PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 3, 1971

BURKILL-CESARI INTEGRALS OF QUASI ADDITIVE INTERVAL FUNCTIONS

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The Burkill-Cesari B-C integrals arising in L. Cesari's theory of quasi additive vector-valued set functions need not be additive as functions of sets. It is shown in the present paper that these integrals satisfy quasi additivity and overadditivity properties. These properties are used to prove Banach-type differentiation theorems for B-C integrals defined on Euclidean spaces. Variants of Cesari's basic quasi additivity hypothesis and some simplifications in the formulation of the general theory are also discussed.

The theory of quasi additive vector-valued interval functions z and associated B-C integrals $\int[z, A]$ over an abstract space A was introduced by Cesari [2]. The integration processes of Cauchy-Riemann and Lebesgue-Stieltjes were shown to be included in this theory. More importantly, it was proved that the property of quasi additivity is preserved by parametric integrands $f(p, q)$ and that Cesari-Weierstrass integrals over a variety T are the B-C integrals $\int[f(T, z), A]$ of the corresponding composite functions. In [3] Cesari extended these concepts to subsets of A and determined conditions under which the Cesari-Weierstrass integral can be represented as a Lebesgue integral $\int_A f(T, \theta)d\mu$ with respect to a suitable measure μ and vector θ of Radon-Nikodym derivatives. Further developments in the theory have been discussed by Nishiura [4], Stoddart [6], and Warner [7, 8, 9]. In particular, Warner extended the theory to include quasi additive functions with values in locally convex spaces and showed that many other integration processes, including those of Perron and Pettis, are contained in the theory.

In this paper we discuss properties of B-C integrals as functions defined on the subsets of a given space A. It is convenient to follow the original setting of Cesari. Thus the B-C integral $\int[z, M]$ of z over an arbitrary subset M of A exists whenever z is quasi additive on M. We differ from Cesari's procedure, however, by formulating all quasi additivity relations relative to a single directed system. This technical device, used by Nishiura [4] in surface area theory, simplifies Cesari's formulation and allows us to prove, in § 1, that if z is quasi additive on M and if the B-C integral $\int[|z|, M]$ is finite, then z is quasi additive on every subset M' of M; hence $\int[z, M']$ exists for every $M' \subset M$.

Structure theorems for B-C integrals are discussed in § 2. In addition to the theorems already proved by Cesari, we prove that if z is quasi additive on M and $\int[|z|, M]$ is finite, then the interval function $\int[z, \cdot]$ is also quasi additive on M and

$$\int[z, M] = \int\left[\int[z, \cdot], M\right];$$

if, in addition, z is real valued and nonnegative, then $\int[z, M]$ is the total variation of $\int[z, \cdot]$ over M.

The structure theorems of § 2 are used in § 3 to prove Banach-type differentiation theorems for the case in which A is an open set in Euclidean space. It is shown that if z is quasi additive on A and $\int[|z|, A]$ is finite, then the interval function $\int[z, \cdot]$ admits a "hard analysis" vector-valued derivative J such that $\int[|z|, A] \geqq \int_A |J(w)|\, dw$; equality holds if and only if $\int[|z|, \cdot]$ is AC in the sense of [1, p. 411]. This result, which we shall use in a later paper, is obtained by extracting essential elements in Cesari's presentation of the theory of generalized Jacobians associated with a parametric surface of finite area [1].

Stronger types of quasi additivity relations are discussed in § 4. Necessary and sufficient conditions in terms of the interval function $\int[|z|, \cdot]$ are determined for the basic quasi additivity hypothesis to be equivalent to a stronger hypothesis used by Cesari [3, p. 130].

The connection between the present formulation and that of Cesari is discussed in § 5. It is observed that Cesari's representation theorem, $\int[f(T, z), A] = \int_A f(T, \theta)d\mu$, holds in the present setting also.

1. Definitions and first properties. Let A be a nonempty set, $\{I\}$ be a nonempty collection of subsets I of A, $\{D\}$ be a nonempty family of nonempty finite systems $D = [I]$ of sets I in $\{I\}$, and δ be a real-valued function defined on $\{D\}$. We refer to the sets in $\{I\}$ as *intervals* and to the function δ as a *mesh*. The axioms

(a): A is a topological space,

(b): each interval I has a nonempty interior, (c): the intervals of each system D are nonoverlapping, i.e., int $(I) \cap \mathrm{cl}(J) = \mathrm{cl}(I) \cap \mathrm{int}(J) = \emptyset$ whenever $I, J \in D$, $I \neq J$,

(d): $0 < \delta(D) < \infty$ for each system D and, given $\varepsilon > 0$, there are systems with $\delta(D) < \varepsilon$,

will be assumed. In order to treat some trivial examples in a uniform manner we allow $\delta(D) = 0$ if only the zero (real or vector-valued) interval function (see below) is being considered.

