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ON POINT-FREE PARALLELISM AND WILCOX LATTICES

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A Wilcox lattice L is constructed from a complemented modular lattice A, by deleting nonzero elements of some ideal of A and by introducing in the remains L the same order as A. The lattice A is called the modular extension of L. Using the theory of parallelism in atomistic lattices, it was proved that any affine matroid lattice is an atomistic Wilcox lattice, that is, an existence theorem of the modular extension in the atomistic case. The main purpose of this paper is to extend this result to the general case, by the use of arguments on point-free parallelism.

A matroid lattice is an upper continuous atmistic lattice with the covering property. In the book [2] of Dubreil-Jacotin, Leisieur and Croisot, a generalized affine geometry is defined as a weakly modular matroid lattice of length $\geqq 4$, satisfying the Euclid's weak parallel axiom. This lattice is called an affine matroid lattice in [4] and [5]. In [2], pp. 311–314, it is proved that any affine matroid lattice has the modular extension and hence it is a Wilcox lattice. One can see that the key theorems in the proof of this result are the transitivity theorem of parallelism and theorems on the incomplete elements.

In this paper, we consider a sectionally semicomplemented lattice L with some join-dense set of modular elements (see § 1). This is a generalization of an atomistic lattice. Instead of the parallelism in matroid lattices, we use the point-free parallelism introduced by F. Maeda [6]. In § 2, we give some fundamental results on point-free parallelism. In § 3, we introduce three axioms (P 1), (P 2) and (P 3) on point-free parallelism in L, which are satisfied if L is a Wilcox lattice. In the subsequent three sections, we assume that L is weakly modular, left complemented and of length $\geqq 4$, and that L satisfies (P 1) and (P 2). The main result in § 4 is the transitivity of point-free parallelism. In § 5, we define the parallel images of incomplete elements which generalize those defined in [5], § 4. In § 6, adding the axiom (P 3) in a special case, we construct the modular extension of L and we get two main theorems 6.1 and 6.2.

1. Preliminaries.

In a lattice L, we write $(a, b)M$ when

$$(c \vee a) \wedge b = c \vee (a \wedge b) \quad \text{for} \quad c \leqq b .$$

An element $a \in L$ is called a *modular* element when $(x, a)M$ for every

$x \in L$. The elements 0, 1, if they exist, are modular elements. We denote by $\mathscr{M}$ the set of modular elements of L except 0 and 1. An element $a \in L$ is called a *strongly modular* element when every element a_1 with $a_1 \leqq a$ is modular. We denote by $\mathscr{M}_s$ the set of strongly modular elements except 0 and 1.

A lattice L is called *M-symmetric* when $(a, b)M$ implies $(b, a)M$ in L. In an M-symmetric lattice L with 0, we write $a \perp b$ when $a \wedge b = 0$ and $(a, b)M$. The following properties are easily verified (see [9]): (1) $a \perp a$ implies $a = 0$, (2) $a \perp b$ implies $b \perp a$, (3) $a \perp b$, $a_1 \leqq a$ imply $a_1 \perp b$, (4) $a \perp b$, $a \vee b \perp c$ imply $a \perp b \vee c$. If $a \perp b$ and $a \vee b \perp c$ then we have $a \perp b \vee c$ and $b \perp a \vee c$ by (4), and then we write $(a, b, c)\perp$.

A lattice L with 0 is called *left complemented* when for every $a, b \in L$ there exists $b_1 \in L$ such that

$$b_1 \leqq b,\ a \vee b = a \vee b_1,\ a \wedge b_1 = 0 \quad \text{and} \quad (b_1, a)M .$$

By [10], Theorem 2, any left complemented lattice L is M-symmetric, and hence if $a \leqq b$ in L then there exists $c \in L$ such that $a \vee c = b$ and $a \perp c$. Moreover it is easy to show that L is relatively complemented.

A lattice with 0 is called *weakly modular* when $a \wedge b \neq 0$ implies $(a, b)M$.

A subset S of a lattice L is called *join-dense* when every $a \in L$ is the join of some elements in S. We write $a \prec b$ when $a < b$ and there is no element c such that $a < c < b$. In a lattice L with 0, an element p is called an *atom* when $0 \prec p$. Evidently any atom is a strongly modular element. The set of atoms of L is denoted by Ω. A lattice L with 0 is called *atomistic* when Ω is join-dense. We say that L has the *covering property* when $p \not\leqq a$, $p \in \Omega$ imply $a \prec a \vee p$. It is easily seen that this property is equivalent to $(p, a)M$ for every $p \in \Omega$ and $a \in L$. Hence any M-symmetric lattice with 0 has the covering property.

A lattice L with 0 is called *semicomplemented* when for every $a \in L$ (with $a \neq 1$ if L has 1) there exists $c \in L$ such that $c \neq 0$ and $c \wedge a = 0$. L is called *sectionally semicomplemented* (in symbols, an *SSC* lattice) when every interval $L[0, b]$ is semicomplemented, that is, for $a, b \in L$ with $a < b$ there exists $c \in L$ such that $0 \neq c \leqq b$ and $c \wedge a = 0$. Let $\mathscr{C}$ be a set of nonzero elements of L. L is called *$\mathscr{C}$-SSC* when for $a, b \in L$ with $a < b$ there exists $c \in \mathscr{C}$ such that $c \leqq b$ and $c \wedge a = 0$. The following statements are easily verified: (1) If p is an atom of a $\mathscr{C}$-*SSC* lattice then p must be in $\mathscr{C}$, (2) L is atomistic if and only if L is Ω-*SSC*, (3) L is $\mathscr{C}$-*SSC* if and only if L is *SSC* and $\mathscr{C}$ is join-dense in L.

The length of a lattice L is the least upper bound of the lengths of the chains in L (see [1], p. 5).

LEMMA 1.1.[1] *Let $a < b$ in an M-symmetric lattice L. If there exists a connected chain $a = a_0 \prec a_1 \prec \cdots \prec a_n = b$ of length n, then all chains from a to b have length $\leqq n$.*

Proof. If $n = 1$, the lemma is trivial. Suppose this is true for $n-1$. Let $a = a_0 \prec a_1 \prec \cdots \prec a_n = b$ and $a = b_0 < b_1 < \cdots < b_m = b$, and we shall show $m \leqq n$. When $a_1 \leqq b_1$, we have $a_1 < b_2 < \cdots < b_m = b$. Hence $m - 1 \leqq n - 1$ by the induction hypothesis. When $a_1 \nleqq b_1$, let $r(<m)$ be greatest such that $a_1 \nleqq b_r$. For $i = 1, \cdots, r$, we have $a_1 \wedge b_i = a$ and $(b_i, a_1)M$, since a_1 covers a. Then $(a_1, b_i)M$ by M-symmetry. If $a_1 \vee b_{i-1} = a_1 \vee b_i$, then

$$b_i = (a_1 \vee b_i) \wedge b_i = (b_{i-1} \vee a_1) \wedge b_i = b_{i-1} \vee (a_1 \wedge b_i) = b_{i-1} ,$$

a contradiction. Hence we have

$$a_1 = a_1 \vee b_0 < a_1 \vee b_1 < \cdots < a_1 \vee b_r \leqq b_{r+1} < b_{r+2} < \cdots < b_m = b .$$

Hence, by the induction hypothesis, we get $m - 1 \leqq n - 1$. Therefore the lemma is true for n.

It follows from this lemma that if the length of $L[a, b] > n$ then there is no connected chain of length n from a to b.

2. Point-free parallelism. Let L be a lattice with 0, and let a and b be nonzero elements of L. If $a \wedge b = 0$ and there exists $m \in \mathscr{M}$ such that $m \leqq a$ and $m \vee b = a \vee b$ then we write $a <|_{(m)} b$. If $a <|_{(m)} b$ and $b <|_{(n)} a$ then we write $a \,||_{(m,n)}\, b$ and we say that a and b are *parallel* with axes m and n. We remark that $a \,||_{(m,n)}\, b$ if and only if $a \wedge b = 0$ and there exist $m, n \in \mathscr{M}$ such that $m \leqq a$, $n \leqq b$ and $m \vee b = a \vee n$.

LEMMA 2.1. *If $a <|_{(m)} b$ then m is maximal in the set $\{n \in \mathscr{M}: n \leqq a\}$.*

Proof. Let n be an element of $\mathscr{M}$ with $m \leqq n \leqq a$. Then since n is modular and $n \wedge b \leqq a \wedge b = 0$, we have

$$n = (a \vee b) \wedge n = (m \vee b) \wedge n = m \vee (b \wedge n) = m .$$

LEMMA 2.2. *Let $a <|_{(m)} b$.*

(i) *If $m \leqq a_1 < a$ then $a_1 <|_{(m)} b$.*

[1] The author is indebted to the referee for this lemma.

(ii) *If* $b < b_1$ *and* $a \wedge b_1 = 0$ *then* $a < |_{(m)}\, b_1$.
(iii) *If* $b < b_1$ *and* $m < b_1$ *then* $a < b_1$.

These statements can easily be proved.

LEMMA 2.3. *In an M-symmetric lattice with* 0, *if* $a \,||_{(m,n)}\, b$ *and* $a = m$ *then* $b = n$.

Proof. Since L is M-symmetric, we have $(m, b)M$. Since $b \leqq a \vee n = m \vee n$ and $m \wedge b = 0$, we have

$$b = (n \vee m) \wedge b = n \vee (m \wedge b) = n\,.$$

LEMMA 2.4. *In an M-symmetric lattice with* 0,
(i) *if* $a \wedge b = 0$ *and* $a \vee b \perp c$ *then* $a \wedge (b \vee c) = 0$,
(ii) *if* $a < |_{(m)}\, b$ *and* $a \vee b \perp c$ *then* $a < |_{(m)}\, b \vee c$.

Proof. (i) Since $c \perp a \vee b$, we have $(b \vee c) \wedge (a \vee b) = b \vee \{c \wedge (a \vee b)\} = b$. Hence $a \wedge (b \vee c) = a \wedge b = 0$.

(ii) We have $a \wedge (b \vee c) = 0$ by (i). Moreover, $m \vee b \vee c = a \vee b \vee c$ by $a < |_{(m)}\, b$. Hence $a < |_{(m)}\, b \vee c$.

LEMMA 2.5. *In a weakly modular lattice,*
(i) *if* $a \wedge b = 0$, $m \leqq a$, $n \leqq b$ *where* $m, n \in \mathscr{M}$ *then* $a \wedge (m \vee b) \,||_{(m,n)}\, b \wedge (n \vee a)$,
(ii) *if* $a < |_{(m)}\, b$ *and* $n \leqq b$ *where* $n \in \mathscr{M}$ *then* $a \,||_{(m,n)}\, b \wedge (n \vee a)$.

Proof. (i) We have $a \wedge (m \vee b) \wedge b \wedge (n \vee a) = a \wedge b = 0$. Since $a \wedge (m \vee b) \geqq m > 0$, we have $(a, m \vee b)M$ and similarly $(b, n \vee a)M$. Hence

$$n \vee \{a \wedge (m \vee b)\} = (n \vee a) \wedge (m \vee b) = m \vee \{b \wedge (n \vee a)\}\,.$$

Thus, $a \wedge (m \vee b) \,||_{(m,n)}\, b \wedge (n \vee a)$. (ii) follows from (i) evidently.

