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CONFORMAL MAPPINGS ONTO ω-SWIRLY DOMAINS

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This paper introduces the concept of ω-swirly domains as a generalization of the notion of starlike domains. After analyzing certain geometric properties of this concept, some analytical properties of conformal mappings of the unit disc onto ω-swirly domains are developed. The main theorem determines the maximum radius r_ω for which $\{z: |z| < r_\omega\}$ is mapped onto an ω-swirly domain by all schlicht functions on the unit disc; thus it gives an estimate of the twistedness of the images of subdiscs centered at the origin under conformal mappings of the unit disc. Comparisons are then made between the concept of ω-swirly and other geometric notions in the theory of conformal mapping.

I. Geometric considerations. Let $\mathscr{D}'$ be a simply connected domain in the complex plane having a boundary $\partial\mathscr{D}'$ which is a continuously differentiable Jordan curve. Consider a point x in the interior of $\mathscr{D}'$ and a point y of the boundary of $\mathscr{D}'$.

DEFINITION. Let $\arg(\mathscr{D}', x, y)$ be the angle between the vector $y - x$ and the outer normal to the domain $\mathscr{D}'$ at y. The argument is chosen to be zero at a point q of the boundary of minimal distance from x and is continued along the boundary.

The function $\arg(\mathscr{D}', x, y)$ can easily be shown to be singlevalued using the following fact:

REMARK 1. As a point p moves once about a continuously differentiable Jordan curve, the argument of the outer normal at p increases 2π while the argument of a vector from a fixed interior point to p also increases 2π.

It is easily seen $\arg(\mathscr{D}', x, y)$ does not depend on the choice of q. Consider a domain bounded by line segments from x to two points q_1 and q_2 and an arc of the boundary joining these points of minimal distance from x. Apply the version of Remark 1 for domains with piecewise continuously differentiable boundary.

DEFINITION. Let $\arg(\mathscr{D}', x) = \sup_{y \in \partial\mathscr{D}'} |\arg(\mathscr{D}', x, y)|$.

The *swirl of $\mathscr{D}'$ with respect to x* is $\arg(\mathscr{D}', x)$, and $\mathscr{D}'$ is *ω-swirly with respect to x* if the swirl of $\mathscr{D}'$ is less than or equal to ω.

The notion of swirl is now extended in a consistent way to domains not having smooth boundaries. Let $\mathscr{D}$ be a simply connected domain containing a point x.

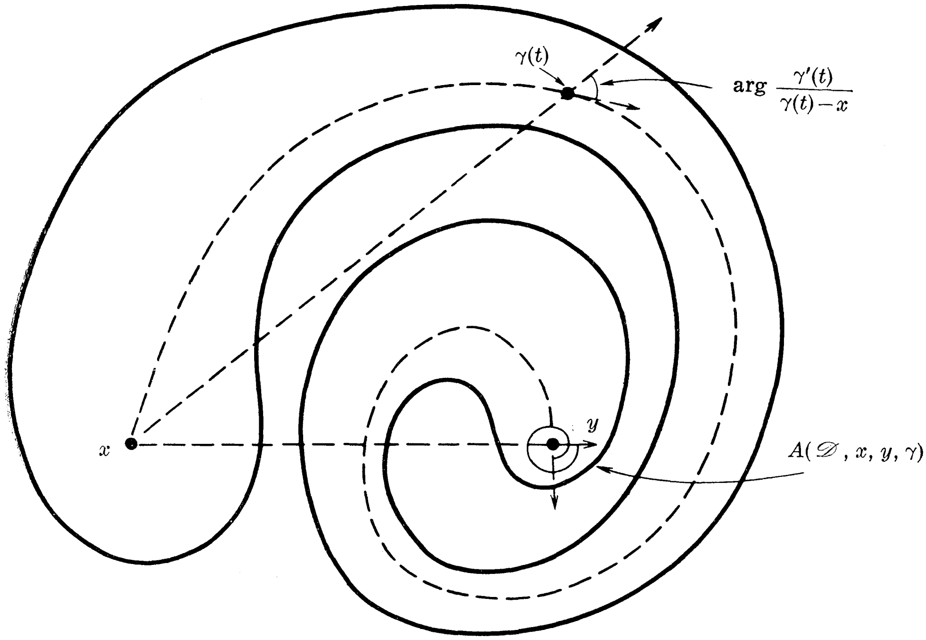
DEFINITION. *The swirl of* $\mathscr{D}$ *with respect to* x is the smallest number ω such that if $\varepsilon > 0$, there exists an expanding sequence $\{\mathscr{D}_n'\}$ of subdomains of $\mathscr{D}$, exhausting $\mathscr{D}$, which is such that each $\mathscr{D}_n'$ has a continuously differentiable boundary, contains x, and has swirl with respect to x less than $\omega + \varepsilon$. And $\mathscr{D}$ is ω-*swirly with respect to* x if the swirl of $\mathscr{D}$ with respect to x is less than or equal to ω.