The norm of a point $q = (q_1, \cdots, q_m)$ in the real Euclidean m-space E_m is denoted by $|q| = [\Sigma_r q_r^2]^{1/2}$, where Σ_r ranges over $r = 1, \cdots, m$. We set $a^{\pm} = (|a| \pm a)/2$ for any real number a.

Let z: $\{I\} \to E_m$, $z = (z_1, \cdots, z_m)$, be an interval function and M be a subset of A. Associated with z are the nonnegative interval functions $|z|$, $|z_r|$, z_r^+,and z_r^-, $r = 1, \cdots, m$. When needed, z' denotes a second interval function having the same range space as z.

Given a system $D_0 = [I]$, let $S[z, M, D_0] = \Sigma_I s(I, M) z(I)$, where Σ_I ranges over all $I \in D_0$ and $s(I, M) = 1$ or 0 according as $I \subset M$ or $I \not\subset M$. If $D = [J]$ is also a system, then $S[z, M, D] - S[z, M, D_0] = \Sigma_I s(I, M) [\Sigma_J s(J, I) z(J) - z(I)] + \Sigma_J s(J, M)[1 - \Sigma_I s(J, I) s(I, M)] z(J)$; the second term on the right is nonnegative whenever z is nonnegative.

The *B-C integral* of z over M is defined as

$$\int [z, M] = \lim S[z, M, D]$$

provided this limit, taken as $\delta(D) \to 0$, exists in E_m. If z is real-valued, then ∞ is also allowed as a value for this integral.

DEFINITION 1.1. *z is quasi additive* on M if for each $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, M) > 0$ such that if $D_0 = [I]$ is any system satisfying $\delta(D_0) < \eta$, then there also exists $\lambda = \lambda(\varepsilon, M, D_0) > 0$ such that the relations

(qa_1): $\Sigma_I s(I, M) | \Sigma_J s(J, I) z(J) - z(I) | < \varepsilon$,

(qa_2): $\Sigma_J s(J, M)[1 - \Sigma_I s(J, I) s(I, M)] \, |z(J)| < \varepsilon$,

hold for every system $D = [J]$ with $\delta(D) < \lambda$. If z is real-valued and if (qa_1) and (qa_2) are replaced by the single relation

(qsa): $\Sigma_I s(I, M)[\Sigma_J s(J, I) z(J) - z(I)]^- < \varepsilon$,

then z is *quasi subadditive* on M.

PROPOSITION 1.2. (i) *If z is quasi additive on M, then the B-C integral $\int[z, M]$ exists in E_m.*

(ii) *If z is nonnegative and quasi subadditive on M, then $\int[z, M]$ exists, $0 \leqq \int[z, M] \leqq \infty$. If, in addition, this B-C integral is finite,* *then z is quasi additive on M.*

(iii) *If z and z' are quasi additive on M, then $az + bz'$ is quasi additive on M for every pair of real numbers a and b, and*

$$\int[az + bz', M] = a \int[z, M] + b \int[z', M] .$$

(iv) *If z and z' are nonnegative and quasi subadditive on M and M', respectively, and if $z' \leqq z$, $M' \subset M$, then $\int[z', M'] \leqq \int[z, M]$.*

(v) *z is quasi additive on M if and only if its components z_r, $r = 1, \cdots, m$, are all quasi additive on M.*

(vi) *If z is quasi additive on M, then $|z|, |z_r|, z_r^+$, and z_r^-, $r = 1, \cdots, m$, are all quasi subadditive on M.*

The proofs of the preceding statements are analogous to the proofs given in [2, pp. 97–99] for the case $M = A$.

If z is quasi additive on M and if there exist systems of arbitrarily small mesh, none of whose intervals are contained in M, then $\int[z, M]$ is the zero vector. In particular, z is automatically quasi additive on the empty set and $\int[z, \varnothing]$ is the zero vector.

It follows from relation (qsa) that if z is quasi subadditive on M, then it is quasi subadditive on every subset of M.

PROPOSITION 1.3. *If z is quasi additive on M and if $\int[|z|, M]$ is finite, then z is quasi additive on every subset of M.*

Proof. Let $M' \subset M$ be given. We shall refer to the statements of (1.2). By (vi), $|z|$ is quasi subadditive on M and therefore also on M'. By (ii) and (iv), $\int[|z|, M']$ exists and is dominated by $\int[|z|, M] < \infty$. Thus $|z|$ is quasi additive on M' by (ii). Given $\varepsilon > 0$ we can determine the parameters of (1.1) so that the relations (qa_1) and (qa_2) are simultaneously satisfied relative to z on M and $|z|$ on M'. Thus

$$\begin{aligned} &\Sigma_I\, s(I, M')\, |\Sigma_J\, s(J, I)z(J) - z(I)| \\ &\leqq \Sigma_I\, s(I, M)\, |\Sigma_J\, s(J, I)z(J) - z(I)| < \varepsilon , \\ &\Sigma_J\, s(J, M')[1 - \Sigma_I\, s(J, I)s(I, M')]\, |z(J)| < \varepsilon , \end{aligned}$$

and we conclude that z is quasi additive on M'.