LEMMA 2.6. *In a weakly modular, M-symmetric lattice L, if* $a \,||_{(m,n)}\, b$, *then there exist mutually inverse, isomorphic mappings between the intervals* $L[m, a]$ *and* $L[n, b]$, *which are defined by* $a_1 \to b \wedge (n \vee a_1)$ *for* $a_1 \in L[m, a]$ *and* $b_1 \to a \wedge (m \vee b_1)$ *for* $b_1 \in L[n, b]$.

Proof. [6], Theorem 2.12.

LEMMA 2.7. *Let* $a < |_{(m)}\, c$ *in a weakly modular, M-symmetric,* $\mathscr{M}$*-SSC lattice L with* 1. *If* $n \in \mathscr{M}$, $n \wedge a = 0$ *and* $n \vee a < 1$ *then there exists* $b \in L$ *such that* $a \,||_{(m,n)}\, b$.

Proof. (i) When $c \nleqq n \vee a$, we have $(n \vee a) \wedge c < c$. There exists $n' \in \mathscr{M}$ such that $n' \leqq c$ and $n' \wedge (n \vee a) = 0$. Putting $c' = c \wedge (n' \vee a)$, we have $a \|_{(m,n')} c'$ by Lemma 2.5 (ii). Since $n \perp a$ and $n \vee a \perp n'$, we have $n \perp a \vee n' = a \vee c'$. Hence, by Lemma 2.4 (ii), we have $a <|_{(m)} c' \vee n$. Putting $b = (c' \vee n) \wedge (n \vee a)$, we have $a \|_{(m,n)} b$ by Lemma 2.5 (ii).

(ii) When $c \leqq n \vee a(<1)$, we take $n' \in \mathscr{M}$ with $(n \vee a) \wedge n' = 0$. Then $c \vee a \perp n'$, since $(c \vee a) \wedge n' \leqq (n \vee a) \wedge n' = 0$. Putting $c' = c \vee n'$, we have $a <|_{(m)} c'$ by Lemma 2.4 (ii). Since $c' \nleqq n \vee a$, by (i) there exists b such that $a \|_{(m,n)} b$.

3. **Parallelism in Wilcox lattices.** A *Wilcox lattice* L is constructed in the following manner (see [9] and [6]). Let Λ be a given complemented modular lattice whose lattice operations are denoted by $\cup$ and $\cap$. Let S be a fixed proper ideal of Λ with 0 deleted (S may be empty). As ordering of the set $L \equiv \Lambda - S$ we use that of Λ. Then L is a lattice having the following properties, where $a, b \in L$ and we denote the lattice operations in L by $\vee$ and $\wedge$:

(W 1) $a \vee b = a \cup b$ for all $a, b \in L$,

(W 2) $a \wedge b = \begin{cases} a \cap b & \text{if } a \cap b \in L \\ 0 & \text{if } a \cap b \in S, \end{cases}$

(W 3) $(a, b)M$ in L if and only if $a \cap b \in L$.

By (W 2) and (W 3), L is weakly modular and M-symmetric. Moreover, $a \perp b$ in L if and only if $a \cap b = 0$.

We call Λ the *modular extension* of L. An element in S is called an *imaginary element* for L, and when S has a greatest element i then it is called the *imaginary unit*. A nonzero element a of L is called *regular* when $a \cap u = 0$ for every $u \in S$. The set of regular elements is denoted by R. Evidently, S is empty if and only if $1 \in R$.

LEMMA 3.1. *Let $L \equiv \Lambda - S$ be a semicomplemented Wilcox lattice, and let a be an element of L with $0 < a < 1$. The following three statements are equivalent.*

(α) *a is regular.* (β) *a is modular.* (γ) *a is strongly modular.*

Proof. $(\alpha) \Rightarrow (\gamma)$. Let $b \in L$ and $a_1 \leqq a$ in L. If $b \cap a_1 \in S$, then, since a is regular, we have $0 = a \cap (b \cap a_1) = b \cap a_1$, a contradiction. Hence $b \cap a_1 \in L$, and then $(b, a_1)M$ by (W 3). Therefore a is strongly modular. $(\gamma) \Rightarrow (\beta)$ is trivial. $(\beta) \Rightarrow (\alpha)$. Since L is semicomplemented, there exists $c \in L$ such that $c \neq 0$ and $c \wedge a = 0$. Since a is modular, we have $c \cap a \in L$ by (W 3). Hence $c \cap a = c \wedge a = 0$ by (W 2). For an arbitrary $u \in S$, we put $b = c \cup (a \cap u)$. Since $b \in L$, we have $(b, a)M$ and hence $b \cap a \in L$. By the modularity of Λ,

$$b \cap a = \{(a \cap u) \cup c\} \cap a = (a \cap u) \cup (c \cap a) = a \cap u \,.$$

Hence $a \cap u \in L$, and then $a \cap u = 0$. Therefore a is regular.

LEMMA 3.2.[2] (i) *A Wilcox lattice $L \equiv \Lambda - S$ is left complemented if and only if it satisfies the following condition:*

(1) *If $a < b$ in L then there exists $c \in L$ such that $0 \neq c \leqq b$, $c \wedge a = 0$ and $(c, a)M$.*

(ii) *If a Wilcox lattice is $\mathscr{M}$-SSC, then it is left complemented.*

Proof. If L is left complemented then evidently (1) is satisfied. Conversely, assume that (1) is satisfied in L. Let $a, b \in L$. We shall show that

(2) There exists $b_1 \in L$ such that $b = (a \cap b) \cup b_1$ and $(a \cap b) \cap b_1 = 0$.

When $a \cap b \in S$, we take a complement b_1 of $a \cap b$ in the interval $\Lambda[0, b]$. Then $b_1 \in L$, since otherwise $b = b_1 \cup (a \cap b) \in S$, a contradiction. Hence b_1 has the desired property. When $a \cap b \in L$, we may assume $a \cap b < b$. By (1) there exists $c \in L$ such that $0 \neq c \leqq b$, $c \wedge (a \cap b) = 0$ and $(c, a \cap b)M$. Then we have $c \cap (a \cap b) = 0$. Let λ be a complement of $(a \cap b) \cup c$ in $\Lambda[0, b]$, and put $b_1 = c \cup \lambda$. Since $(a \cap b, c, \lambda)\perp$ in Λ, b_1 is a complement of $a \cap b$ in $\Lambda[0, b]$. Moreover, since $0 \neq c \in L$, we have $b_1 \in L$. Thus (2) has been proved.

By (2), we have $a \vee b_1 = a \cup b_1 = a \cup b = a \vee b$, and since $a \cap b_1 = a \cap b \cap b_1 = 0$, we have $a \wedge b_1 = 0$ and $(b_1, a)M$. Therefore L is left complemented. The second statement of the lemma follows from the first one.

REMARK 3.1. Let $L \equiv \Lambda - S$ be a Wilcox lattice with the imaginary unit i, and assume that L is *SSC*. For any nonzero element a of L, there exists $a_1 \in \mathscr{M}$ such that $a_1 \leqq a$; because a complement a_1 of $a \cap i$ in $\Lambda[0, a]$ is regular since $a_1 \cap i = 0$, and hence $a_1 \in \mathscr{M}$ by Lemma 3.1. Therefore L is $\mathscr{M}$-*SSC* and then it is left complemented.

DEFINITION. An element a of a Wilcox lattice $L \equiv \Lambda - S$ is called *irregular* when there exist $m \in R$ and $u \in S$ such that $a = m \cup u$ (see [6]). We call u an *imaginary part* of a, and denote it by $\iota(a)$; while m is called a *regular part* of a. For a regular element a we put $\iota(a) = 0$. It is easy to show that if a Wilcox lattice L has the imaginary unit i, then every nonzero element of L is either regular or irregular.

LEMMA 3.3. *If a is an irregular element of a Wilcox lattice $L \equiv \Lambda - S$, then an imaginary part $\iota(a)$ is uniquely determined by*

[2] This lemma is suggested by the referee.

a, and it is the greatest element of S contained in a.

Proof. Let $a = m \cup u$, $m \in R$ and $u \in S$. If $v \in S$ and $v \leqq a$, then since $u \cup v \in S$, we have $m \cap (u \cap v) = 0$. Hence by the modularity of Λ,

$$u = u \cup \{m \cap (u \cup v)\} = (u \cup m) \cap (u \cup v) = a \cap (u \cup v) \geqq v .$$

REMARK 3.2. A regular part of an irregular element a is a complement of $\iota(a)$ in $\Lambda[0, a]$. This is not necessarily unique. It is easy to show by Lemma 3.3 that a Wilcox lattice L has the imaginary unit if and only if the unit 1 of L is irregular.

LEMMA 3.4. *Let a and b be nonzero elements of a Wilcox lattice $L \equiv \Lambda - S$. The following statements are equivalent.*
(α) *$a <|_{(m)} b$ with $a \neq m \in R$.*
(β) *$a \cap b \in S$ and $a = m \cup (a \cap b)$ with $m \in R$.*
Each of (α) and (β) implies that a is irregular and $\iota(a) = a \cap b$.

Proof. [6], Theorem 3.8.

REMARK 3.3. In a Wilcox lattice, if $a <|_{(m)} b$ with $m \in R$ then $a = m \cup \iota(a)$ by Lemma 3.4 and by the fact that $a = m$ implies $\iota(a) = 0$.

LEMMA 3.5. *Let a and b be irregular elements of a Wilcox lattice $L \equiv \Lambda - S$. If $a <|_{(m)} b$ with $m \in R$ then $\iota(a) \leqq \iota(b)$, and if $a \,||_{(m,n)}\, b$ with $m, n \in R$ then $\iota(a) = \iota(b)$.*

Proof. This lemma is evident by Lemmas 3.3 and 3.4.

LEMMA 3.6. *In a Wilcox lattice $L \equiv \Lambda - S$, let $a = m \cup u$ and $b = n \cup u$ where $m, n \in R$ and $u \in S$. If either $a \wedge n = 0$ or $b \wedge m = 0$ then $a \,||_{(m,n)}\, b$.*

Proof. [6], Lemma 3.10.

PARALLEL AXIOMS. Let $\mathscr{C}$ be a subset of $\mathscr{M}$. We consider the following three parallel axioms with respect to $\mathscr{C}$.

(P 1) If $a \,||_{(m_1,n)}\, b_1$ and $a \,||_{(m_2,n)}\, b_2$ where $m_1, m_2, n \in \mathscr{C}$, then $b_1 = b_2$.

(P 2) If $a_1 \,||_{(m,n)}\, b_1$ and $a_2 \,||_{(m,n)}\, b_2$ where $m, n \in \mathscr{C}$ and if $(a_1 \vee a_2) \wedge n = 0$, then $a_1 \vee a_2 \,||_{(m,n)}\, b_1 \vee b_2$.

(P 3) If $a_1 <|_{(m)} b_1$ and $a_2 <|_{(m)} b_2$ where $m \in \mathscr{C}$ and if $a_1 \vee a_2 = 1$, then for any a with $m < a < 1$ there exists b such that $a <|_{(m)} b$.

LEMMA 3.7. *If $L \equiv \Lambda - S$ is a semicomplemented Wilcox lattice* *then* (P 1), (P 2) *and* (P 3) *are satisfied for any subsets* $\mathscr{C}$ *of* $\mathscr{M}$.

Proof. (i) We have $\mathscr{C} \subset R$ by Lemma 3.1. Let $a \|_{(m_1, n)} b_1$ and $a \|_{(m_2, n)} b_2$ where $m_1, m_2, n \in \mathscr{C} \subset R$. It follows from Remark 3.3 and Lemma 3.5 that $b_i = n \cup \iota(b_i)$ $(i = 1, 2)$ and $\iota(b_1) = \iota(a) = \iota(b_2)$. Hence $b_1 = b_2$.