The geometric interest in the notion of swirl comes from the following observation. If x and y are in $\mathscr{D}$, in order to move in $\mathscr{D}$ from x to y, it may be necessary to wind around y and around x. The swirl of $\mathscr{D}$ with respect to x is essentially an upper bound on the magnitude of the winding around y minus the winding around x. (See Figures 1 and 2.) This observation is now made precise and proved.

Let x and y be distinct points in $\mathscr{D}$. Let J be the set of all continuously differentiable Jordan arcs γ: $[0, 1] \to \mathscr{D}$ with $\gamma(0) = x$ and $\gamma(1) = y$.

DEFINITION. Let

$$A(\mathscr{D}, x, y, \gamma) = \sup_{0<t\leq 1} \left| \arg \frac{\gamma'(t)}{\gamma(t) - x} \right|$$

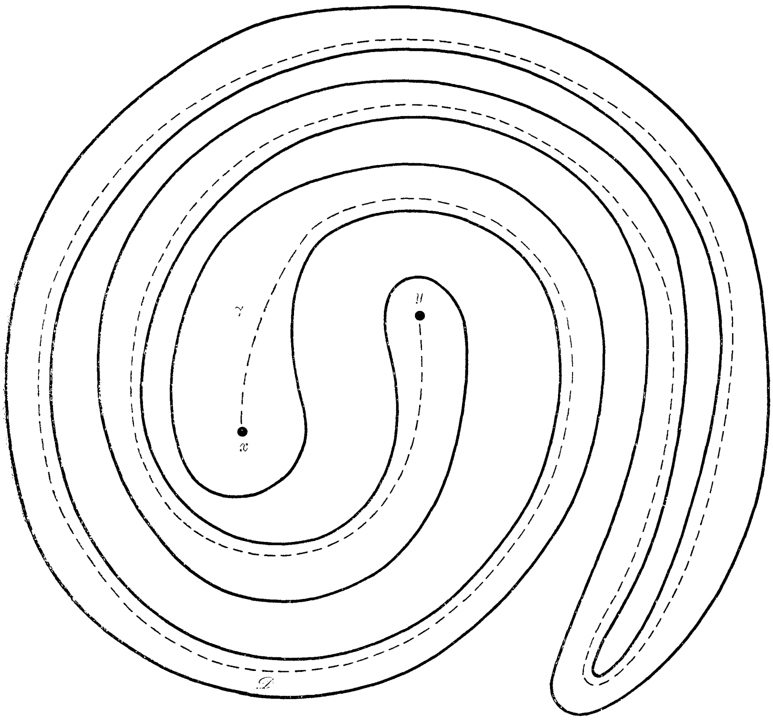


$A(\mathscr{D}, x) \geq \frac{9\pi}{4}$

FIGURE 1

where $\gamma'(t)$ denotes the left hand derivative and the branch of the argument is chosen to tend to zero as t approaches zero. Let $A(\mathscr{D}, x, y) = \inf_{\gamma \in J} A(\mathscr{D}, x, y, \gamma)$ and

$$A(\mathscr{D}, x) = \sup_{y \in \mathscr{D}, y \neq x} A(\mathscr{D}, x, y)$$



$A(\mathscr{D}, x, y) < 2\pi$

FIGURE 2

The relation between the swirl of the domain $\mathscr{D}$ with respect to x and $A(\mathscr{D}, x)$ will first be proved for the case that $\mathscr{D}$ has a smooth boundary.

Since $\mathscr{D}'$ has a boundary which is a continuously differentiable Jordan curve, $A(\mathscr{D}', x, y, \gamma)$ can be defined for y on the boundary of $\mathscr{D}'$ as well as in the interior and γ having points on the boundary as well as the interior of $\mathscr{D}'$. It is easily seen that the value of $A(\mathscr{D}', x)$ is not changed by enlarging the classes of points y and curves γ considered in its definition.

The regularity of the boundary of $\mathscr{D}'$ allows the use of the following fact.

REMARK 2. Suppose D is a simply connected domain having a boundary consisting of a rectifiable Jordan curve. If x and y are distinct points of D or the boundary, there exists a unique shortest arc in the closure of D joining x and y.