It is convenient to set

$$F(M) = \int[|z|, M] , \qquad F_r(M) = \int[|z_r|, M] ,$$

$$F_r^+(M) = \int[z_r^+, M] , \qquad F_r^-(M) = \int[z_r^-, M] ,$$

$$\mathscr{F}_r(M) = \int[z_r, M] ,$$

whenever these B-C integrals exist and are finite, and to set $F(M) = 0$, etc., otherwise. We also define

$$\mathscr{F}(M) = (\mathscr{F}_1(M), \cdots, \mathscr{F}_m(M)) .$$

As a consequence of the definitions and preceding propositions we have the following result.

PROPOSITION 1.4. *Suppose that z is quasi additive on M and that $\int[|z|, M]$ is finite. Let $M' \subset M$ be given. Then $z, z_r, |z|, |z_r|, z_r^+$, and z_r^-, $r = 1, \cdots, m$, are all quasi additive on M', and*

$$F_r^+(M') - F_r^-(M') = \mathscr{F}_r(M') ,$$
$$F_r^+(M') + F_r^-(M') = F_r(M') ,$$
$$|\mathscr{F}_r(M')| \leqq F_r(M') \leqq F(M') ,$$
$$|\mathscr{F}(M')| \leqq [\Sigma_r F_r^2(M')]^{1/2} \leqq F(M') \leqq \Sigma_r F_r(M') ,$$

for each r. Given $\varepsilon > 0$, there exists $\mu = \mu(\varepsilon, M')$, $0 < \mu \leqq \varepsilon$, such that if $D_0 = [I]$ is any system with $\delta(D_0) < \mu$, then

$$|\mathscr{F}(M') - S[z, M', D_0]| < \varepsilon, |F(M') - S[|z|, M', D_0]| < \varepsilon ,$$

and analogously for $\mathscr{F}_r$, F_r, F_r^+ and *F_r^- for each r. Finally, there exists $\lambda = \lambda(\varepsilon, M', D_0)$, $0 < \lambda \leqq \mu$, such that the relations (qa_1) and (qa_2) of* (1.1) *(applied to M') hold simultaneously for $z, z_r, |z|, |z_r|, z_r^+$,* and *$z_r^-$ for every r and every system $D = [J]$ with $\delta(D) < \lambda$.*

2. B-C integrals as interval functions. The *total variation* (relative to $\{D\}$) of z over M is defined as

$$V[z, M] = \sup S[|z|, M, D]$$

where the supremum is taken over all systems $D \in \{D\}$. We have $0 = V[z, \emptyset] \leqq V[z, M'] \leqq V[z, M] \leqq V[z, A] \leqq \infty$ whenever $M' \subset M$. If z is quasi additive on M, then $\int[|z|, M] \leqq V[z, M]$ and strict inequality may hold.

PROPOSITION 2.1. *If $\{D\}$ is the family of all nonempty finite systems of nonoverlapping intervals $I \in \{I\}$, then*

(i) *$V[z, M] \geqq \sum_{n=1}^{\infty} V[z, M_n]$ for every sequence $\{M_n\}$ of nonoverlapping subsets M_n of M,*

(ii) *if each interval $I \in \{I\}$ is connected, then*

$$V[z, G] = \sum_{n=1}^{\infty} V[z, G_n]$$

whenever $\{G_n\}$ is a sequence of disjoint open subsets of A such that $G = \bigcup_{n=1}^{\infty} G_n$.

These two properties of the total variation are well-known (cf. [1, 9.3]). The connectedness of the intervals is assumed in (ii) to assure that if $I \subset G$, then $I \subset G_n$ for one and only one value of n.

In the next two results we assume that z is nonnegative (real-valued). In this case we have $\mathscr{F} = F$.

THEOREM 2.2. *If z is nonnegative and quasi additive on M, then the interval function F is also quasi additive on M, and*

$$F(M') = V[F, M'] = \int[F, M'] \tag{1}$$

for every subset M' of M.

