PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 3, 1971

ARCWISE CONNECTIVITY OF SEMI-APOSYNDETIC PLANE CONTINUA

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Suppose M is a bounded semi-aposyndetic plane continuum and for any positive real number ε there are at most a finite number of complementary domains of M of diameter greater than ε. In this paper it is proved that M is arcwise connected.

Let M be a continuum (a closed connected point set) and let x and y be distinct points of M. If M contains a continuum H and an open set G such that $x \in G \subset H \subset M - \{y\}$, then M is said to be *aposyndetic* at x with respect to y [4]. M is said to be *semi-aposyndetic* if for each pair of distinct points x and y of M, M is aposyndetic either at x with respect to y or at y with respect to x. In [3] it is proved that every bounded semi-aposyndetic plane continuum which does not have infinitely many complementary domains is arcwise connected. For other results concerning semi-aposyndetic plane continua see [1] and [2].

Let x and y be distinct points of a metric space S. A finite collection $\{A_1, A_2, \cdots, A_m\}$ of sets in S is a *chain* in S from x to y provided A_1 contains x, A_m contains y, and for i and j belonging to $\{1, 2, \cdots, m\}$, $A_i \cap A_j \neq \phi$ if and only if $|i - j| \leqq 1$. If each element of a chain $\mathscr{A}$ has diameter less than r (a positive real number) then $\mathscr{A}$ is said to be an *r-chain*. Suppose $\mathscr{A} = \{A_1, A_2, \cdots, A_m\}$ and $\mathscr{B} = \{B_1, B_2, \cdots, B_n\}$ are chains in S from x to y. The chain $\mathscr{B}$ is said to *run straight through* $\mathscr{A}$ provided the closure of each element of $\mathscr{B}$ is contained in an element of $\mathscr{A}$ and if B_i and B_k $(1 \leqq i \leqq k \leqq n)$ both lie in an element A_s of $\mathscr{A}$, then for each integer j $(i < j < k)$, B_j is contained in an element of $\mathscr{A}$ whose intersection with A_s is nonvoid.

If M is a bounded plane continuum and for any positive real number ε there are at most a finite number of complementary domains of M of diameter greater than ε, then M is said to be an *E-continuum* [6, p. 112].

The boundary of a set A is denoted by Bd A.

THEOREM 1. *Suppose M is a semi-aposyndetic E-continuum is S (a 2-sphere with metric φ), U is a disk in S, x and y are distinct points which belong to the same component of $M \cap U$, and V is an open disk in S containing U. Then for any positive real number r less than both $\varphi(x, y)/5$ and $\varphi(\text{Bd } U, \text{Bd } V)/5$ there exists an r-chain $\{H_1, H_2, \cdots, H_n\}$ $(n > 3)$ in S from x to y such that for each positive* *integer i less than or equal n, H_i is a continuum in $M \cap V$ and $\varphi(H_i, \text{Bd } V)$ is greater than $4r$.*

Proof. Let G be the union of all components of $S - M$ which have diameter less than $r/3$. Since M is a semi-aposyndetic E-continuum, $M \cup G$ is a semi-aposyndetic continuum which does not have infinitely many complementary domains [5, Th. 2 (proof)]. Let F be the x-component of $U \cap (M \cup G)$. F is a semi-aposyndetic continuum in S which does not have infinitely many complementary domains [3, Th. 1] (D and M in [3] are $S - U$ and $M \cup G$ respectively). Hence F is arcwise connected [3, Th. 2]. Let A be an arc in F from x to y. There exists a finite point set B in $A - \{x, y\}$ such that each component of $A - B$ has diameter less than $r/3$. For each component C of $A - B$, let $G(C)$ be C union all components of G which intersect C and let $Z(C)$ be the boundary (relative to S) of $G(C)$. For each component C of $A - B$, since the boundary of each component of G is a continuum [6, Th. 2.1, p. 105] and each point of C that is not in G belongs to $Z(C)$, $Z(C)$ is a continuum of diameter less than r in M. Let $\mathscr{K}$ be the finite coherent collection of continua $\{Z(C) \mid C \text{ is a component of } A - B\}$. The points x and y each belong to an element of $\mathscr{K}$ and each element of $\mathscr{K}$ intersects U. It follows that any chain from x to y whose elements are members of $\mathscr{K}$ has the specified conditions.

THEOREM 2. *If M is a semi-aposyndetic E-continuum, then M is arcwise connected.*

Proof. Let S be a 2-sphere which contains M and let φ be a distance function on S. Let p and q be distinct points of M. Define r_1 to be a positive real number less than both $1/8$ and $\varphi(p, q)/5$ and let $s_1 = 4r_1$. According to Theorem 1, there exists an r_1-chain $\{H_1^1, H_2^1, \cdots, H_{n_1}^1\}$ ($n_1 > 3$) in S from p to q such that for each positive integer i less than or equal n_1, H_i^1 is a continuum in M. Let m_1 be the smallest integer greater than or equal to $(n_1 - 1)/2$. There exist a set of disks $\{U_1^1, U_1^2, \cdots, U_{m_1}^1\}$ and a set of open disks $\{V_1^1, V_2^1, \cdots, V_{m_1}^1\}$ such that $\{V_1^1, V_2^1, \cdots, V_{m_1}^1\}$ is an s_1-chain in S from p to q and for each positive i less than or equal m_1, $H_{2i-1}^1 \cup H_{2i}^1 \cup H_{2i+1}^1 \subset U_i^1 \subset V_i^1$ (if n_1 is even, let $H_{n_1+1}^1 = \phi$).