Proof. Since the boundary is rectifiable, it is easy to show there is an arc in the closure of D of finite length joining x to y. Let γ_n be a sequence of such arcs having their lengths tending to the infimum length of arcs in the closure of D joining x to y. A subsequence $\{\gamma_{n_k}\}$ can be chosen such that the points at half the infimum length converge. From this sequence, a further subsequence can be chosen such that points at one quarter the infimum length converge, $\cdots$ *et cetera*, for all points at a rational part of the distance between zero and the infimum length. Diagonalizing in the usual way and using properties of length, a sequence of arcs is obtained which converges pointwise to a Jordan arc Γ in the closure of D that is of minimal length and that joins x to y.

The uniqueness is now shown. Suppose there is another arc Σ in the closure of D joining x to y and having minimal length. Clearly Σ is a Jordan arc and there are distinct points x' and y' in $\Gamma \cap \Sigma$ such that the arc $\Gamma_{x'y'}$ of Γ from x' to y' and the arc $\Sigma_{x'y'}$ of Σ from y' to x' form a Jordan curve. This Jordan curve bounds a simply connected subdomain d of D. Consider a point p on $\Gamma_{x'y'} \cup \Sigma_{x'y'}$ at the maximal perpendicular distance from the line $x'y'$. Let L be a straight line parallel to $x'y'$ having p on one side and $x'y'$ on the other. Then the intersection of L and the interior of d has an open interval for a component. If both and end points of such an interval were on $\Gamma_{x'y'}$ or both on $\Sigma_{x'y'}$, then one of the arcs could be shortened by utilizing that arc. Hence we can assume that every such segment has one end point on $\Gamma_{x'y'}$ and the other on $\Sigma_{x'y'}$. As the line L approaches p, a subsequence of interior segments can be chosen so that the associated closed intervals converge to an interval (possibly a single point) on the line parallel to $x'y'$ and through p. This interval consists of points of the boundary of d. It has at least one point of $\Gamma_{x'y'}$ and at least one point of $\Sigma_{x'y'}$. And thus it has a point in both $\Gamma_{x'y}$ and $\Sigma_{x'y'}$. But $\Gamma_{x'y'}$ and $\Sigma_{x'y'}$ were supposed to have only the end points x' and y' in common. This contradiction implies there must be only one shortest arc.

As a corollary of the proof, if the boundary of D is continuously differentiable, then Γ is a Jordan arc consisting of straight line segments separated by arcs of the boundary of D which are met tangentially. Thus Γ is a continuously differentiable Jordan arc.

A relation between $\arg(\mathscr{D}', x)$ and $A(\mathscr{D}', x)$ is established by the following theorem.

THEOREM 1'. $|\arg(\mathscr{D}', x) - A(\mathscr{D}', x)| \leqq \pi/2$.

Proof. Let $\gamma(t)$ be a continuously differentiable Jordan arc from x to a point y on the boundary of $\mathscr{D}'$. Consider $\arg \gamma'(t_0)/(\gamma(t_0) - x)$ at a point $P = \gamma(t_0)$ of contact with the boundary of $\mathscr{D}'$. Let σ be a straight line segment joining x to a point $\sigma(t_1)$ of the boundary of $\mathscr{D}'$ at the minimal distance from x. Clearly $\arg \sigma'(t_1)/\sigma(t_1) - x = 0$ is between $\arg (\mathscr{D}, x, \sigma(t_1)) + \pi/2 = \pi/2$ and $\arg (\mathscr{D}', x, \sigma(t_1)) - \pi/2 = -\pi/2$. Consider a closed curve consisting of σ, γ and an arc of the boundary of $\mathscr{D}'$. This closed curve separates into Jordan curves to which the version of Remark 1 for domains with piecewise continuously differentiable boundary can be applied, thereby proving $\arg \gamma'(t_0)/\gamma(t_0) - x$ is between $\arg (\mathscr{D}', x, P) + \pi/2$ and $\arg (\mathscr{D}', x, P) - \pi/2$. It easily follows that $A(\mathscr{D}', x) \geqq \arg (\mathscr{D}', x) - \pi/2$.

The estimate of $A(\mathscr{D}', x)$ from above is now shown.