Proof. In view of (1.3), it suffices to take $M' = M$. We first prove the equalities in (1). For any two systems $D_0 = [I]$ and $D = [J]$ we have

$$\begin{aligned} 0 &\leqq \textstyle\sum_J s(J, M)[1 - \sum_I s(J, I)s(I, M)]z(J) \\ &= \textstyle\sum_J s(J, M)z(J) - \sum_J s(J, M) \sum_I s(J, I)s(I, M)z(J) \\ &= S[z, M, D] - \textstyle\sum_I s(I, M) \sum_J s(J, I)z(J) \\ &= S[z, M, D] - \textstyle\sum_I s(I, M)S[z, I, D] \,. \end{aligned}$$

As $\delta(D) \rightarrow 0$ we obtain

$$0 \leqq F(M) - \textstyle\sum_I s(I, M)F(I) = F(M) - S[F, M, D_0] \,.$$

Thus

$$F(M) \geqq V[F, M] \geqq S[F, M, D_0] \tag{2}$$

for every system D_0. If $\varepsilon > 0$ and if D_0 and D are as in (1.1), then we also have $0 \leqq S[z, M, D] - \sum_I s(I, M)S[z, I, D] < \varepsilon$. As $\delta(D) \rightarrow 0$ we thus obtain

$$0 \leqq F(M) - S[F, M, D_0] \leqq \varepsilon \tag{3}$$

for all systems D_0 of sufficiently small mesh. The equalities (1) follow from (2) and (3) since $\varepsilon > 0$ was arbitrary.

It remains to show that F is quasi additive on M. Let $\varepsilon > 0$ be given and let $D_0 = [I]$ be a system. If $\int [z, M] = 0$, then the problem is trivial. We thus assume that $\int [z, M] > 0$ and that $\delta(D_0)$ is small enough that the set M contains at least one interval $I \in D_0$. Let N be the number of intervals $I \in D_0$ with $I \subset M$. By (1) there exists $\lambda = \lambda(\varepsilon, M, D_0) > 0$ such that

$$0 \leqq V[F, I] - S[F, I, D] < \varepsilon/N \tag{4}$$

for every system $D = [J]$ with $\delta(D) < \lambda$ and for every interval $I \in D_0$ with $I \subset M$. Since $V[F, I] = F(I)$ for each of these intervals, we have

$$\sum_I s(I, M)[\sum_J s(J, I)F(J) - F(I)]^- < N(\varepsilon/N) = \varepsilon$$

by (4). This proves that F is quasi subadditive on M. By (1) we have $F(M) = \int [F, M]$ and this B-C integral is finite. Thus F is quasi additive on M by statement (ii) of (1.2).

PROPOSITION 2.3. *If z is nonnegative and quasi additive on M, then $F(M) \geqq \sum_{n=1}^{\infty} F(M_n)$ for every sequence $\{M_n\}$ of nonoverlapping subsets M_n of M.*

Proof. Let $\{D'\}$ be the family of all nonempty finite systems of nonoverlapping intervals $I \in \{I\}$ and let δ' be the mesh on $\{D'\}$ defined by $\delta'(D') = \delta(D')$ if $D' \in \{D\}$ and $\delta'(D') = 1$ if $D' \in \{D'\} - \{D\}$. z is obviously quasi additive on M relative to $(\{D'\}, \delta')$, and $F(M') = \lim S[z, M', D']$ as $\delta'(D') \to 0$ for every set $M' \subset M$. The proposition is now a consequence of (2.1) and (2.2) applied to $(\{D'\}, \delta')$.

Examples (see [1, p. 400]) show that strict inequality may hold in the above proposition even if M is the union of extensively overlapping sets M_n.

We now return to the case in which $z = (z_1, \cdots, z_m)$ is vector-valued.

PROPOSITION 2.4. *Assume that each interval $I \in \{I\}$ is connected. Let $\{G_n\}$ be a sequence of disjoint open subsets of A and let $G = \cup_{n=1}^{\infty} G_n$. If z is quasi additive on G and if $\int [|z|, G]$ is finite, then $\mathscr{F}(G) = \sum_{n=1}^{\infty} \mathscr{F}(G_n)$ and the series is absolutely convergent. Analogous statements hold for $\mathscr{F}_r, F, F_r, F_r^+$, and F_r^-, $r = 1, \cdots, m$.*

Proof. As shown in the proof of the preceding proposition, it is not restrictive to assume that $\{D\}$ is the family of all finite systems of nonoverlapping intervals. The desired equalities thus hold for F, F_r, F_r^+, and F_r^- by (2.1) and (2.2). These equalities extend to $\mathscr{F}_r$ and $\mathscr{F}$ by virtue of the relations $\mathscr{F}_r = F_r^+ - F_r^-$ and $\mathscr{F} = (\mathscr{F}_1, \cdots, \mathscr{F}_m)$. Absolute convergence for the latter series holds since $|\mathscr{F}_r| \leqq F_r$ and $|\mathscr{F}| \leqq F$.

Another proof of (2.4) has been given under slightly different hypotheses by Cesari [3, p. 118].