(ii) Let $a_1 \|_{(m,n)} b_1$ and $a_2 \|_{(m,n)} b_2$ where $m, n \in \mathscr{C}$, and assume that $(a_1 \vee a_2) \wedge n = 0$. We have $a_i = m \cup \iota(a_i)$, $b_i = n \cup \iota(b_i)$ and $\iota(a_i) = \iota(b_i)$ $(i = 1, 2)$. We put $u = \iota(a_1) \cup \iota(a_2) = \iota(b_1) \cup \iota(b_2)$. Then we have $u \in S$, $a_1 \vee a_2 = a_1 \cup a_2 = m \cup u$ and $b_1 \vee b_2 = b_1 \cup b_2 = n \cup u$. Hence $a_1 \vee a_2 \|_{(m,n)} b_1 \vee b_2$ by Lemma 3.6.

(iii) Let $a_1 <|_{(m)} b_1$ and $a_2 <|_{(m)} b_2$ where $m \in \mathscr{C}$, and assume that $a_1 \vee a_2 = 1$. Since $a_i = m \cup \iota(a_i)$, putting $u = \iota(a_1) \cup \iota(a_2)$, we have $u \in S$ and $m \cup u = a_1 \cup a_2 = a_1 \vee a_2 = 1$. If $m < a < 1$, then we have $m \cup (u \cap a) = (m \cup u) \cap a = a$, and moreover $u \cap a \in S$, since otherwise $u \cap a = 0$ and then $m = a$. Since L is semicomplemented, there exists $c \in L$ with $c \neq 0$ and $c \wedge a = 0$. Then $c \cap a \in S \smallsetminus \{0\}$. Putting $b = c \cup (u \cap a)$, we have

$$a \cap b = \{(u \cap a) \cup c\} \cap a = (u \cap a) \cup (c \cap a) \in S\ .$$

Moreover,

$$a = m \cup (u \cap a) = m \cup (u \cap a) \cup (c \cap a) = m \cup (a \cap b)\ .$$

Hence $a <|_{(m)} b$ by Lemma 3.4.

4. **Transitivity of parallelism.** We consider the following condition for a subset $\mathscr{C}$ of $\mathscr{M}$:

(C 1) If $m \in \mathscr{C}$ and $0 < m_1 < m$ then $m_1 \in \mathscr{C}$.

For instance, in a lattice with 0, both the set Ω of atoms and the set $\mathscr{M}_s$ of strongly modular elements satisfy (C 1), and also the set of regular elements of a Wilcox lattice does. Evidently, if $\mathscr{C}$ satisfies (C 1) then $\mathscr{C} \subset \mathscr{M}_s$.

In this and the next sections, let L be a weakly modular, left complemented lattice, with 0 and 1, of length $\geqq 4$ (may be infinite), and assume that for some fixed subset $\mathscr{C}$ of $\mathscr{M}$, satisfying (C 1), L is $\mathscr{C}$-*SSC* and L satisfies the axioms (P 1) and (P 2) with respect to $\mathscr{C}$.

For any $m \in \mathscr{C}$, $L[m, 1]$ is a modular lattice, since L is weakly modular. We put $\mathscr{C}' = \{m \in \mathscr{C};$ the length of $L[m, 1] \geqq 3\}$. Evidently, $\mathscr{C}'$ satisfies (C 1). Since the length of $L \geqq 4$, it is easily seen that for any $m \in \mathscr{C}$ there exists $m_1 \in \mathscr{C}'$ such that $m_1 \leqq m$. Hence L is $\mathscr{C}'$-*SSC*. Therefore, taking $\mathscr{C}'$ instead of $\mathscr{C}$, we may assume that

(C 2) the length of $L[m, 1] \geqq 3$ for every $m \in \mathscr{C}$.

LEMMA 4.1. *In* L, *if* $a \,||_{(m_1,n)}\, b_1$, $a < |_{(m_2)}\, b_2$ *and* $n \leqq b_2$ *where* m_1, m_2, $n \in \mathscr{C}$ *then* $b_1 \leqq b_2$.

Proof. We have $a \,||_{(m_2,n)}\, b_2 \wedge (n \vee a)$ by Lemma 2.5 (ii). Hence by (P 1), $b_1 = b_2 \wedge (n \vee a) \leqq b_2$.

LEMMA 4.2. *In* L, *if* $a \,||_{(m_1,n_1)}\, b_1$, $a \,||_{(m_2,n_2)}\, b_2$ *where* $m_1, m_2, n_1, n_2 \in \mathscr{C}$ *and* $(a, n_1, n_2)\perp$ *then* $b_1 \,||_{(n_1,n_2)}\, b_2$ *and* $a \wedge (b_1 \vee b_2) = 0$.

Proof. Since $n_1 \perp a \vee n_2$, we have $b_1 \wedge b_2 \leqq (a \vee n_1) \wedge (a \vee n_2) = a \vee \{n_1 \wedge (a \vee n_2)\} = a$. Hence $b_1 \wedge b_2 = a \wedge b_1 \wedge b_2 = 0$. Next, since $a \vee b_1 = a \vee n_1$, we have $a \vee b_1 \perp n_2$. By Lemma 2.4 (ii), we have $a < |_{(m_1)}\, b_1 \vee n_2$. Hence $b_2 \leqq b_1 \vee n_2$ by Lemma 4.1. Similarly $b_1 \leqq b_2 \vee n_1$. Therefore, $b_1 \vee n_2 = b_2 \vee n_1$, and hence $b_1 \,||_{(n_1,n_2)}\, b_2$. Moreover, $a \wedge (b_1 \vee b_2) = a \wedge (b \vee n_2) = 0$.

DEFINITION. Let $m \in \mathscr{C}$. An element a of L is called an *incomplete* element over m when there exists $b \in L$ such that $a < |_{(m)}\, b$. The set of all incomplete elements over m is denoted by I_m. It follows from Lemma 4.2 (i) that if $a \in I_m$ and $m \leqq a_1 \leqq a$ then $a_1 \in I_m$. The following theorem shows a fundamental property of incomplete elements.

THEOREM 4.1. *If* $a \in I_m$ *and* $n \wedge a = 0$ *where* $m, n \in \mathscr{C}$, *then there exists a unique element* b *such that* $a \,||_{(m,n)}\, b$.

Proof. The uniqueness follows from (P 1). We shall show the existence. When $a = m$, we may take $b = n$. When $n \vee a < 1$, the existence of b follows from Lemma 2.7. Now we assume that $m < a$ and $n \vee a = 1$.

When n is not an atom, there exist $n_1, n_2 \in \mathscr{C}$ such that $n = n_1 \vee n_2$ and $n_1 \perp n_2$, since L is left complemented and $\mathscr{C}$ satisfies (C 1). Since $(a, n_1, n_2)\perp$, we have $n_i \vee a < 1 (i = 1, 2)$, and hence there exist b_1, b_2 such that $a \,||_{(m,n_i)}\, b_i$ by Lemma 2.7. By Lemma 4.2 we have $a \wedge (b_1 \vee b_2) = 0$. Moreover, $a \vee n = a \vee n_1 \vee n_2 = m \vee b_1 \vee b_2$. Hence we have $a \,||_{(m,n)}\, b_1 \vee b_2$.

When n is an atom, we have $a \prec a \vee n = 1$ by the covering property. Since $L[m, 1]$ is a modular lattice of length $\geqq 3$ by (C 2), there exists a_1 such that $m < a_1 < a$. Since L is left complemented, there exist nonzero elements c_1 and c_2 such that $m \vee c_1 = a_1$, $m \perp c_1$, $a_1 \vee c_2 = a$ and $a_1 \perp c_2$. Since $(a_1, c_2, n)\perp$, we have $n \vee a_1 < 1$. Putting $a_2 = m \vee c_2$, since $(a_2, c_1, n)\perp$, we have $n \vee a_2 < 1$. Since a_1, $a_2 \in I_m$, there exist b_1, b_2 such that $a_i \,||_{(m,n)}\, b_i$. Then by (P 2) we have $a \,||_{(m,n)}\, b_1 \vee b_2$.

COROLLARY. *If* $a_1, a_2 \in I_m$ *where* $m \in \mathscr{C}$ *and if* $a_1 \vee a_2 < 1$, *then* $a_1 \vee a_2 \in I_m$.

Proof. Since L is $\mathscr{C}$-*SSC*, there exists $n \in \mathscr{C}$ such that $(a_1 \vee a_2) \wedge n = 0$. By Thorem 4.1, there exist b_1 and b_2 such that $a_i \,||_{(m,n)}\, b_i (i = 1, 2)$. Hence $a_1 \vee a_2 \,||_{(m,n)}\, b_1 \vee b_2$ by (P 2).

LEMMA 4.3. *If* $a \,||_{(m,n_1)}\, b_1$ *and* $a \,||_{(m,n_2)} b_2$ *where* $m, n_1, n_2 \in \mathscr{C}$ *and* $n = n_1 \vee n_2 \in \mathscr{C}$ *and if* $a \wedge n = 0$, *then* $a \,||_{(m,n)}\, b_1 \vee b_2$.

Proof. It follows from Theorem 4.1 that there exists b such that $a \,||_{(m,n)}\, b$. Since $a \,||_{(m,n_1)}\, b_1$, $a <|_{(m)}\, b$ and $n_1 \leqq n \leqq b$, we have $b_1 \leqq b$ by Lemma 4.1. Similarly $b_2 \leqq b$. Hence $a \wedge (b_1 \vee b_2) \leqq a \wedge b = 0$. Moreover $a \vee n = a \vee n_1 \vee n_2 = m \vee b_1 \vee b_2$. Hence $a \,||_{(m,n)}\, b_1 \vee b_2$.

LEMMA 4.4. *If* $a \,||_{(m,n_1)}\, b_1$ *and* $a \,||_{(m,n_2)}\, b_2$ *where* $m, n_1, n_2 \in \mathscr{C}$ *and if* $b_1 \wedge n_2 = 0$ *then* $b_2 \wedge n_1 = 0$.

Proof. If $b_2 \wedge n_1 \neq 0$, then putting $n_1' = b_2 \wedge n_1$, we have $n_1' \in \mathscr{C}$, $n_1' \leqq b_1$ and $n_1' \leqq b_2$. Since $a <|_{(m)}\, b_i \ (i = 1, 2)$, we have $a \,||_{(m,n_1')}\, b_i \wedge (n_1' \vee a)$ by Lemma 2.5 (ii). Hence $b_1 \wedge (n_1' \vee a) = b_2 \wedge (n_1' \vee a)$ by (P 1), and hence

$$n_2 \wedge (n_1' \vee a) = n_2 \wedge b_2 \wedge (n_1' \vee a) = n_2 \wedge b_1 \wedge (n_1' \vee a) = 0 .$$

Thus we have $(a, n_1', n_2)\perp$. It follows from Lemma 4.2 that $b_2 \wedge (n_1' \vee a) \,||_{(n_1',n_2)}\, b_2$, which is a contradiction since $0 < b_2 \wedge (n_1' \vee a) \leqq b_2$.

THEOREM 4.2. (Transitivity of parallelism) *If* $a \,||_{(m_1,n_1)}\, b_1$ *and* $a \,||_{(m_2,n_2)}\, b_2$ *where* $m_1, m_2, n_1, n_2 \in \mathscr{C}$ *and if* $b_1 \wedge n_2 = 0$, *then* $b_1 \,||_{(n_1,n_2)}\, b_2$.