Let Γ_y be the shortest arc in the closure of $\mathscr{D}'$ joining x to y. By definition $A(\mathscr{D}', x, y, \Gamma_y) \geqq A(\mathscr{D}', x, y)$. Observe that for z in the interior of $\mathscr{D}'$, Γ_z approaches z in a straight line which can be extended until it intersects the boundary of $\mathscr{D}'$. There is a point y_0 on the boundary of $\mathscr{D}'$ such that Γ_{y_0} contains Γ_z. Thus the value of $\sup A(\mathscr{D}', x, y, \Gamma_y)$ is the same whether the supremum is taken over all y in $\mathscr{D}'$ or over all y on the boundary of $\mathscr{D}'$. Furthermore the extremum values of $\arg \Gamma'_y/y - x$ are taken on at points of contact with the boundary of $\mathscr{D}'$. Hence, by the first part of the proof, $\arg (\mathscr{D}', x) + \pi/2 \geqq A(\mathscr{D}', x)$, and the theorem is proved.

The previous work suggests that $\arg (\mathscr{D}', x)$ and $A(\mathscr{D}', x)$ are generalizations of starlikeness with respect to x. The relationships are easily seen to be as stated in the following theorem.

THEOREM 2'. These three statements are equivalent:

(a) $\mathscr{D}'$ is starlike with respect to x

(b) $\arg (\mathscr{D}', x) \leqq \pi/2$

(c) $A(\mathscr{D}', x) = 0$.

Theorems 1' and 2' can be generalized to domains without a smooth boundary.

THEOREM 1. *If* $A(\mathscr{D}, x)$ *is finite,*
$|A(\mathscr{D}, x) -$ *the swirl of* $\mathscr{D}$ *with respect to* $x| \leqq \pi/2$.

THEOREM 2. *The following statements are equivalent:*

(a) $\mathscr{D}$ is starlike with respect to x

(b) *the swirl of* $\mathscr{D}$ *with respect to* $x \leqq \pi/2$

(c) $A(\mathscr{D}, x) = 0$.

To prove Theorems 1 and 2 it suffices to prove that (1) if $\varepsilon > 0$, exists an expanding sequence $\{\mathscr{D}_n'\}$ of subdomains of $\mathscr{D}$ exhausting $\mathscr{D}$ and such that each $\mathscr{D}_n'$ has a continuously differentiable Jordan curve for a boundary, contains x, and has $A(\mathscr{D}_n', x) \leqq A(\mathscr{D}, x) + \varepsilon$ and (2) for a sequence as described in (1), $\lim \inf_{n\to\infty} A(\mathscr{D}_n' x) \geqq A(\mathscr{D}, x)$.

A sequence such as described in (1) is now constructed.

Consider $\varepsilon > 0$. Let the rational points in the plane be put into a sequence $\{z_n\}_{n=1}^{\infty}$ such that every rational point appears infinitely often. Let $\mathscr{D}_0^*$ be the maximal circular disc with center x which is in $\mathscr{D}$. If z_n is not in $\mathscr{D}_{n-1}^*$, let $\mathscr{D}_n^* = \mathscr{D}_{n-1}^*$. If z_n is in $\mathscr{D}_{n-1}^*$, consider the circular disc d_r with center z_n and radius r. Let $\mathscr{D}_{n,r}^{\dagger}$ be the minimal simply connected set containing $\mathscr{D}_{n-1}^* \cup d_r$. Consider the maximal number r_n such that $A(\mathscr{D}_{n,r_n}^{\dagger}, x) < A(\mathscr{D}, x) + \varepsilon$ and $\mathscr{D}_{n,r_n}^{\dagger} \subset \mathscr{D}$. Let $\mathscr{D}_n^* = \mathscr{D}_{n,r_n}^*$.

It is shown that $\bigcup_{n=1}^{\infty} \mathscr{D}_n^* = \mathscr{D}$ by an inductive argument. Any point y in $\mathscr{D}$ can be joined to x by an arc γ in $\mathscr{D}$ such that $A(\mathscr{D}, x, y, \gamma) < A(\mathscr{D}, x) + \varepsilon$. There exists a polygonal approximation γ_N to γ such that γ_N is in $\mathscr{D}$, joins x to y, and satisfies

$$A(\mathscr{D}, x, y, \gamma_N) < A(\mathscr{D}, x) + \varepsilon .$$

(While $A(\mathscr{D}, x, y, \gamma)$ was defined only for continuously differentiable arcs, it is clear how to extend the definition to include polygonal paths.) The induction is on the number of segments in the polygonal path.