Let $\{p_1^1, p_2^1, \cdots, p_{m_1+1}^1\}$ be a point set such that $p_1^1 = p$, $p_{m_1+1}^1 = q$, and for each positive integer i less than or equal m_1, p_i^1 belongs to H_{2i-1}^1. Let t_1 be the smallest number in the set $\{\varphi(\text{Bd } U_i^1, \text{Bd } V_i^1) \mid i \leq m_1\} \cup \{\varphi(p_i^1, p_{i+1}^1) \mid i \leq m_1\}$. Let r_2 be a positive real number less than both $t_1/5$ and $1/16$. Define s^2 to be $4r_2$. For each positive integer i less than or equal m_1, there exists an r_2-chain $\mathscr{C}_i$ in S from p_i^1 to p_{i+1}^1 such that each element of $\mathscr{C}_i$ is a continuum in $M \cap V_i^1$ and at a distance greater than $4r_2$ from Bd V_i^1 (Theorem 1). There exists an r_2-chain $\{H_1^2, H_2^2, \cdots, H_{n_2}^2\}$ in S from p to q whose elements belong to $\bigcup_{i=1}^{m_1} \mathscr{C}_i$ such that for each positive integer i less than or equal m_1, $\mathscr{C}_i \cap \{H_1^2, H_2^2, \cdots, H_{n_2}^2\}$ is a coherent collection. Let m_2 be the smallest integer greater than or equal to $(n_2 - 1)/2$. There exist a set of disks $\{U_1^2, U_2^2, \cdots, U_{m_2}^2\}$ and a set of open disks $\{V_1^2, V_2^2, \cdots, V_{m_2}^2\}$ such that $\{V_1^2, V_2^2, \cdots, V_{m_2}^2\}$ is an s_2-chain in S from p to q and for each positive integer i less than or equal m_2, $H_{2i-1}^2 \cup H_{2i}^2 \cup H_{2i+1}^2 \subset U_i^2 \subset V_i^2$ (if n_2 is even, let $H_{n_2+1}^2 = \varnothing$). Note that $\{V_1^2, V_2^2, \cdots, V_{m_2}^2\}$ runs straight through $\{V_1^1, V_2^1, \cdots, V_{m_1}^1\}$.

Continue this process. For $i = 3, 4, 5, \cdots$, there exists a chain $\{H_1^i, H_2^i, \cdots, H_{n_i}^i\}$ in S from p to q whose elements are continua in M, and there exists an s_i-chain $\{V_1^i, V_2^i, \cdots, V_{m_i}^i\}$ ($s_i < 1/2^i$) in S from p to q whose elements are open disks in S such that $\bigcup_{j=1}^{m_i} V_j^i$ contains $\bigcup_{j=1}^{n_i} H_j^i$ and $\{V_1^i, V_2^i \cdots, V_{m_i}^i\}$ runs straight through $\{V_1^{i-1}, V_2^{i-1}, \cdots, V_{m_{i-1}}^{i-1}\}$. For each positive integer i, let L_i be the continuum $\bigcup_{j=1}^{n_i} H_j^i$. The limiting set L of the sequence $L_1, L_2, L_3, \cdots$ is a continuum in M containing p and q. Note that for each positive integer i, L is contained in $\bigcup_{j=1}^{m_i} V_j^i$.

Let x be a point of $L - \{p, q\}$. For each positive integer i, let $V_{j_i}^i$ be an element of $\{V_1^i, V_2^i, \cdots, V_{m_i}^i\}$ which contains x. Assume without loss of generality that $4 < j_1 < m_1 - 4$. For each positive integer i, let P_i be $\{V_1^i, V_2^i, \cdots, V_{j_i-4}^i\}$ and let F_i be $\{V_{j_i+4}^i, V_{j_i+5}^i, \cdots, V_{m_i}^i\}$. Let $P = \bigcup_{i=1}^{\infty} (P_i \cap L)$ and $F = \bigcup_{i=1}^{\infty} (F_i \cap L)$. P and F are nonempty disjoint relatively open subsets of L and $P \cup F = L - \{x\}$. Hence x is a separating point of L. It follows that L has only two nonseparating points. Therefore L is an arc [6, Th. 6.2, p. 54]. Hence M is arcwise connected.

REMARK. Using [3, Th. 1] and Theorem 2 one can easily prove that if M is a semi-aposyndetic E-continuum, then M has Jones's cyclic property (that is, if p and q are distinct points of M and no point cuts p from q in M, then there exists a simple closed curve lying in M which contains p and q).

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Received December 4, 1970 and in revised form April 28, 1971.

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