Proof. (i) When $(a \vee b_1) \wedge n_2 = 0$, we have $(a, n_1, n_2)\perp$. Hence it follows from Lemma 4.2 that $b_1 \,||_{(n_1,n_2)}\, b_2$.

(ii) When $n_2 \leqq a \vee b_1 < 1$, there exists $n \in \mathscr{C}$ such that $a \vee b_1 \perp n$. Since $n \wedge a = 0$, it follows from Theorem 4.1 that there exists b such that $a \,||_{(m_1,n)}\, b$. Since $(a \vee b_1) \wedge n = 0$, we have $b_1 \,||_{(n_1,n)}\, b$ by (i). On the other hand, since $a \vee b_2 = a \vee n_2 \leqq a \vee b_1$, we have $(a \vee b_2) \wedge n = 0$. Hence $b_2 \,||_{(n_2,n)}\, b$ by (i). Now, we have $(n_2, b_1, n)\perp$, since $n_2 \vee b_1 \leqq a \vee b_1 \perp n$. Since $b \vee n_1 = b_1 \vee n$, we have $(b, n_1, n_2)\perp$. Hence, $b_1 \,||_{(n_1,n_2)}\, b_2$ by Lemma 4.2.

(iii) When $a \vee b_1 < 1$, we put $n_1' = (a \vee b_1) \wedge n_2$. Since L is left complemented, there exists n_2'' such that $n_2 = n_2' \vee n_2''$ and $n_2' \perp n_2''$. If $n_2' = 0$ or $n_2'' = 0$ then we have $b_1 \,||_{(n_1,n_2)}\, b_2$ by (i) or (ii). Let $n_2' \neq 0$ and $n_2'' \neq 0$. Then $n_2', n_2'' \in \mathscr{C}$, and by Theorem 4.1 there exist b_2' and b_2'' such that $a \,||_{(m_2,n_2')}\, b_2'$ and $a \,||_{(m_2,n_2'')}\, b_2''$. Since $a \,||_{(m_2,n_2)}\, b_2' \vee b_2''$ by

Lemma 4.3, we have $b_2 = b_2' \vee b_2''$ by (P 1). Since $n_2' \leqq a \vee b_1$, we have $b_1 \|_{(n_1,n_2')} b_2'$ by (ii), and since $n_2'' \wedge (a \vee b_1) = n_2'' \wedge (a \vee b_1) \wedge n_2 = n_2'' \wedge n_2' = 0$, we have $b_1 \|_{(n_1,n_2'')} b_2''$ by (i). By Lemma 4.3 again, we get $b_1 \|_{(n_1,n_2)} b_2' \vee b_2'' = b_2$.

(iv) Let $a \vee b_1 = 1$. When n_1 is not an atom, there exist n_1', $n_1'' \in \mathscr{C}$ such that $n_1 = n_1' \vee n_1''$ and $n_1' \perp n_1''$. As above there exist b_1' and b_1'' such that $a \|_{(m_1,n_1')} b_1'$, $a \|_{(m_1,n_1'')} b_1''$ and $b_1 = b_1' \vee b_1''$. Since $b_1' \wedge n_2 = 0$ and $a \vee b_1' = a \vee n_1' < 1$, we have $b_1' \|_{(n_1',n_2)} b_2$. Similarly, $b_1'' \|_{(n_1',n_2)} b_2$. Since $b_2 \wedge n_1 = 0$ by Lemma 4.4, we have $b_1 \|_{(n_1,n_2)} b_2$ by Lemma 4.3. When n_1 is an atom, we have $a \prec a \vee n_1 = 1$. Then, by the same way as in the proof of Theorem 4.1, there exist elements a' and a'' such that $m_1 \leqq a' \wedge a''$, $a' \vee a'' = a$, $a' \vee n_1 < 1$ and $a'' \vee n_1 < 1$. Since $a' < |_{(m_1)} b_i$ and $a'' < |_{(m_1)} b_i (i = 1, 2)$, by Lemma 2.5 (ii) there exist $b_i', b_i'' \leqq b_i$ such that $a' \|_{(m_1,n_i)} b_i'$ and $a'' \|_{(m_1,n_i)} b_i''$. Since $a \wedge (b_i' \vee b_i'') \leqq a \wedge b_i = 0$ and

$$a \vee n_i = a' \vee a'' \vee n_i = m_1 \vee b_i' \vee b_i'',$$

we have $a \|_{(m_1,n_i)} b_i' \vee b_i''$. Hence $b_i = b_i' \vee b_i''$ by (P 1). On the other hand, since $a' \vee b_1' = a' \vee n_1 < 1$, it follows from (iii) that $b_1' \|_{(n_1,n_2)} b_2'$. Similarly, $b_1'' \|_{(n_1,n_2)} b_2''$. Hence $b_1 \|_{(n_1,n_2)} b_2$ by (P 2).

COROLLARY. *If $a \|_{(m,n)} b$, $b \|_{(n,r)} c$ and $c \|_{(r,m)} d$ where $m, n, r \in \mathscr{C}$ then $a = d$.*

Proof. Since $b \|_{(n,r)} c$, $b \|_{(n,m)} a$ and $c \wedge m = 0$, it follows from Theorem 4.2 that $c \|_{(r,m)} a$. Hence $a = d$ by (P 1).

LEMMA 4.5. *If $a \|_{(m,n_1)} b_1$, $b_1 \|_{(n_1,r)} c_1$, $a \|_{(m,n_2)} b_2$ and $b_2 \|_{(n_2,r)} c_2$ where $m, n_1, n_2, r \in \mathscr{C}$ then $c_1 = c_2$.*

Proof. (i) When $r \wedge a = 0$, it follows from Theorem 4.2 that $a \|_{(m,r)} c_1$ and $a \|_{(m,r)} c_2$. Hence $c_1 = c_2$ by (P 1).

(ii) When $n_1 \leqq n_2$ and $r \leqq a$, then since $b_1 \|_{(n_1,r)} b$, $b_1 < |_{(n_1)} a$ and $r \leqq a$, we have $c_1 \leqq a$ by Lemma 4.1. Moreover we have $b_1 \leqq b_2$, since $a \|_{(m,n_1)} b_1$, $a < |_{(m)} b_2$ and $n_1 \leqq n_2 \leqq b_2$. Since $c_1 \|_{(r,n_1)} b_1$ and $c_1 \wedge b_2 \leqq a \wedge b_2 = 0$, we have $c_1 < |_{(r)} b_2$ by Lemma 2.2 (ii). Putting $b_2' = b_2 \wedge (n_2 \vee c_1)$, we have $c_1 \|_{(r,n_2)} b_2'$ by Lemma 2.5 (ii). Since $c_1 \leqq a$ and $a \wedge b_2' \leqq a \wedge b_2 = 0$, we have $b_2' < |_{(n_2)} a$ by Lemma 2.2 (ii). Moreover, since $b_1 \leqq n_1 \vee c_1 \leqq n_2 \vee c_1$, we have $b_1 \leqq b_2'$, and hence $a \leqq m \vee b_1 \leqq m \vee b_2'$. Therefore $a \|_{(m,n_2)} b_2'$. By (P 1), we have $b_2 = b_2'$, whence $b_2 \|_{(n_2,r)} c_1$. By (P 1) again, we have $c_1 = c_2$.

(iii) When $n_1 \leqq n_2$, we put $r' = r \wedge a$ and take r'' such that $r = r' \vee r''$ and $r' \wedge r'' = 0$. If $r' = 0$ or $r'' = 0$, then we have $c_1 = c_2$ by (i) or (ii). Let $r' \neq 0$ and $r'' \neq 0$. Then $r', r'' \in \mathscr{C}$. By Theorem

4.1 there exist c'_i and $c''_i (i = 1, 2)$ such that $b_i \|_{(n_i, r')} c'_i$ and $b_i \|_{(n_i, r'')} c''_i$. Then $b_i \|_{(n_i, r)} c'_i \vee c''_i$ by Lemma 4.3. Hence $c_i = c'_i \vee c''_i$ by (P 1). Since $r' \leqq a$, it follows from (ii) that $c'_1 = c'_2$, and since $r'' \wedge a = 0$, it follows from (i) that $c''_1 = c''_2$. Hence $c_1 = c_2$.

(iv) When $n_1 \nleqq b_2$, then $n_1 \wedge b_2 < n_1$, and hence there exists $n'_1 \in \mathscr{C}$ such that $n'_1 \leqq n_1$ and $n'_1 \wedge b_2 = 0$. Putting $b'_1 = b_1 \wedge (n'_1 \vee a)$, we have $a \|_{(m, n'_1)} b'_1$ by Lemma 2.5 (ii). Since $b'_1 \wedge r \leqq b_1 \wedge r = 0$, by Theorem 4.1 there exists c'_1 such that $b'_1 \|_{(n'_1, r)} c'_1$. Since $n'_1 \leqq n_1$, we have $c'_1 = c_1$ by (iii). On the other hand, since $b_2 \wedge n'_1 = 0$, it follows from Theorem 4.2 that $b_2 \|_{(n_2, n'_1)} b'_1$. Since $c_2 \|_{(r, n_2)} b_2$ and $b'_1 \|_{(n'_1, r)} c'_1$, we have $c_2 = c'_1$ by Corollary of Theorem 4.2. Hence $c_1 = c_2$. When $n_2 \nleqq b_1$, then we have $c_1 = c_2$ by the same way.

(v) When $n_1 \leqq b_2$ and $n_2 \leqq b_1$, we have $b_1 \leqq b_2$ and $b_2 \leqq b_1$ by Lemma 4.1. Hence $b_1 = b_2$, which implies $c_1 = c_2$ by (P 1).

5. **Parallel images of incomplete elements.** Let $m \in \mathscr{C}$ and let a be an incomplete element over m, that is, $a \in I_m$. For any $n \in \mathscr{C}$, we define the *parallel image* of a at n, denoted by $\varphi_n(a)$, as follows:

(i) When $n \wedge a = 0$, it follows from Theorem 4.1 that there exists a unique element b such that $a \|_{(m, n)} b$. We define $\varphi_n(a) = b$.

(ii) When $n \leqq a$, there exists $n_0 \in \mathscr{C}$ such that $n_0 \wedge a = 0$, since $a < 1$. Then there exists b such that $a \|_{(m, n_0)} b$, and since $b \wedge n \leqq b \wedge a = 0$, there exists c such that $b \|_{(n_0, n)} c$. It follows from Lemma 4.5 that c is independent of the choice of n_0. We define $\varphi_n(a) = c$. Note that we have $\varphi_n(a) \leqq a$ by Lemma 4.1, since $b \|_{(n_0, n)} \varphi_n(a)$, $b < |_{(n_0)} a$ and $n \leqq a$.

(iii) When $n \wedge a \neq 0$ and $n \nleqq a$, we have $n \wedge a \in \mathscr{C}$ and hence we get $\varphi_{n \wedge a}(a)$ by (ii). We define $\varphi_n(a) = \varphi_{n \wedge a}(a) \vee n$.

REMARK 5.1. Let $a \in I_m$. Evidently $\varphi_m(a) = a$, and $\varphi_n(a) \leqq a \vee n$ for every $n \in \mathscr{C}$. When $n \wedge a \neq 0$ and $n \nleqq a$, putting $n_1 = n \wedge a$, there exists $n_2 \in \mathscr{C}$ such that $n = n_1 \vee n_2$ and $n_1 \wedge n_2 = 0$. Since $n_2 \wedge a = 0$, we have $\varphi_{n_1}(a) \|_{(n_1, n_2)} \varphi_{n_2}(a)$ by the definition of $\varphi_{n_1}(a)$. Hence $\varphi_{n_1}(a) \vee \varphi_{n_2}(a) = \varphi_{n_1}(a) \vee n_2 = \varphi_{n_1}(a) \vee n = \varphi_n(a)$.