Consider a point y in $\mathscr{D}^* - \mathscr{D}_0^*$ such that the straight line interval xy is in $\mathscr{D}$. There is an isosceles triangle δ in $\mathscr{D}$ with x being the midpoint of its smallest side, this side being in $\mathscr{D}_0^*$, and y being in the interior of δ, on the altitude to the smallest side. Picking z_m near the foot of that altitude, it is clear that $\mathscr{D}_m^*$ will contain a circular disc that contains a portion of each of the equal sides of δ and which covers at least a certain fraction of the altitude of δ to the smallest side. Consider a similar triangle which is smaller, has the same vertex as δ, has sides parallel to the sides of δ, and has its smallest side in $\mathscr{D}_m^*$ and y on the altitude to the smallest side. The same argument applied to the new triangle shows the same fraction on the new altitude being covered. After finitely many steps, the point y will be covered by some $\mathscr{D}_n^*$. Thus any point in $\mathscr{D}$ that can be seen from x will be $\bigcup_{n=1}^{\infty} \mathscr{D}_n^*$.

Consider a point y in $\mathscr{D}$ that can not be seen from x, but can be joined to x by a polygonal arc γ_2 consisting of two straight intervals in $\mathscr{D}$ such that $A(\mathscr{D}, x, y, \gamma_2) < A(\mathscr{D}, x) + \varepsilon$. Let y' be the common point of the two intervals of γ_2. There exists a circular disc c with center y' containing only points of $\mathscr{D}$ that can be seen from x. Let

δ' be an isosceles triangle in $\mathscr{D}$ with y' being the midpoint of its smallest side and y being on the altitude to that side and in the interior of δ'. Pick the smallest side of δ' so small that it is in c, and so small that every point y_0' in the triangle has the following property: The inequality $A(\mathscr{D}, x, y_0', \gamma_2^*) < A(\mathscr{D}, x) + \varepsilon$ is satisfied for the arc γ_2^* consisting of a straight line interval from x to the base of δ' and another straight line interval running to y_0' parallel to the altitude dropped to the smallest side of δ' from y. The disc c will be covered by some $\mathscr{D}_N^*$ Pick z_m with $m > N$, and z_m near y'. Proceed as in the first case.

Continuing in this way, every point of $\mathscr{D}$ is seen to be in $\bigcup_{n=1}^{\infty} \mathscr{D}_n^*$. By definition, $\mathscr{D}_n^* \subset \mathscr{D}$. Thus $\bigcup_{n=1}^{\infty} \mathscr{D}_n^* = \mathscr{D}$.

It is clear that the boundary of each $\mathscr{D}_n^*$ can be smoothed in such a way that the resulting domains $\{\mathscr{D}_n'\}$ form the desired sequence.

Part (2) is now proved. Let $\{\mathscr{D}_n'\}_{n=1}^{\infty}$ be any approximating sequence of $\mathscr{D}$ as described in part (1). If $\eta > 0$, there exists y in $\mathscr{D}$ such that $A(\mathscr{D}, x, y) > A(\mathscr{D}, x) - \eta$. Since the sets $\{\mathscr{D}_n'\}$ are increasing and $\cup \mathscr{D}_n' = \mathscr{D}$, there exists N such that if $n > N$, then $\mathscr{D}_n'$ contains y. Thus $A(\mathscr{D}_n', x, y)$ is defined for $n > N$, and $A(\mathscr{D}_n', x) \geqq A(\mathscr{D}_n', x, y) \geqq A(\mathscr{D}, x, y) > A(\mathscr{D}, x) - \eta$. Hence $\liminf_{n\to\infty} A(\mathscr{D}_n', x) \geqq A(\mathscr{D}, x)$.

From part (1) and part (2), Theorems 1 and 2 easily follow.

II. **Analytic considerations.** Let $f(z)$ be a conformal mapping of the open unit disc onto a domain $\mathscr{D}$ that is ω-swirly with respect to the point $f(0)$ with $f'(0)$ positive. Given $\varepsilon > 0$, let $\{\mathscr{D}_n'\}$ be a sequence of approximating sets with smooth boundaries as constructed in § I. Let $f_n(z)$ be the conformal mapping of the closed unit disc onto the closure of $\mathscr{D}_n'$ such that $f_n(0) = f(0)$ and $f_n'(0)$ is positive. At any point of the boundary of $\mathscr{D}_n'$, the outer normal has the same argument as $z f_n'(z)$ for the appropriate z on the unit circle. The radial vector has the same argument as $f_n(z) - f_n(0)$ for the appropriate z. Hence

$$\left| \arg \frac{z f_n'(z)}{f_n(z) - f_n(0)} \right| \leqq \omega + \varepsilon \tag{1}$$