THEOREM 2.5. *If z is quasi additive on M and if $\int[|z|, M]$ is finite, then the interval function $\mathscr{F}$ is quasi additive on M and $\mathscr{F}(M') = \int[\mathscr{F}, M']$ for every subset M' of M.*

Proof. The interval functions F_r^+ and F_r^-, $r = 1, \cdots, m$, are quasi additive on M by (2.2). The functions $\mathscr{F}_r = F_r^+ - F_r^-$ are thus quasi additive on M by (1.2) (iii). Hence $\mathscr{F}$ is quasi additive on M by (1.2) (v). From (1.4) and (2.2) we have

$$\mathscr{F}_r(M') = F_r^+(M') - F_r^-(M') = \int[F_r^+, M'] - \int[F_r^-, M'] = \int[\mathscr{F}_r, M']$$

and we conclude that $\mathscr{F}(M') = \int[\mathscr{F}, M']$ for every subset M' of M.

THEOREM 2.6. *If z is quasi additive on M and if $\int[|z|, M]$ is finite, then*

$$F(M') = \lim_{\delta(D_0)\to 0} \textstyle\sum_I s(I, M')[\sum_r F_r^2(I)]^{1/2}$$

for every subset M' of M. Here, $D_0 = [I] \in \{D\}$.

Proof. By (1.3) it suffices to take $M' = M$. Let $\varepsilon > 0$ be given. Let $\mu = \mu(\varepsilon, M)$ with $0 < \mu \leqq \varepsilon$, $D_0 = [I]$ with $\delta(D_0) < \mu$, $\lambda = \lambda(\varepsilon, M, D_0)$ with $0 < \lambda \leqq \mu$, and $D = [J]$ with $\delta(D) < \lambda$ be as in (1.4). Then

$$|F(M) - \textstyle\sum_I s(I, M)|\, z(J)\,| - |z_r(I)|| < \varepsilon\ ,$$
$$\textstyle\sum_I s(I, M)\, |\sum_J s(J, I)\, |z_r(J)| - |z_r(I)|| < \varepsilon\ ,$$

for each $r = 1, \cdots, m$. For each r and $I \in D_0$, let

$$a_r(I) = \textstyle\sum_J s(J, I)\, |z_r(J)| - |z_r(I)|\ .$$

By substitution and Minkowski's inequality we obtain

$$
\begin{aligned}
F(M) &< \sum_I s(I, M)\,|z(I)| + \varepsilon \\
&= \sum_I s(I, M)\{\sum_r |z_r(I)|^2\}^{1/2} + \varepsilon \\
&= \sum_I s(I, M)\{\sum_r [\sum_J s(J, I)\,|z_r(J)| - a_r(I)]^2\}^{1/2} + \varepsilon \\
&\leqq \sum_I s(I, M)\{\sum_r [\sum_J s(J, I)\,|z_r(J)| + |a_r(I)|]^2\}^{1/2} + \varepsilon \\
&\leqq \sum_I s(I, M)\{\sum_r [\sum_J s(J, I)\,|z_r(J)|]^2\}^{1/2} \\
&\quad + \sum_I s(I, M)\{\sum_r |a_r(I)|^2\}^{1/2} + \varepsilon \\
&\leqq \sum_I s(I, M)\{\sum_r [\sum_J s(J, I)\,|z_r(J)|]^2\}^{1/2} \\
&\quad + \sum_r \sum_I s(I, M)\,|a_r(I)| + \varepsilon \\
&< \sum_I s(I, M)\{\sum_r [\sum_J s(J, I)\,|z_r(J)|]^2\}^{1/2} + (m + 1)\varepsilon\ .
\end{aligned}
$$

As $\delta(D) \to 0$ and with the help of (2.2) we obtain

$$F(M) \leqq \sum_I s(I, M)[\sum_r F_r^2(I)]^{1/2} + (m + 1)\varepsilon$$

for all systems $D_0 = [I]$ with $\delta(D_0) < \mu(\varepsilon, M) \leqq \varepsilon$. As $\varepsilon > 0$ was arbitrary, we have

$$F(M) \leqq \underline{\lim}_{\delta(D_0) \to 0} \sum_I s(I, M)[\sum_r F_r^2(I)]^{1/2},\ D_0 = [I]\ .$$

By (2.2) and (1.4), on the other hand, we have

$$F(M) \geqq \sum_I s(I, M)F(I) \geqq \sum_I s(I, M)[\sum_r F_r^2(I)]^{1/2}$$

for every system $D_0 = [I]$. This competes the proof.

Note that we have also proved

$$F(M) = \sup \sum_I s(I, M)[\sum_r F_r^2(I)]^{1/2}$$

where the supremum is taken over all systems $D_0 = [I]$.