LEMMA 5.1. *Let $a \in I_m$ and $n_1, n_2 \in \mathscr{C}$. If $(a \vee n_1) \wedge n_2 = 0$ then $\varphi_{n_1}(a) \|_{(n_1, n_2)} \varphi_{n_2}(a)$.*

Proof. (i) When $n_1 \wedge a = 0$, we have $(a, n_1, n_2) \perp$. Since $a \|_{(m, n_i)} \varphi_{n_i}(a) (i = 1, 2)$, we have $\varphi_{n_1}(a) \|_{(n_1, n_2)} \varphi_{n_2}(a)$ by Lemma 4.2. When $n_1 \leqq a$, then $\varphi_{n_1}(a) \|_{(n_1, n_2)} \varphi_{n_2}(a)$ by the definition of $\varphi_{n_1}(a)$.

(ii) When $n_1 \wedge a \neq 0$ and $n_1 \nleqq a$, we put $n'_1 = n_1 \wedge a$ and take n''_1 such that

$$n_1 = n_1' \vee n_1'' \text{ and } n_1' \wedge n_1'' = 0 .$$

By (i) we have $\varphi_{n_1''}(a) \|_{(n_1'',n_2)} \varphi_{n_2}(a)$ and $\varphi_{n_1'}(a) \|_{(n_1',n_2)} \varphi_{n_2}(a)$. Now we shall show that $\varphi_{n_2}(a) \wedge n_1 = 0$. We have $n_1 = n_1' \vee n_1'' \leqq a \vee n_1''$ and $\varphi_{n_2}(a) \leqq a \vee n_2$. Moreover $(a, n_1'', n_2)\perp$. Hence

$$\varphi_{n_2}(a) \wedge n_1 \leqq (a \vee n_2) \wedge (a \vee n_1'') = a \vee \{n_2 \wedge (a \vee n_1'')\} = a .$$

Therefore $\varphi_{n_2}(a) \wedge n_1 \leqq \varphi_{n_2}(a) \wedge a = 0$. By Remark 5.1, we have $\varphi_{n_1'}(a) \vee \varphi_{n_1''}(a) = \varphi_{n_1}(a)$. Hence we have $\varphi_{n_1}(a) \|_{(n_1,n_2)} \varphi_{n_2}(a)$ by Lemma 4.3.

LEMMA 5.2. *Let $a \in I_m(m \in \mathscr{C})$ and $n \in \mathscr{C}$. Then n is a maximal element of $\mathscr{C}$ contained in $\varphi_n(a)$.*

Proof. (i) When $n \wedge a = 0$, we have $\varphi_n(a) \|_{(n,m)} a$. By Lemma 2.1, n is maximal in the set $\{n' \in \mathscr{M}; n' \leqq \varphi_n(a)\}$. Since $\mathscr{C} \subset \mathscr{M}$, n is maximal in $\{n' \in \mathscr{C}; n' \leqq \varphi_n(a)\}$.

(ii) When $n \leqq a$, taking $n_0 \in \mathscr{C}$ with $a \wedge n_0 = 0$, we have $\varphi_n(a) \|_{(n,n_0)} \varphi_{n_0}(a)$. Hence n is maximal as in (i).

(iii) When $n \wedge a \neq 0$ and $n \not\leqq a$, we put $n \wedge a = n_1$ and take n_2 such that $n = n_1 \vee n_2$ and $n_1 \wedge n_2 = 0$. Then $n_1, n_2 \in \mathscr{C}$. Let n' be an element of $\mathscr{C}$ such that $n \leqq n' \leqq \varphi_n(a)$. Since $\varphi_n(a) = \varphi_{n_1}(a) \vee n$ and $\varphi_{n_1}(a) \leqq a$, we have $n' \wedge a \leqq (\varphi_{n_1}(a) \vee n) \wedge a = \varphi_{n_1}(a) \vee (n \wedge a) = \varphi_{n_1}(a)$. Since $n_1 \leqq n' \wedge a \in \mathscr{C}$, we have $n' \wedge a = n_1$ by (ii). If we had $n < n'$, then there would exists $n_0 \in \mathscr{C}$ such that $n_0 \leqq n'$ and $n_0 \wedge n = 0$. Since $(n_1, n_2, n_0)\perp$, we have

$$a \wedge (n_2 \vee n_0) = a \wedge n' \wedge (n_2 \vee n_0) = n_1 \wedge (n_2 \vee n_0) = 0 .$$

Hence $(a, n_2, n_0)\perp$. But $n_0 \leqq n' \leqq \varphi_n(a) \leqq a \vee n = a \vee n_2$, a contradiction. Therefore $n = n'$, and hence n is maximal.

LEMMA 5.3. *Let $a \in I_m(m \in \mathscr{C})$ and $n_1 \leqq n$ where $n_1, n \in \mathscr{C}$. Then $\varphi_n(a) = \varphi_{n_1}(a) \vee n$.*

Proof. We may assume $n_1 < n$. Take $n_2 \in \mathscr{C}$ such that $n = n_1 \vee n_2$ and $n_1 \wedge n_2 = 0$.

(i) When $n \wedge a = 0$, we have $a \|_{(m,n)} \varphi_n(a)$ and $a \|_{(m,n_i)} \varphi_{n_i}(a)$ $(i = 1, 2)$. By Lemma 4.3 and (P 1) we have $\varphi_n(a) = \varphi_{n_1}(a) \vee \varphi_{n_2}(a)$. Since $n_1 \leqq \varphi_{n_1}(a) \wedge n \leqq \varphi_{n_1}(a)$ and $\varphi_{n_1}(a) \wedge n \in \mathscr{C}$, we have $\varphi_{n_1}(a) \wedge n = n_1$ by Lemma 5.2, and hence $\varphi_{n_1}(a) \wedge n_2 = \varphi_{n_1}(a) \wedge n \wedge n_2 = n_1 \wedge n_2 = 0$. Hence $\varphi_{n_1}(a) \|_{(n_1,n_2)} \varphi_{n_2}(a)$ by Theorem 4.2. Therefore

$$\varphi_n(a) = \varphi_{n_1}(a) \vee \varphi_{n_2}(a) = \varphi_{n_1}(a) \vee n_2 = \varphi_{n_1}(a) \vee n .$$

(ii) When $n \leqq a$, taking $n_0 \in \mathscr{C}$ with $a \wedge n_0 = 0$, we have

$\varphi_{n_0}(a) \|_{(n_0,n)} \varphi_n(a)$ and $\varphi_{n_0}(a) \|_{(n_0,n_i)} \varphi_{n_i}(a) (i=1, 2)$. Hence by the same way as (i), we have $\varphi_n(a) = \varphi_{n_1}(a) \vee n$.

(iii) When $n \wedge a \neq 0$ and $n_1 \wedge a = 0$, we have $\varphi_{n\wedge a}(a) \|_{(n\wedge a, n_1)} \varphi_{n_1}(a)$. Hence

$$\varphi_n(a) = \varphi_{n\wedge a}(a) \vee n = \varphi_{n\wedge a}(a) \vee n_1 \vee n = \varphi_{n_1}(a) \vee (n \vee a) \vee n = \varphi_{n_1}(a) \vee n .$$

When $n_1 \wedge a \neq 0$, it follows from (ii) that $\varphi_{n\wedge a}(a) = \varphi_{n_1\wedge a}(a) \vee (n \wedge a)$. Hence

$$\varphi_n(a) = \varphi_{n\wedge a}(a) \vee n = \varphi_{n_1\wedge a}(a) \vee n = \varphi_{n_1\wedge a}(a) \vee n_1 \vee n = \varphi_{n_1}(a) \vee n .$$

THEOREM 5.1. *Let $a \in I_m (m \in \mathscr{C})$ and $n \in \mathscr{C}$. The mapping $x \to \varphi_n(x)$ is an isomorphism of the interval $L[m, a]$ onto $L[n, \varphi_n(a)]$.*

Proof. (i) When $n \wedge a = 0$, we have $a \|_{(m,n)} \varphi_n(a)$. For any $x \in L[m, a]$, we have $x \in I_m$ and $x \|_{(m,n)} \varphi_n(x)$. Hence $\varphi_n(x) = \varphi_n(a) \wedge (n \vee x)$ by Lemma 2.5 (ii) and (P 1). It follows from Lemma 2.6 that φ_n is an isomorphism of $L[m, a]$ onto $L[n, \varphi_n(a)]$.

(ii) When $n \leqq a$, taking $n_0 \in \mathscr{C}$ with $n_0 \wedge a = 0$, we have $a \|_{(m,n_0)} \varphi_{n_0}(a)$ and $\varphi_{n_0}(a) \|_{(n_0,n)} \varphi_n(a)$. It follows from (i) and Lemma 2.6 that the composed mapping $x \to \varphi_{n_0}(x) \to \varphi_n(x) \wedge (n \vee \varphi_{n_0}(x))$ is an isomorphism of $L[m, a]$ onto $L[n, \varphi_n(a)]$. On the other hand, since $(x \vee n) \wedge n_0 \leqq a \wedge n_0 = 0$, we have $\varphi_n(x) \|_{(n,n_0)} \varphi_{n_0}(x)$ by Lemma 7.1. Hence $\varphi_n(x) = \varphi_n(a) \wedge (n \vee \varphi_{n_0}(x))$ by (P 1).

(iii) When $n \wedge a \neq 0$ and $n \not\leqq a$, we put $n_1 = n \wedge a$. Then we have $\varphi_n(a) = \varphi_{n_1}(a) \vee n$, and moreover $\varphi_{n_1}(a) \wedge n = n_1$ since $n_1 \leqq \varphi_{n_1}(a) \wedge n \leqq a \wedge n = n_1$. Since $L[n_1, 1]$ is modular, the mapping $y \to y \vee n$ is an isomorphism of $L[n_1, \varphi_{n_1}(a)]$ onto $L[n, \varphi_n(a)]$. By (ii), the composed mapping $x \to \varphi_{n_1}(x) \to \varphi_{n_1}(x) \vee n$ is an isomorphism of $L[m, a]$ onto $L[n, \varphi_n(a)]$. Moreover $\varphi_{n_1}(x) \vee n = \varphi_n(x)$ by Lemma 5.3.

LEMMA 5.4. *Let $a \in I_m (m \in \mathscr{C})$ and $n = n_1 \vee n_2$ where $n, n_1, n_2 \in \mathscr{C}$. Then $\varphi_n(a) = \varphi_{n_1}(a) \vee \varphi_{n_2}(a)$.*

Proof. Since $n = n_1 \vee n_2 \leqq \varphi_{n_1}(a) \vee \varphi_{n_2}(a)$, it follows from Lemma 5.3 that $\varphi_n(a) = \varphi_{n_1}(a) \vee n \vee \varphi_{n_2}(a) \vee n = \varphi_{n_1}(a) \vee \varphi_{n_2}(a)$.