for all points z on the unit circle. The function $z f_n'(z)/(f_n(z) - f_n(0))$ is analytic in the open unit disc, except for a removable singularity at the origin. Thus (1) also holds for all z in the open, punctured unit disc. Taking the limit as $n \to \infty$,

$$\left| \arg \frac{z f'(z)}{f(z) - f(0)} \right| \leqq \omega + \varepsilon$$

for z in the open, punctured unit disc and for all $\varepsilon > 0$. Hence

$$\left| \arg \frac{z f'(z)}{f(z) - f(0)} \right| \leqq \omega \tag{2}$$

for z in the punctured unit disc, where the branch of the argument is chosen so that the value tends to zero as z tends to zero.

From inequality (2), for $0 < r < 1$, the image under f of the disc $\{z: |z| < r\}$ is an ω-swirly domain with respect to $f(0)$. The images obtained in this way as $r \to 1^-$ could be used for an approximating sequence of $\mathscr{D}$. Therefore, inequality (2) is a characterization of those univalent functions f that map the unit disc onto a domain that is ω-swirly with respect to $f(0)$.

The preceding discussion justifies the following definitions. Definitions: A function f is *ω-swirly* if it is a schlicht function of the unit disc which satisfies (2).

The *radius r_ω of ω-swirliness* is the largest number r such that if $f(z)$ is a schlicht function on the unit disc, then $f(rz)$ is an ω-swirly function.

Following the method for determining the radius of starlikeness [2], we consider

$$\phi(z) = \frac{f\left(\dfrac{z_0 + z}{1 + \bar{z}_0 z}\right) - f(z_0)}{f'(z_0)} .$$

Then ϕ is schlicht on the unit disc. Consider the inequality

$$\left| \arg \frac{g(z)}{z} \right| \leqq \log \frac{1 + |z|}{1 - |z|} \tag{3}$$

for any schlicht function g of the unit disc with $g'(0)$ positive and $g(0) = 0$. Applying (3) to ϕ for $z = -z_0$,

$$\left| \arg \frac{f(z_0) - f(0)}{z_0 f'(z_0)} \right| \leqq \log \frac{1 + |z_0|}{1 - |z_0|} ,$$

and this result is sharp since (3) in sharp for all z. Hence r_ω is determined by

$$\log \frac{1 + r_\omega}{1 - r_\omega} = \omega .$$

The following observation is thus proved.

THEOREM 3. $r_\omega = \tanh \omega/2$.

A final property of ω-swirly functions is left unproved: If f_1 and f_2 are ω_1 and ω_2-swirly functions respectively, and $f_1(0) = 0$, and the range of f_1 is contained in the unit disc, then $f_2 \circ f_1$ is $\omega_1 + \omega_2$-swirly.

III. ω-Tortuous domains and functions. The concept of ω-tortuous is a generalization of the notion of convex just as the concept of ω-swirly is a generalization of the notion of starlike.

DEFINITION. A simply connected domain is *ω-tortuous* if it is ω-swirly with respect to each point in the domain.

(Note that the only ω-tortuous domain with $\omega < \pi/2$ is the entire plane.)

Using essentially the same proof as for (2), it follows that a schlicht function f of the unit disc has an image that is ω-swirly with respect to $f(z_0)$ for some point z_0, $|z_0| < 1$, if and only

$$\limsup_{|z|\to 1} \left| \arg \frac{zf'(z)}{f(z) - f(z_0)} \right| \leqq \omega \tag{4}$$

where the branch of the argument is chosen so that as $|z| \to |z_0|^+$ along the ray from the origin through z_0, the $\arg zf'(z)/(f(z) - f(z_0)) \to 0$.

A schlicht function f maps the unit disc onto an ω-tortuous domain if and only if (4) holds for *all* z_0, $|z_0| < 1$, or equivalently

$$\limsup_{|z|\to 1^-} \left| \arg \frac{zf'(z)}{f(z) - f(z\zeta)} \right| \leqq \omega \tag{5}$$

for all z with $0 < |z| < 1$, and for all ζ, $|\zeta| < 1$, where the branch of the argument is chosen so that, if z is fixed, ζ is real and $\zeta \to 1^-$, then $\arg zf'(z)/(f(z) - f(z\zeta))$ tends to zero.