3. **Derivatives.** Points of E_k, $k \geqq 1$, will be denoted by $w = (w_1, \cdots, w_k)$. The interior and frontier of a set E in E_k will be denoted by E° and E^*, respectively. The term a.e. (almost everywhere) will be used relative to k-dimensional Lebesgue measure L_k on E_k.

Throughout this section A will denote a nonempty open subset of E_k and $\{I\}$ the collection of all nondegenerate closed intervals $I = \{w \in E_k: a_i \leqq w_i \leqq b_i,\ i = 1, \cdots, k\}$ contained in A. $\{D\}$ will denote the family of all non-empty finite systems $D = [I]$ of nonoverlapping sets $I \in \{I\}$. We assume that a mesh δ on $\{D\}$ is given. The definitions and results of this section may also be used if $\{I\}$ is replaced by the collection of all polyhedral regions or simple polyhedral regions (see [4]) contained in A.

We recall some definitions. A real-valued interval function z is said to be

(i) *overadditive* if $z(I) \geqq \sum_j z(I_j)$ for each set $I \in \{I\}$ and each finite subdivision $I = \bigcup_j I_j$ of I into nonoverlapping sets $I_j \in \{I\}$,

(ii) *additive* if equality holds in (i),

(iii) *BV* if the total variation $V[z, A]$ (relative to $\{D\}$) is finite,

(iv) *AC* if (a) z is additive, and (b) for each $\varepsilon > 0$ there exists $\eta > 0$ such that $S[|z|, A, D] < \varepsilon$ for each system D with $S[L_k, A, D] < \eta$. Conditions (a) and (b) in (iv) are independent [1, p. 216].

Given $w \in A$, let Q be generic for a closed k-cube, $Q \subset A$, with faces parallel to the coordinate hyperplanes of E_k and with $w \in Q^\circ$. The *derivative* of a real-valued interval function z at the point w is defined as

$$D(w, z) = \lim z(Q)/L_k(Q) \tag{1}$$

provided this limit, taken as $L_k(Q) \to 0$, exists and is finite; otherwise we set $D(w, z) = 0$. For the following theorem, see [1, Section 27] and [5, Section III. 2].

THEOREM 3.1. *Suppose z is nonnegative, overadditive, and BV. Then the limit* (1) *exists and is finite a.e. in A, $D(w, z)$ is Borel measurable and L-integrable on A, and $V[z, A] \geqq \int_A D(w, z)dw$. The same inequality holds if A is replaced by any open set $G \subset A$ or by any set $I \in \{I\}$. The equality $V[z, A] = \int_A D(w, z)dw$ holds if and only if z is AC and, in this case, the same equality holds if A is replaced by any G or I as above.*

Let $z = (z_1, \cdots, z_m)$ be vector-valued. If z is quasi additive on A and if $\int[|z|, A]$ is finite, then the nonnegative interval functions F, F_r, F_r^+, and F_r^-, $r = 1, \cdots, m$, are overadditive and BV by (2.2) and (2.3). It is convenient to use the notations

$$\begin{aligned} D(w) &= D(w, F)\,, & D_r(w) &= D(w, F_r)\,,\\ D_r^+(w) &= D(w, F_r^+)\,, & D_r^-(w) &= D(w, F_r^-)\,,\\ J_r(w) &= D_r^+(w) - D_r^-(w)\,, & J(w) &= (J_1(w), \cdots, J_m(w))\,. \end{aligned}$$

From (1.4) we obtain $|J_r| = |D_r^+ - D_r^-| \leqq D_r^+ + D_r^- = D_r \leqq D$ a.e. in A for each r.

In the next two results we assume that z is real-valued.

PROPOSITION 3.2. *If z is real-valued and quasi additive on A and if $\int[|z|, A]$ is finite, then $D^+(w)D^-(w) = 0$ and $|J(w)| = D(w)$ a.e. in A.*

Proof. Let $M_1 = \{w \in A\colon D^+(w) > 0\}$ and $M_2 = \{w \in A\colon D^-(w) > 0\}$. For each integer $n = 1, 2, \cdots$, let $M_{1n} = \{w \in A\colon D^+(w) > 1/n\}$ and $M_{2n} = \{w \in A\colon D^-(w) > 1/n\}$.

Let $\varepsilon > 0$ and n be given. Let $\mu = \mu(\varepsilon/2n, A)$ be as in (1.4) and let $D_0 = [I]$ satisfy $\delta(D_0) < \mu$. Let A_1 be the union of all $I \in D_0$ such that $z^+(I) > 0$, and let A_2 be the union of all $I \in D_0$ such that $z^-(I) > 0$. Then the sets $G_1 = A - A_1$ and $G_2 = A - A_2$ are open. We show first that $L_k(G_1 \cap M_{1n}) \leqq \varepsilon$ and that $L_k(G_2 \cap M_{2n}) \leqq \varepsilon$.