THEOREM 5.2. *If $a \|_{(m,n)} b$ where $m, n \in \mathscr{C}$ then $\varphi_r(a) = \varphi_r(b)$ for every $r \in \mathscr{C}$.*

Proof. (i) When $r \leqq a$, we have $b \|_{(n,r)} \varphi_r(b)$ since $r \wedge b \leqq a \wedge b = 0$. On the other hand, since $n \wedge a = 0$, we have $\varphi_r(a) \|_{(r,n)} b$. Hence $\varphi_r(a) = \varphi_r(b)$ by (P 1). When $r \leqq b$, similarly we have $\varphi_r(a) = \varphi_r(b)$.

(ii) When $r \wedge a = r \wedge b = 0$, we have $\varphi_r(a) \|_{(r,m)} a$, $a \|_{(m,n)} b$ and $b \|_{(n,r)} \varphi_r(b)$. Hence $\varphi_r(a) = \varphi_r(b)$ by Corollary Theorem 4.2.

(iii) When $r \wedge a = 0$, we put $r_1 = r \wedge b$ and take r_2 such that $r = r_1 \vee r_2$ and $r_1 \wedge r_2 = 0$. If $r_1 = 0$ or $r_2 = 0$ then $\varphi_r(a) = \varphi_r(b)$ holds by (ii) or (i). Hence we may assume $r_1, r_2 \in \mathscr{C}$. Then $\varphi_r(a) = \varphi_{r_1}(a) \vee \varphi_{r_2}(a)$ and $\varphi_r(b) = \varphi_{r_1}(b) \vee \varphi_{r_2}(b)$ by Lemma 5.4. We have $\varphi_{r_i}(a) = \varphi_{r_i}(b) (i = 1, 2)$ by (i) and (ii). Hence $\varphi_r(a) = \varphi_r(b)$.

(iv) When $r \wedge a \neq 0$ and $r \nleqq a$, we put $r_1 = r \wedge a$ and take r_2 such that $r = r_1 \vee r_2$ and $r_1 \wedge r_2 = 0$. Then $r_1, r_2 \in \mathscr{C}$, and then $\varphi_r(a) = \varphi_{r_1}(a) \vee \varphi_{r_2}(a)$ and $\varphi_r(b) = \varphi_{r_1}(b) \vee \varphi_{r_2}(b)$ by Lemma 5.4. We have $\varphi_{r_i}(a) = \varphi_{r_i}(b)$ by (i) and (iii). Hence $\varphi_r(a) = \varphi_r(b)$.

LEMMA 5.5. *Let* $a \in I_m (m \in \mathscr{C})$ *and* $n \in \mathscr{C}$. *If* $\varphi_n(a) < 1$, *then* $\varphi_n(a) \in I_n$ *and* $\varphi_r(\varphi_n(a)) = \varphi_r(a)$ *for every* $r \in \mathscr{C}$.

Proof. (i) When $a \wedge n = 0$, we have $a \|_{(m,n)} \varphi_n(a)$. Hence $\varphi_n(a) \in I_n$, and $\varphi_r(a) = \varphi_r(\varphi_n(a))$ by Theorem 5.2. When $a \vee n < 1$, we take $n_0 \in \mathscr{C}$ such that $(a \vee n) \wedge n_0 = 0$. Then $\varphi_n(a) \|_{(n,n_0)} \varphi_{n_0}(a)$ by Lemma 5.1. Hence $\varphi_n(a) \in I_n$. Moreover, since $a \|_{(m,n_0)} \varphi_{n_0}(a)$, we have $\varphi_r(a) = \varphi_r(\varphi_{n_0}(a)) = \varphi_r(\varphi_n(a))$ by Theorem 5.2.

(ii) When $a \wedge n \neq 0$ and $a \vee n = 1$, we put $n_1 = a \wedge n$ and take n_2 such that $n = n_1 \vee n_2$ and $n_1 \wedge n_2 = 0$. Since $a < 1$, we have $n \nleqq a$, and hence $n_1, n_2 \in \mathscr{C}$. We have $\varphi_{n_1}(a) < a$, since otherwise $\varphi_{n_1}(a) = a$ and then $\varphi_n(a) = \varphi_{n_1}(a) \vee n = a \vee n = 1$, a contradiction. Hence there exists $n_0 \in \mathscr{C}$ such that $n_0 \leqq a$ and $\varphi_{n_1}(a) \wedge n_0 = 0$. Since $n_1, n_0 \leqq a$ and $a \wedge n_2 = 0$, we have $\varphi_{n_1}(a) \|_{(n_1,n_2)} \varphi_{n_2}(a)$ and $\varphi_{n_0}(a) \|_{(n_0,n_2)} \varphi_{n_2}(a)$. Moreover $\varphi_{n_1}(a) \|_{(n_1,n_0)} \varphi_{n_0}(a)$ by Theorem 4.2. Since $(\varphi_{n_0}(a) \vee n_1) \wedge n_2 \leqq a \wedge n_2 = 0$, we have $(\varphi_{n_0}(a), n_1, n_2) \perp$, and hence $\varphi_{n_0}(a) \wedge (n_1 \vee n_2) = 0$. Hence, by Lemma 4.3, we have $\varphi_{n_0}(a) \|_{(n_0,n)} \varphi_{n_1}(a) \vee \varphi_{n_2}(a) = \varphi_n(a)$. Therefore $\varphi_n(a) \in I_n$. Since $a \|_{(m_1,n_2)} \varphi_{n_2}(a)$, we have $\varphi_r(a) = \varphi_r(\varphi_{n_2}(a)) = \varphi_r(\varphi_{n_0}(a)) = \varphi_r(\varphi_n(a))$ by Theorem 5.2.

LEMMA 5.6. *Let* $a \in I_m (m \in \mathscr{C})$ *and* $n_1, n_2 \in \mathscr{C}$. *If* $\varphi_{n_1}(a) \wedge n_2 = 0$ *then* $\varphi_{n_1}(a) \|_{(n_1,n_2)} \varphi_{n_2}(a)$.

Proof. Since $\varphi_{n_1}(a) < 1$, we have $\varphi_{n_1}(a) \in I_{n_1}$ by Lemma 5.5. Hence $\varphi_{n_2}(\varphi_{n_1}(a)) \|_{(n_2,n_1)} \varphi_{n_1}(a)$. By Lemma 5.5, we have $\varphi_{n_2}(\varphi_{n_1}(a)) = \varphi_{n_2}(a)$.

DEFINITION. Let $\mathscr{C}_0$ be the subset of $\mathscr{C}$ deleting maximal elements in $\mathscr{C}$ which are not atoms, that is, $\mathscr{C}_0 = (\mathscr{C} - \{\text{maximal elements}\}) \smile \Omega$. Then, it is evident that for any $m \in \mathscr{C}$ there exists $m_1 \in \mathscr{C}_0$ such that $m_1 \leqq m$. Hence L is $\mathscr{C}_0$-*SSC*.

LEMMA 5.7. *If* $a \in I_m (m \in \mathscr{C})$ *and* $n \in \mathscr{C}_0$, *then* $\varphi_n(a) \in I_n$.

Proof. If we had $\varphi_n(a) = 1$, then n would be maximal in $\mathscr{C}$ by Lemma 5.2. Hence n would be an atom by the definition of $\mathscr{C}_0$. If $a \wedge n = 0$, then $a \|_{(m,n)} \varphi_n(a) = 1$, a contradiction. If $a \wedge n \neq 0$, then $n \leqq a$, and then $1 = \varphi_n(a) \leqq a \in I_m$, a contradiction. Therefore, we have $\varphi_n(a) < 1$, and hence $\varphi_n(a) \in I_n$ by Lemma 5.5.

LEMMA 5.8. *If $m, n \in \mathscr{C}_0$, then I_m and I_n are isomorphic by mutually inverse mappings φ_n and φ_m.*

Proof. We have $\varphi_n(I_m) \subset I_n$ and $\varphi_m(I_n) \subset I_m$ by Lemma 5.7. It follows from Lemma 5.5 that $\varphi_m(\varphi_n(a)) = \varphi_m(a) = a$ for $a \in I_m$ and $\varphi_n(\varphi_m(b)) = \varphi_n(b) = b$ for $b \in I_n$. Moreover φ_m and φ_n are order-preserving by Theorem 5.1. Hence I_m and I_n are isomorphic.

LEMMA 5.9. *Let $m \in \mathscr{C}$. If m is not an atom and if there exist $a_1, a_2 \in I_m$ such that $a_1 \vee a_2 = 1$, then $m \notin \mathscr{C}_0$.*

Proof. Since L is left complemented, there exists $b \leqq a_2$ such that $a_1 \vee b = a_1 \vee a_2$ and $a_1 \perp b$. Putting $a_2' = m \vee b$, we have

$$a_2' \in I_m, \; a_1 \vee a_2' = a_1 \vee b = a_1 \vee a_2 = 1$$

and

$$a_1 \wedge a_2' = (m \vee b) \wedge a_1 = m \vee (b \wedge a_1) = m \; .$$

Hence we may assume that $a_1 \wedge a_2 = m$. Since m is not an atom, there exists $m_1 \in \mathscr{C}$ such that $m_1 \leqq m$. Since $\varphi_{m_1}(a_1) \leqq a_1$, we have

$$\varphi_{m_1}(a_1) \wedge a_2 = \varphi_{m_1}(a_1) \wedge a_1 \wedge a_2 = \varphi_{m_1}(a_1) \wedge m = m_1$$

by Lemma 5.2. Since L is weakly modular and $\varphi_{m_1}(a_1) \wedge a_2 \neq 0$, we have $(\varphi_{m_1}(a_2) \vee \varphi_{m_1}(a_1)) \wedge a_2 = \varphi_{m_1}(a_2) \vee (\varphi_{m_1}(a_1) \wedge a_2) = \varphi_{m_1}(a_2) \wedge m_1 = \varphi_{m_1}(a_2)$. Since $\varphi_{m_1}(a_2) \wedge m = m_1$, we have $\varphi_{m_1}(a_2) < a_2$, and hence the above equation implies that $\varphi_{m_1}(a_1) \vee \varphi_{m_1}(a_2) < 1$. Putting

$$a_0 = \varphi_{m_1}(a_1) \vee \varphi_{m_1}(a_2) \; ,$$

we have $a_0 \in I_{m_1}$ by Corollary of Theorem 4.1, and by Theorem 5.1 we have

$$\begin{aligned} \varphi_m(a_0) &= \varphi_m\varphi_{m_1}(a_1) \vee \varphi_m\varphi_{m_1}(a_2) = \varphi_m(a_1) \vee \varphi_m(a_2) = a_1 \vee a_2 \\ &= 1 \notin I_m \; . \end{aligned}$$

By Lemma 5.7, we have $m \notin \mathscr{C}_0$.

LEMMA 5.10. *If for some $m \in \mathscr{C}_0$ there exist $a_1, a_2 \in I_m$ such that $a_1 \vee a_2 = 1$, then every element n of $\mathscr{C}_0$ is an atom* (hence L is atomistic) *and $\varphi_n(a_1) \vee \varphi_n(a_2) = 1$.*

Proof. It follows from Lemma 5.9 that m is an atom. Let $n \in \mathscr{C}_0$. If we had $\varphi_n(a_1) \vee \varphi_n(a_2) < 1$, then as in the proof of Lemma 5.9, we would have $m \notin \mathscr{C}_0$. Hence $\varphi_n(a_1) \vee \varphi_n(a_2) = 1$, and then n is an atom by Lemma 5.9.

6. Construction of modular extensions. As in §§ 4 and 5, let L be a weakly modular, left complemented lattice, with 0 and 1, of length $\geqq 4$, and assume that for fixed subset $\mathscr{C}$ of $\mathscr{M}$, satisfying (C 1), L is $\mathscr{C}$-*SSC* and L satisfies the axioms (P 1) and (P 2) with respect to $\mathscr{C}$. (We may assume that $\mathscr{C}$ satisfies (C 2).)

