implies that $L \in \overline{\mathscr{C}}_n$).

COROLLARY 4.7.   *$\overline{\mathscr{C}}_3$ is generated by its finite members.*

Proof.   Suppose not.   Let V be the variety generated by the finite members of $\overline{\mathscr{C}}_3$.   Then there is a subdirectly irreducible member of $\overline{\mathscr{C}}_3 \sim V$, say L.   Thus there is some identity which holds in V but fails in L.   The failure of L to satisfy this identity depends only on a finite number of elements of L.   Thus we may assume L is finitely generated.   However, this would imply that L has a least member giving us L a finite Boolean algebra.

COROLLARY 4.8.   *Let $\mathscr{B}_m$ denote the variety generated by B_m where B_m is the 2^m-element Boolean algebra.   If $m < n$, then $\mathscr{B}_m \subset \mathscr{B}_n$.   Also we have $\bigvee_{n \in \omega} \mathscr{B}_n = \overline{\mathscr{C}}_3$.*

COROLLARY 4.9   *If B is any infinite Boolean algebra, then $\hat{B}$ generates $\overline{\mathscr{C}}_3$.*

5.   H, S, P for implicative semilattices.

DEFINITION 5.1.   If Q and Q' are two compositions of the operators H, S, P, we let $Q \leqq Q'$ mean that $Q(K) \subseteqq Q'(K)$ for every class K of

implicative semilattices.   Also $Q < Q'$ mean $Q \leqq Q'$, but $Q(K) \subset Q'(K)$ for some class K.

It is well known that for any class K of algebras we have $SH(K) \subseteqq HS(K)$, $PH(K) \subseteqq HP(K)$, and $PS(K) \subseteqq SP(K)$. Pigozzi [8] has shown using these inequalities that any composition of H, S, P is equal to one of the compositions without a repeated term or to either $SHPS$ or $SPHS$. He has also shown that no further reductions are possible. For implicative semilattices, we shall show that $HS = SH$ and that the reductions which follow from this equality are the only one possible for implicative semilattices.

LEMMA 5.1.   *If L and L' are implicative semilattices and if L' is a homomorphic image of a subalgebra of L, then L' is a subalgebra of a homomorphic image of L.*

Proof. Suppose $L'' \leqq_s L$ and $f: L'' \to L'$ is an epimorphism.   Let $J = \ker(f)$. Then $L' \cong L''/J$. Let K be the filter of L generated by J. It is clear that K restricted to L'' is J so $L' \cong L''/J \cong L''/(K \cap L'') \leqq_s L/K$.

THEOREM 5.2.   *For implicative semilattices,*

$$HS = SH, HSP = SHP, PHS = PSH, SHPS = HSP$$

and $SPHS = SPH$.   No further reduction is possible.

Proof. All of the equalities follow easily from the first which is immediate from Lemma 5.1.

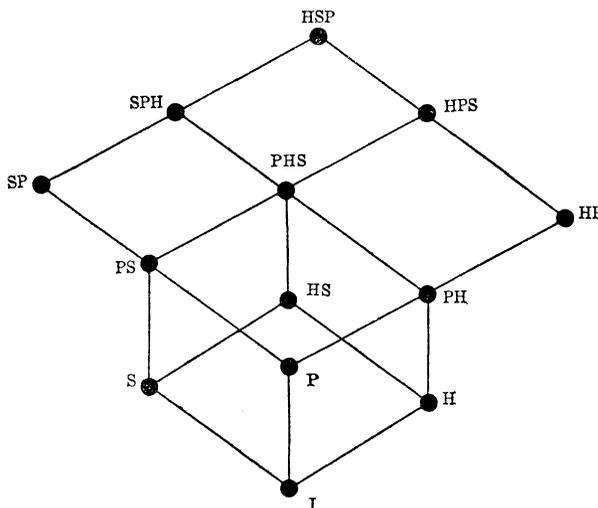To establish that no further reduction is possible we show that

FIGURE 2

(i) $S \not\leq HS$, (ii) $H \not\leq SP$, (iii) $P \not\leq HS$, (iv) $HP \not\leq SPH$, and (v) $SP \not\leq HPS$. To see that this will suffice we note that the ordering indicated by the diagram in Figure 2 is consistent with $\leq$ and that any further collapse in the diagram would contradict one of (i)-(v).