Let $\lambda = \lambda(\varepsilon/2n, A, D_0)$ with $0 < \lambda \leqq \mu$ be as in (1.4) and let $D = [J]$ be any system with $\delta(D) < \lambda$. Let H_1 be the closure of G_1 in A. Thus $G_1 \subset H_1 \subset A$ and $z^+(I) = 0$ for every interval $I \in D_0$ with $I \subset H_1$. Also, if $J \in D, J \subset H_1$, and $J \not\subset I$ for any $I \in D_0$ with $I \subset H_1$, then $J \not\subset I$ for any $I \in D_0$. Hence,

$$\begin{aligned}
0 &\leqq S[z^+, G_1, D] \leqq S[z^+, H_1, D] = \textstyle\sum_J s(J, H_1)z^+(J) \\
&= \textstyle\sum_I s(I, H_1)[\sum_J s(J, I)z^+(J) - z^+(I)] \\
&\quad + \textstyle\sum_J s(J, H_1)[1 - \sum_I s(J, I)s(I, H_1)]z^+(J) \\
&\leqq \textstyle\sum_I s(I, A)\,|\sum_J s(J, I)z^+(J) - z^+(I)| \\
&\quad + \textstyle\sum_J s(J, A)[1 - \sum_I s(J, I)s(I, A)]z^+(J) \\
&< \varepsilon/2n + \varepsilon/2n = \varepsilon/n
\end{aligned}$$

and, as $\delta(D) \to 0$, we obtain $0 \leqq F^+(G_1) \leqq \varepsilon/n$. From (2.2) and (3.1) we now have

$$\begin{aligned}
0 &\leqq L_k(G_1 \cap M_{1n}) \leqq n\int_{G_1} D^+(w)dw \leqq n\ V[F^+, G_1] \\
&= n\,F^+(G_1) \leqq \varepsilon.
\end{aligned}$$

An analogous argument shows that $L_k(G_2 \cap M_{2n}) \leqq \varepsilon$.

Since $A \subset [G_1 \cup G_2 \cup (A_1^* \cap A_2^*)]$ and $L_k(A_1^* \cap A_2^*) = 0$, we have

$$\begin{aligned}
L_k(M_{1n} \cap M_{2n}) &= L_k(M_{1n} \cap M_{2n} \cap A) \\
&\leqq L_k[M_{2n} \cap (M_{1n} \cap G_1)] + L_k[M_{1n} \cap (M_{2n} \cap G_2)] \\
&\quad + L_k(M_{1n} \cap M_{2n} \cap A_1^* \cap A_2^*) \\
&\leqq L_k(M_{1n} \cap G_1) + L_k(M_{2n} \cap G_2) < 2\varepsilon\ .
\end{aligned}$$

As $\varepsilon > 0$ and n were arbitrary, we conclude that $L_k(M_{1n} \cap M_{2n}) = 0$ for each n. From the construction of the sets M_{1n} and M_{2n} we conclude further that $L_k(M_1 \cap M_2) = 0$. Therefore $D^+D^- = 0$ a.e. in A and $|J| = |D^+ - D^-| = D^+ + D^- = D$ a.e. in A. This completes the proof.

THEOREM 3.3. *Suppose that z is real-valued and quasi additive on A and that $\int[|z|, A]$ is finite. Then*

$$F(A) \geqq \int_A |J(w)|dw\ , \qquad F^+(A) \geqq \int_A D^+(w)dw\ ,$$
$$F^-(A) \geqq \int_A D^-(w)dw\ ,$$

and the same inequalities hold if A is replaced by any open subset G of A or by any set $I \in \{I\}$. The equality $F(A) = \int_A |J(w)|\, dw$ holds if and only if F is AC. If F is AC, then

$$F(A) = \int_A |J(w)|dw\ , \qquad F^+(A) = \int_A D^+(w)dw\ ,$$
$$F^-(A) = \int_A D^-(w)dw\ , \qquad \mathscr{F}(A) = \int_A J(w)dw\ ,$$

and the same equalities hold if A is replaced by any G or I as above.

Proof. Recalling that F, F^+, and F^- are nonnegative and overadditive and that $F = F^+ + F^-$, it is easily verified that F is AC if and only if F^+ and F^- are both AC. The theorem now follows from (2.2), (3.1), (3.2), and the relation $\mathscr{F} = F^+ - F^-$.

We return now to the case in which $z = (z_1, \cdots, z_m)$ is vector-valued.