Let $\mathscr{C}_0$ be the subset of $\mathscr{C}$ given in § 5. We say that L is of *type A* when for some $m \in \mathscr{C}_0$ there exist $a_1, a_2 \in I_m$ such that $a_1 \vee a_2 = 1$. It follows from Lemma 5.8, Corollary of Theorem 4.1 and Lemma 5.10 that we have the following results.

(1) For any $m, n \in \mathscr{C}_0$, I_m and I_n are isomorphic by the mappings φ_n and φ_m.

(2) If L is not of type A then I_m is an ideal of the lattice $L[m, 1]$ for every $m \in \mathscr{C}_0$.

(3) If L is of type A then L is atomistic and $\mathscr{C}_0 = \Omega$. If moreover L satisfies the axiom (P 3) with respect to Ω then $I_m = L[m, 1] - 1$ for every $m \in \mathscr{C}_0$.

Hereafter, whenever L is of type A, we assume that L satisfies (P 3) with respect to Ω. In this case, it is convenient that we set $1 \in I_m$ and $\varphi_m(1) = 1$ for every $m \in \mathscr{C}_0$. Then $I_m = L[m, 1]$ and moreover I_m and I_n are isomorphic by φ_n and φ_m.

Definition. For an incomplete element a, we denote by $[a]$ the set of parallel images of a at all elements of $\mathscr{C}_0$, that is, $[a] = \{\varphi_m(a); m \in \mathscr{C}_0\}$. We denote by S the set of all $[a]$ deleting $[m]$. For $[a]$, $[b] \in S$, we define $[a] \leqq [b]$ by $\varphi_m(a) \leqq \varphi_m(b)$ for some $m \in \mathscr{C}_0$ (and hence for every $m \in \mathscr{C}_0$). Hence S is isomorphic to $I_m - \{m\}$ for every $m \in \mathscr{C}_0$. If L is of type A, then S has the greatest element $[1]$ and $S \cong L[m, 1] - \{m\}$.

Lemma 6.1. *In the set $\Lambda \equiv L \smile S$, we define a partial order by the following conventions*:

(O 1) *For $a, b \in L$, we have $a \leqq b$ in Λ when $a \leqq b$ in L. For $[a], [b] \in S$, we have $[a] \leqq [b]$ in Λ when $[a] \leqq [b]$ in S.*

(O 2) *For $[a] \in S$ and $b \in L$, we have $[a] < b$ when $\varphi_m(a) \leqq b$ for $m \in \mathscr{C}_0$ with $m \leqq b$.* (Especially, $[1] < b$ only when $b = 1$.)

(O 3) *For $[a] \in S$ and $b \in L$, we have $b < [a]$ only when $b = 0$.*

Then Λ is a lattice where the lattice operations $\cup$ and $\cap$ have the following properties:

(1) *If $a, b \in L$ then $a \cup b = a \vee b$.*

(2) *If* $0 \neq a \in L$ *and* $[b] \in S$ *then for* $m \in \mathscr{C}_0$ *with* $m \leqq a$ *we have* $a \cup [b] = a \vee \varphi_m(b)$.

(3) *If* $[a], [b] \in S$ *then for* $m \in \mathscr{C}_0$ *we have* $[a] \cup [b] = [\varphi_m(a) \vee \varphi_m(b)]$.

(4) *If* $a, b \in L$ *and* $a \wedge b \neq 0$ *then* $a \cap b = a \wedge b$.

(5) *If* $a, b \in L$ *and* $a \wedge b = 0$ *and if* $m, n \in \mathscr{C}_0$ *such that* $m \leqq a$ *and* $n \leqq b$ *then* $a \cap b = [a \wedge (m \vee b)] = [b \wedge (n \vee a)]$.

(6) *If* $a \in L$, $[b] \in S$ *and if* $m \in \mathscr{C}_0$ *such that* $m \leqq a$ *then* $a \cap [b] = [a \wedge \varphi_m(b)]$.

(7) *If* $[a], [b] \in S$ *and* $m \in \mathscr{C}_0$ *then* $[a] \cap [b] = [\varphi_m(a) \wedge \varphi_m(b)]$.
Note that in (5), (6) *and* (7) *we set* $0 = [m]$ *for convenience.* (If L is of type A then especially we have $a \cup [1] = 1$, $[a] \cup [1] = [1]$, $a \cap [1] = [a]$ and $[a] \cap [1] = [a]$.)

Proof. (1) Let $a, b \in L$. To prove (1), we may assume $a \neq 0$ and $b \neq 0$. By (O 3), any upper bound of $\{a, b\}$ in Λ belongs to L. Hence $a \cup b$ exists and (1) holds.

(2) We have $[b] < \varphi_m(b)$ by (O 2). Hence $a \vee \varphi_m(b)$ is an upper bound of $\{a, [b]\}$ in Λ. If c is an upper bound of $\{a, [b]\}$ then $c \in L$ by (O 3). Since $m \leqq a \leqq c$ and $[b] < c$, we have $\varphi_m(b) \leqq c$. Thus $a \cup [b]$ exists and (2) holds.

(3) This follows from (O 1) evidently.

(4) Take $m \in \mathscr{C}_0$ such that $m \leqq a \wedge b$. If $c \in L$ is a lower bound of $\{a, b\}$ in Λ then $c \leqq a \wedge b$ by (O 1). If $[d] \in S$ is a lower bound of $\{a, b\}$ in Λ, then $\varphi_m(d) \leqq a \wedge b$ and hence $[d] \leqq a \wedge b$. Therefore $a \cap b$ exists and (4) holds.

(5) We have $a \wedge (m \vee b) \,||_{(m,n)}\, b \wedge (n \vee a)$ by Lemma 2.5 (i). Hence, $[a \wedge (m \vee b)] = [b \wedge (n \vee a)]$ is a lower bound of $\{a, b\}$. Since $a \wedge b = 0$, any lower bound is either an element of S or 0, and hence it has the form $[c]$ where $c \geqq m$. Since $[c] \leqq a, b$, we have $\varphi_m(c) \leqq a$ and $\varphi_n(c) \leqq b$. Since $\varphi_m(c) \wedge n \leqq a \wedge b = 0$, we have $\varphi_m(c) \,||_{(m,n)}\, \varphi_n(c)$ by Lemma 5.6. Hence $\varphi_m(c) \leqq a \wedge (m \vee \varphi_n(c)) \leqq a \wedge (m \vee b)$. Therefore $[c] \leqq [a \wedge (m \vee b)]$. Thus $a \cap b$ exists and (5) holds.

(6) It is evident that $[a \wedge \varphi_m(b)]$ is a lower bound of $\{a, [b]\}$. Any lower bound of $\{a, [b]\}$ has the form $[c]$ where $c \geqq m$. We have $[c] \leqq [a \wedge \varphi_m(b)]$ since $\varphi_m(c) \leqq a, \varphi_m(b)$. Hence $a \cap [b]$ exists and (6) holds.

(7) For $[a], [b] \in S$, taking $m \in \mathscr{C}_0$, $[\varphi_m(a) \wedge \varphi_m(b)]$ is a lower bound of $\{[a], [b]\}$. For any lower bound $[c]$ of $\{[a], [b]\}$, we have $[c] \leqq [\varphi_m(a) \wedge \varphi_m(b)]$, since $\varphi_m(c) \leqq \varphi_m(a), \varphi_m(b)$. Hence $[a] \cap [b]$ exists and (7) holds.

LEMMA 6.2. *The lattice* Λ *constructed in Lemma* 6.1 *is complemented.*

Proof. (i) Let $0 \neq a \in L$. Since L is left complemented, there exists $b \in L$ such that $a \vee b = 1$ and $a \perp b$. We have $a \cup b = a \vee b = 1$ by Lemma 6.1 (1). We take $m \in \mathscr{C}_0$ with $m \leqq a$. Since $(b, a)M$, we have $a \wedge (m \vee b) = (m \vee b) \wedge a = m \vee (b \wedge a) = m$. Hence, by Lemma 6.1 (5), $a \cap b = [a \wedge (m \vee b)] = [m] = 0$. Therefore, b is a complement of a in Λ.

(ii) Let $[a] \in S$. We take $m \in \mathscr{C}_0$ with $m \leqq a$. Since L is left complemented, it is relatively complemented. Hence there exists $b \in L$ such that $a \vee b = 1$ and $a \wedge b = m$. By Lemma 6.1 (2), we have $[a] \cup b = \varphi_m(a) \vee b = a \vee b = 1$. By Lemma 6.1 (6), we have $[a] \cap b = [\varphi_m(a) \wedge b] = [a \wedge b] = [m] = 0$. Hence b is a complement of $[a]$ in Λ. (Especially, if L is of type A, then m is a complement of $[1]$.)

LEMMA 6.3. *The lattice Λ constructed in Lemma 6.1 is modular.*

Proof. (i) Let $a, b \in L$ and we shall show $(a, b)M$ in Λ. When $a \wedge b \neq 0$, we have $a \cap b = a \wedge b$ by Lemma 6.1 (4), and hence the interval $\Lambda[a \cap b, 1]$ of Λ coinsides with $L[a \wedge b, 1]$ which is modular since L is weakly modular. Hence $(a, b)M$ in $\Lambda[a \cap b, 1]$ and then $(a, b)M$ in Λ. When $a \wedge b = 0$, we may assume $a \neq 0$ and $b \neq 0$. Let λ be an element of Λ with $0 < \lambda < b$. If $\lambda \in L$, then we put $\lambda = b_1$ and take $n \in \mathscr{C}_0$ with $n \leqq b_1$. Since $(n \vee a) \wedge b \geqq n > 0$ and since L is weakly modular, we have $(n \vee a, b)M$ in L. Hence

$$(b_1 \cup a) \cap b = (b_1 \vee a) \cap b = (b_1 \vee a) \wedge b = (b_1 \vee n \vee a) \wedge b$$
$$= b_1 \vee \{(n \vee a) \wedge b\} .$$

On the other hand, by Lemma 6.1 (5) and (2), we have

$$b_1 \cup (a \cap b) = b_1 \cup [b \wedge (n \vee a)] = b_1 \vee \{b \wedge (n \vee a)\} .$$

Hence $(b_1 \cup a) \cap b = b_1 \cup (a \cap b)$.