For fixed ζ, as $z \to 0$, $|\arg zf'(z)/(f(z) - f(z\zeta))|$ has a limit less than $\pi/2$. Thus, by the maximum principle, (5) can be replaced by

$$\left| \arg \frac{zf'(z)}{f(z) - f(z\zeta)} \right| \leqq \omega\ . \tag{6}$$

For fixed z, observing the behavior as $\zeta \to 1$, it is clear that the maximum principle can be applied to show that (6) is equivalent to

$$\left| \arg \frac{zf'(z)}{f(z) - f(ze^{i\theta})} \right| \leqq \omega \tag{7}$$

for all θ, $0 < \theta < 2\pi$, and all z, $0 < |z| < 1$, where the branch of the argument is chosen so that as $z \to 0$, $\arg zf'(z)/(f(z) - f(ze^{i\theta}))$ tends to $\pi/2 - \theta/2$.

DEFINITION. An analytic function f on the unit disc is *ω-tortuous* if (7) holds.

(That f is univalent follows from the argument in (7) being defined.)

From the definition, if $f(z)$ is ω-tortuous and $0 < \rho < 1$, then $f(\rho z)$ is ω-tortuous. Thus the following definition is suggested.

DEFINITION. ρ_ω is the largest number ρ such that, for all schlicht functions f on the unit disc, the image of $\{z: |z| < \rho\}$ is ω-tortuous.

THEOREM 4. *For $\omega \geqq \pi/2$, ρ_ω is the largest number ρ such that for all θ, $0 < \theta < 2\pi$,*

$$\log \frac{1 + \left| \dfrac{1 - e^{i\theta}}{1 - \rho^2 e^{i\theta}} \right| \rho}{1 - \left| \dfrac{1 - e^{i\theta}}{1 - \rho^2 e^{i\theta}} \right| \rho} + \left| \arg \frac{1 - e^{i\theta}}{1 - \rho^2 e^{i\theta}} \right| \leqq \omega \,. \tag{8}$$

Proof. Let f be a schlicht function on the unit disc. Consider two points z_1 and z_2 with $|z_1| = |z_2| = \rho < 1$. Let

$$\varphi(\zeta) = f\left(\frac{\zeta + z_1}{1 + \bar{z}_1 \zeta}\right).$$

Then $f(z_1) = \varphi(0)$ and $f(z_2) = \varphi(\zeta_0)$ where

$$\zeta_0 = \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} = \frac{e^{i\theta} - 1}{1 - \rho^2 e^{i\theta}} z_1,\ z_2 = e^{i\theta} z_1 \,,$$

and $f'(z_1) = (1 - |z_1|^2)^{-1} \varphi'(0)$. Thus by (7), $\rho \leqq \rho_\omega$ if and only if for every schlicht function φ

$$\left| \arg \frac{z_1 \varphi'(0)}{\varphi(0) - \varphi\left(\dfrac{e^{i\theta} - 1}{1 - \rho^2 e^{i\theta}} z_1\right)} \right| \leqq \omega \tag{9}$$

for $0 < \theta < 2\pi$, where the argument is chosen to be of minimum magnitude as $z_1 \to 0$.

Let $\psi(z) = (\varphi(z) - \varphi(0))/\varphi(0)$. Then $\psi(0) = 0$, $\psi'(0) = 1$, and $\psi(z)$ is a schlicht function on the unit disc. Because of the freedom in the choice of f, it is sufficient to examine (9) for $z_1 = -\rho$. Then (9) becomes

$$\left| \arg \psi\left(\frac{1 - e^{i\theta}}{1 - \rho^2 e^{i\theta}} \rho\right) \right| \leqq \omega \,,$$

for all θ, $0 < \theta < 2\pi$. Using the estimate (3) for $|\arg \psi(z)/z|$ cited in § II, and the fact that the estimate gives sharp upper and lower bounds, the proof of Theorem 4 is completed. (See the graph of ω as a function of ρ_ω obtained by computation in Figure 3.)