(i) $S \not\leq HP$: Let $K = \{C\}$ where $C = \{1\} \cup \{x_n : n = 1, 2, \cdots\}$ and $1 > x_1 > x_2 \cdots$. Clearly the two-element chain is a subalgebra of C so it is in $S(K)$. Suppose $C_2 \in HP(K)$. Let $f : \prod_{i \in I} C \to C_2$ be an epimorphism. Suppose $f(x) = 0$. Since C has no smallest element, we take $y \in \prod_{i \in I} C$ such that $y_i < x_i$ for each $i \in I$. We must have $f(y) = 0$. Then $(x * y)_i = x_i * y_i = y_i$ so $x * y = y$. But this gives $0 = f(y) = f(x * y) = f(x) * f(y) = 0 * 0 = 1$ a contradiction.

(ii) $H \not\leq SP$: Let $K = \{L\}$ where $L = \Pi(C_n; n = 1, 2, \cdots)$. Let $C_n = \{1, y^2, \cdots, y^n\}$ with $y^1 > y^2 > \cdots > y^n$. Let

$$J = \{x \in L : x_i = 1 \text{ for all but a finite number of } i\text{'s}\}.$$

Clearly J is a filter. For $x \in L$ we let $[x]$ denote the equivalence class in L/J which contains x. For each n we let $\bar{n}$ be the element of L defined by

$$\bar{n}_i = \begin{cases} y^n & \text{if } i \geq n \\ 1 & \text{if } i < n. \end{cases}$$

Now if $n > m$, then for any $i \geq n$ we have $\bar{n}_i = y^n < y^m = \bar{m}_i$. Thus $\bar{n}_i < \bar{m}_i$ for all but a finite number of i's. Thus $[\bar{n}] < [\bar{m}]$. Also for all but a finite number of i's we have $\bar{m}_i * \bar{n}_i = \bar{n}_i$ so $[\bar{m}] * [\bar{n}] = [\bar{n}]$. Thus we have an infinite chain $C = \{[\bar{1}], [\bar{2}], \cdots\} \leq_s L/J$. Thus $C \in SH(K)$. Now if $H(K) \subsetneq SP(K)$ we get $C \in SH(K) \subsetneq SSP(K) = SP(K)$. Since C is subdirectly irreducible, Lemma 2.6 gives $C \in S(K)$. But if $C \leq_s \Pi C_n$, we apply Lemma 2.6 again to get $C \leq_s C_n$ for some n, a contradiction.

(iii) $P \not\leq HS$: Let $K = \{C_2\}$. Then $C_2 \times C_2 \in P(K)$, but $C_2 \times C_2 \in HS(K)$.

(iv) $HP \not\leq SPH$: If $HP \leq SPH$, then $SHP \leq SPH$. We show that this is not the case. Let $K = \{C_n : n = 1, 2, \cdots\}$. Clearly $H(K) = K$ so $SPH(K) = SP(K)$. Now in (ii) we saw that $SH(\Pi C_n)$ has an infinite subdirectly irreducible member C. Thus $C \in SHP(K)$. Now if $C \in SPH(K)$ we have by Lemma 2.6 that $C \in S(K) = K$, but this is not the case. Hence $C \in SHP(K)$, but $C \notin SPH(K)$.

(v) $SP \not\leq HPS$: Let $K = \{C_2\}$. Let L consist of all cofinite subsets of an infinite set. Then $L \in SP(K)$. We note that L has no least element. It is clear that any member of $HPS(K)$ has a least element.

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