If $\lambda \in S$, then we put $\lambda = [c]$ and take $m, n \in \mathscr{C}_0$ with $m \leqq a$ and $n \leqq b$. We shall prove $(\varphi_m(c) \vee a) \wedge b = 0$. Since $[c] < b$, we have $\varphi_n(c) \leqq b$, whence $m \wedge \varphi_n(c) \leqq a \wedge b = 0$. Hence $\varphi_m(c) \parallel_{(m,n)} \varphi_n(c)$ by Lemma 5.6, while $a \wedge (m \vee b) \parallel_{(m,n)} b \wedge (n \vee a)$ by Lemma 2.5 (i). Since $(\varphi_n(c) \vee \{b \wedge (n \vee a)\}) \wedge m \leqq b \wedge a = 0$, by (P 2) we have

$$(*) \qquad \varphi_m(c) \vee \{a \wedge (m \vee b)\} \parallel_{(m,n)} \varphi_n(c) \vee \{b \wedge (n \vee a)\} .$$

Now we have $(a, m \vee b)M$ since L is weakly modular. Since $\varphi_m(c) \leqq m \vee \varphi_n(c) \leqq m \vee b$, we have

$$\varphi_m(c) \vee \{a \wedge (m \vee b)\} = (\varphi_m(c) \vee a) \wedge (m \vee b) .$$

Moreover, since $(n \vee a, b)M$, we have

$$\varphi_n(c) \vee \{b \wedge (n \vee a)\} = (\varphi_n(c) \vee n \vee a) \wedge b = (\varphi_n(c) \vee a) \wedge b \,.$$

Hence, by (*), we have $0 = (\varphi_m(c) \vee a) \wedge (m \vee b) \wedge (\varphi_n(c) \vee a) \wedge b = (\varphi_m(c) \vee a) \wedge (\varphi_n(c) \vee a) \wedge b$. Since

$$\varphi_m(c) \vee a \leqq m \vee \varphi_n(c) \vee a = \varphi_n(c) \vee a \,,$$

we get $(\varphi_m(c) \vee a) \wedge b = 0$. By Lemma 6.1 (2) and (5), we have

$$\begin{aligned}([c] \cup a) \cap b = (\varphi_m(c) \vee a) \cap b &= [(\varphi_m(c) \vee a) \wedge (m \vee b)] \\ &= [\varphi_m(c) \vee \{a \wedge (m \vee b)\}] \,.\end{aligned}$$

On the other hand, by Lemma 6.1 (5) and (3),

$$[c] \cup (a \cap b) = [c] \cup [a \wedge (m \vee b)] = [\varphi_m(c) \vee \{a \wedge (m \vee b)\}] \,.$$

Hence $([c] \cup a) \cap b = [c] \cup (a \cap b)$. Therefore, $(\lambda \cup a) \cap b = \lambda \cup (a \cap b)$ for any $\lambda \in \Lambda$ with $\lambda \leqq b$.

(ii) Let $[a] \in S$ and $b \in L$, and we shall show $([a], b)M$. We may assume $b \neq 0$. Let $\lambda \in \Lambda$ with $0 < \lambda < b$. If $\lambda = b_1 \in L$, then we take $n \in \mathscr{C}_0$ with $n \leqq b_1$. Since $\varphi_n(a) \wedge b \neq 0$, we have

$$(b_1 \cup [a]) \cap b = (b_1 \vee \varphi_n(a)) \cap b = (b_1 \vee \varphi_n(a)) \wedge b = b_1 \vee (\varphi_n(a) \wedge b) \,,$$

while by Lemma 6.1 (6) we have

$$b_1 \cup ([a] \cap b) = b_1 \cup [\varphi_n(a) \wedge b] = b_1 \vee (\varphi_n(a) \wedge b) \,.$$

If $\lambda = [c] \in S$, then we take $n \in \mathscr{C}_0$ with $n \leqq b$. Since $\varphi_n(c) \leqq b$, we have

$$\begin{aligned}([c] \cup [a]) \cap b = [\varphi_n(c) \vee \varphi_n(a)] \cap b &= [(\varphi_n(c) \vee \varphi_n(a)) \wedge b] \\ &= [\varphi_n(c) \vee (\varphi_n(a) \wedge b)] \,,\end{aligned}$$

while

$$[c] \cup ([a] \cap b) = [c] \cup [\varphi_n(a) \wedge b] = [\varphi_n(c) \vee (\varphi_n(a) \wedge b)] \,.$$

Hence $(\lambda \cup [a]) \cap b = \lambda \cup ([a] \cap b)$ for any $\lambda \in \Lambda$ with $\lambda \leqq b$.

(iii) Let $a \in L$ and $[b] \in S$, and we shall show $(a, [b])M$. We may assume $a \neq 0$. Let $\lambda \in \Lambda$ with $0 < \lambda < [b]$. Then, since $\lambda \in S$, we put $\lambda = [c]$. We take $m \in \mathscr{C}_0$ with $m \leqq a$. Since $\varphi_m(c) \leqq \varphi_m(b)$, we have

$$\begin{aligned}([c] \cup a) \cap [b] = (\varphi_m(c) \vee a) \cap [b] &= [(\varphi_m(c) \vee a) \wedge \varphi_m(b)] \\ &= [\varphi_m(c) \vee (a \wedge \varphi_m(b))] \,,\end{aligned}$$

while

$$[c] \cup (a \cap [b]) = [c] \cup [a \wedge \varphi_m(b)] = [\varphi_m(c) \vee (a \wedge \varphi_m(b))] \,.$$

(iv) Let $[a], [b] \in S$ and let $[c] < [b]$. We take $m \in \mathscr{C}_0$ with $m \leqq a$. Since $\varphi_m(c) \leqq \varphi_m(b)$, we have

$$([c] \cup [a]) \cap [b] = [\varphi_m(c) \vee a] \cap [b] = [(\varphi_m(c) \vee a) \wedge \varphi_m(b)]$$
$$= [\varphi_m(c) \vee (a \wedge \varphi_m(b))] ,$$

while

$$[c] \cup ([a] \cap [b]) = [c] \cup [a \wedge \varphi_m(b)] = [\varphi_m(c) \vee (a \wedge \varphi_m(b))] .$$

Hence $([a], [b])M$.

REMARK 6.1. We shall show that if an $\mathscr{M}$-*SSC* lattice, with 0 and 1, has a finite connected chain from 0 to 1, then it is atomistic. It suffices to prove that any nonzero element a includes an atom, since the lattice is *SSC*. Let $0 = a_0 \prec a_1 \prec \cdots \prec a_n = 1$ and let $r(<n)$ be greatest such that $a \not\leqq a_r$. Since $a \wedge a_r < a$, there exists $m \in \mathscr{M}$ such that $m \leqq a(\leqq a_{r+1})$ and $m \wedge a_r = 0$. If $0 < x \leqq m$, then $a_r < a_r \vee x \leqq a_{r+1}$, whence $a_r \vee x = a_{r+1}$. Hence

$$m = m \wedge a_{r+1} = (x \vee a_r) \wedge m = x \vee (a_r \wedge m) = x .$$

Therefore m is an atom included in a.

THEOREM 6.1. (Non-atomistic case) *Let L be an $\mathscr{M}_s$-SSC lattice with 0 and 1 which is not atomistic. Then, L is a Wilcox lattice if and only if L is weakly modular, left complemented and satisfies two parallel axioms* (P 1) *and* (P 2) *with respect to $\mathscr{M}_s$.*

Proof. If L is a Wilcox lattice, then evidently L is weakly modular and it is left complemented by Lemma 3.2. Moreover L satisfies the parallel axioms by Lemma 3.7. We shall prove the converse statement. We remark that L is of infinite length by Remark 6.1 and that L is not of type A. Putting $\mathscr{C} = \mathscr{M}_s$, L satisfies all the conditions stated at the beginning of this section. Hence it follows from Lemmas 6.1, 6.2 and 6.3 that L is a Wilcox lattice.

THEOREM 6.2. (Atomistic case) *Let L be an atomistic lattice, with 0 and 1, of length $\geqq 4$. Then, L is a Wilcox lattice if and only if L is weakly modular, left complemented and satisfies three parallel axioms* (P 1), (P 2) *and* (P 3) *with respect to Ω.*

Proof. As the proof of Theorem 6.1, the "only if" part follows from Lemmas 3.2 and 3.7 and the "if" part follows from Lemmas 6.1, 6.2 and 6.3, by putting $\mathscr{C} = \Omega$.

REMARK 6.2. We remark that two axioms (P 2) and (P 3) can be replaced by the following one axiom (assuming that L satisfies (P 1)):
(P 4) If $a_1 < |_{(m)}\, b_1$, $a_2 < |_{(m)}\, b_2$ where $m \in \mathscr{C}$ and if $m < a \leqq a_1 \vee a_2$

then for any $n \in \mathscr{C}$ with $a \wedge n = 0$ there exists b such that $a \,||_{(m,n)}\, b$.

Proof. Since L is $\mathscr{C}$-*SSC*, it is evident that (P 4) implies (P 3). We shall show that (P 4) implies (P 2). Let $a_i \,||_{(m,n)}\, b_i (i = 1, 2)$ and $(a_1 \vee a_2) \wedge n = 0$. By (P 4) there exists b such that $a_1 \vee a_2 \,||_{(m,n)}\, b$. Then, $a_i \,||_{(m,n)}\, b \wedge (n \vee a_i)$ by Lemma 2.5 (ii), whence $b_i = b \wedge (n \wedge a_i)$ by (P 1). Therefore, $b_1 \vee b_2 \leqq b$, and hence $(a_1 \vee a_2) \wedge (b_1 \vee b_2) = 0$. On the other hand, we have $a_1 \vee a_2 \vee n = m \vee b_1 \vee b_2$. Hence $a_1 \vee a_2 \,||_{(m,n)}\, b_1 \vee b_2$. Conversely, we assume that (P 2) and (P 3) are satisfied. Then we may use the results in § 4. Let $a_i < |_{(m)}\, b_i (i = 1, 2)$, $m < a \leqq a_1 \vee a_2$ and $a \wedge n = 0$. If $a_1 \vee a_2 = 1$ then $a \in I_m$ by (P 3). If $a_1 \vee a_2 < 1$ then $a_1 \vee a_2 \in I_m$ by Corollary of Theorem 4.1 and hence $a \in I_m$. Therefore it follows from Theorem 4.1 that there exists b such that $a \,||_{(m,n)}\, b$.

REMARK 6.3. We can show that an application of Theorem 6.2 to the upper continuous atomistic case implies the theorem on affine matroid lattices given in [2], p. 314. A matroid lattice is defined as an upper continuous atomistic lattice with the covering property ([4], Definition 1.8), which is left complemented and M-symmetric (see [8]). In an matroid lattice we write $a < | b$ when $a < |_{(p)}\, b$ for some atom $p \leqq a$, and write $a \,||\, b$ when $a < | b$ and $b < | a$. (In [2], p. 272, it is written $b \,||\, a$ instead of $a < | b$.) A weakly modular matroid lattice L of length $\geqq 4$ is called an *affine matroid lattice* ([4], Definition 3.3) when L satisfies the following axiom (the join of two different atoms is called a line):

(EP) If l, k_1 and k_2 are lines such that $l \,||\, k_1$, $l \,||\, k_2$ and if $k_1 \wedge k_2 \neq 0$ then $k_1 = k_2$.

THEOREM. *Let L be an upper continuous atomistic lattice of length $\geqq 4$. Then, L is a Wilcox lattice if and only if L is weakly modular, M-symmetric and satisfies the axiom (EP), that is, L is an affine matroid lattice.*

This theorem follows from Theorem 6.2, by Remark 6.2 and by the results given in [2], pp. 307–309 (Theorem 5 and Prop. 9). Note that our definition of an incomplete element ($\neq 1$) coinsides with that in [2], p. 307.

REMARK 6.4. About the uniqueness of the modular extension of a Wilcox lattice, we can prove the following results (see [11]). Here the proofs are omitted.

(1) Let $L \equiv \Lambda - S$ be a semicomplemented Wilcox lattice of

length $\geqq 3$. Then L is modular if and only if S is empty.

(2) Let L be an $\mathscr{M}$-*SSC* Wilcox lattice of length $\geqq 3$. The modular extension of L is uniquely determined up to isomorphism.

We remark that any $\mathscr{M}$-*SSC* Wilcox lattice of length 2 has exactly two modular extensions and that a Wilcox lattice of length $\geqq 3$ may have two modular extensions if it is not $\mathscr{M}$-*SSC*.

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