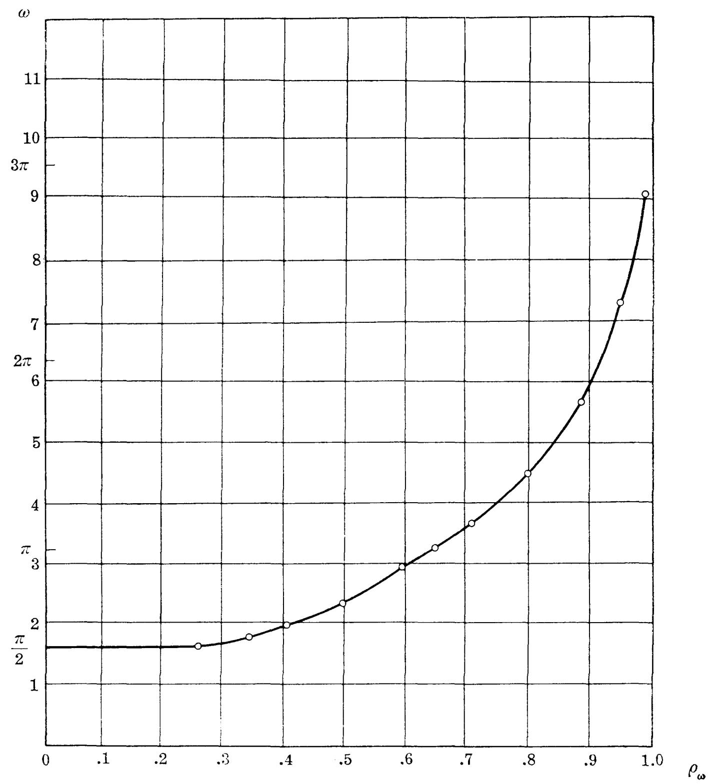


FIGURE 3

IV. Relations to previous concepts. Goluzin [2] suggested that the notion of starlikeness should be generalized in the following fashion. Given a simply connected domain $\mathscr{D}$ containing a distinguished point x, let $n(\mathscr{D})$ be the smallest integer N such that if y is a point of $\mathscr{D}$, there exists a polygonal arc in $\mathscr{D}$, consisting of N straight line intervals, joining x to y. This measure of a domain has the desirable property of being a natural geometrical concept. Also, by using the radius of starlikeness, it is easy to estimate from below the maximum number r_N such that, for any schlicht function f of the unit disc, the image d_r of $\{z\colon |z| < r < r_N\}$ under f satisfies $n(d_r) \leqq N$.

On the other hand, it has several undesirable characteristics. For example, f may be a conformal mapping of the unit disc onto a domain $\mathscr{D}$ for which $n(\mathscr{D}) = N$, but for some r, $0 < r < 1$, f may map $\{z\colon |z| < r\}$ onto a domain d for which $n(d) > N$. (See Figure 4.)

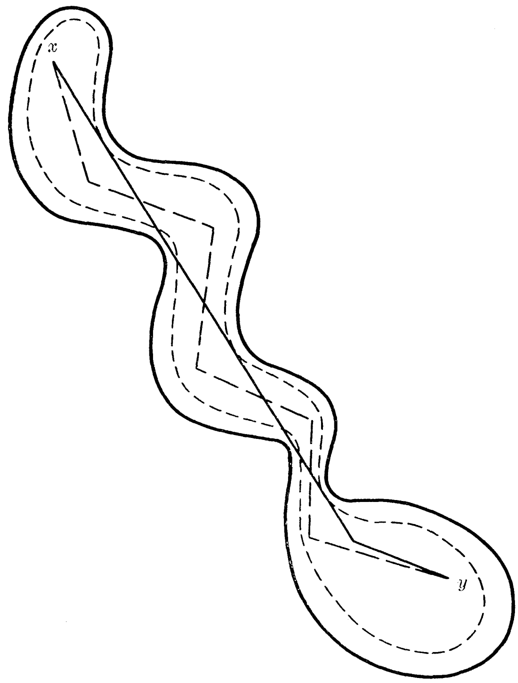


FIGURE 4

Secondly, there does not appear to be a simple analytic characterization of the function mapping the unit disc onto a domain $\mathscr{D}$ such that $n(\mathscr{D}) = N$. Thirdly, there seems little hope of finding the exact value of r_N.

While the notion of ω-swirly in two dimensions has none of these disadvantages of Golusin's concept, its geometrical interpretation is certainly more difficult. For higher dimensions, where the level surfaces of Green's functions are being considered, the notion of ω-swirly has all the disadvantages listed for Goluzin's concept.

Another class of domains that has been introduced is that of plosive domains [1]. A simply connected domain $\mathscr{D}$ containing a distinguished point x is plosive with respect to x if for every point y in $\mathscr{D}$, there exists an arc γ in $\mathscr{D}$ joining x to y such that if p is a point moving along γ, then the straight ling distance from x to y is a monotonic function of the position of p. A domain $\mathscr{D}$ is plosive with respect to x if and only if it is π-swirly with respect to x. A domain $\mathscr{D}$ is a flare domain if it is plosive with respect to each point in $\mathscr{D}$. The flare domains are exactly the π-tortuous domains.

The relations of ω-swirly to starlike and ω-tortuous to convex have already been discussed.

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