PROPOSITION 3.4. *Suppose that z is quasi additive on A and that $\int[|z|, A]$ is finite. Then F is AC if and only if the functions F_r, $r = 1, \cdots, m$, are all AC.*

Proof. It is clear that F satisfies condition (b) in the definition of AC if and only if each F_r also satisfies this condition. It remains to show that F is additive if and only if each F_r is additive. Let I be an interval and $I = \cup_j I_j$ be a finite subdivision of I into nonoverlapping intervals I_j. Let

$$d = F(I) - \textstyle\sum_j F(I_j)\ , \qquad d_r = F_r(I) - \sum_j F_r(I_j)\ .$$

For each system $D = [J]$ let

$$\begin{aligned} d(D) &= \textstyle\sum_J s(J, I)[1 - \sum_j s(J, I_j)]F(J) \\ &= S[F, I, D] - \textstyle\sum_j s[F, I_j, D]\ , \\ d_r(D) &= \textstyle\sum_J s(J, I)[1 - \sum_j s(J, I_j)]F_r(J) \\ &= S[F_r, I, D] - \textstyle\sum_j S[F_r, I_j, D]\ . \end{aligned}$$

As $\delta(D) \to 0$, we have $d(D) \to d$ and $d_r(D) \to d_r$ by (2.2). From the inequalities $F_r(J) \leqq F(J) \leqq \sum_r F_r(J)$ we obtain $d_r(D) \leqq d(D) \leqq \sum_r d_r(D)$

and, after a passage to the limit, $d_r \leqq d \leqq \sum_r d_r$. Since F_r is overadditive and $d_r \geqq 0$, we conclude that $d = 0$ if and only if $d_r = 0$ for each r. This completes the proof.

PROPOSITION 3.5. *Suppose that z is quasi additive on A and that $\int[|z|, A]$ is finite. Then $D(w) \geqq |J(w)|$ a.e. in A and equality holds if F is AC.*

The proof is essentially the same as given in parts (a) and (b) of the proof of [1, 30.1 (ii)]: simply replace the letter V by F and the references to [1, 9.1] and [1, 12.1] by a reference to (2.6) in the present paper.

THEOREM 3.6. *Suppose that z is quasi additive on A and that $\int[|z|, A]$ is finite. Then $F(A) \geqq \int_A |J(w)|dw$ and the same inequalities hold if A is replaced by any open subset G of A or by any set $I \in \{I\}$. The equality $F(A) = \int_A |J(w)|dw$ holds if and only if F is AC and, in this case, the same equality holds if A is replaced by any set G or I as above.*

This theorem is a consequence of (2.2), (3.1), and (3.5).

4. **t-quasi additivity.** We assume axioms (a)-(d) of § 1 throughout this section. In addition, let there be associated with each subset E of A a set E^t satisfying the condition

(t_1): E^t is contained in the interior of E,

(t_2): $E^t \subset G^t$ whenever $E \subset G \subset A$.

DEFINITION 4.1. *z is t-quasi additive* on M if, under the circumstances of Definition (1.1), z satisfies

(tqa_1): $\sum_I s(I, M) \,|\sum_J s(J, I^t)z(J) - z(I)| < \varepsilon$,

(tqa_2): $\sum_J s(J, M)[1 - \sum_I s(J, I^t)s(I, M)]\, |z(J)| < \varepsilon$.

An analogous definition of "t-quasi subadditivity" may be formulated if z is real-valued. The statements of § 1 remain valid if the terms "quasi additive", "quasi subadditive", and "$s(J, I)$" are consistently replaced by "t-quasi additive", "t-quasi subadditive", and "$s(J, I^t)$", respectively. (We do not modify the definition of the B–C integral.)

PROPOSITION 4.2. *If z is t-quasi additive on M, then z is also quasi additive on M.*

Proof. For any two systems $D_0 = [I]$ and $D = [J]$ we have

$$\begin{aligned} &\sum_I s(I, M)[\sum_J s(J, I)z(J) - z(I)] \\ (1) \qquad &= \sum_I s(I, M)[\sum_J s(J, I^t)z(J) - z(I)] \\ &\quad + \sum_I s(I, M)\sum_J s(J, I)[1 - s(J, I^t)]z(J) \,. \end{aligned}$$

Let $\varepsilon > 0$ be given and let $\eta = \eta(\varepsilon/2, M) > 0$, $D_0 = [I]$ with $\delta(D_0) < \eta$, $\lambda = \lambda\ (\varepsilon/2, M, D_0) > 0$, and $D = [J]$ with $\delta(D) < \lambda$ be as in the definition of t-quasi additivity. Then

$$(2) \qquad \sum_I s(I, M)\,|\sum_J s(J, I^t)z(J) - z(I)| < \varepsilon/2 \,,$$

$$(3) \qquad \sum_J s(J, M)[1 - \sum_I s(J, I^t)s(I, M)]\,|z(J)| < \varepsilon/2 \,.$$

The last term in (1) is less inclusive than the term in (3) and from (1)–(3) we obtain

$$(2)' \qquad \sum_I s(I, M)\,|\sum_J s(J, I)z(J) - z(I)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \,.$$

The term in (3)′ below is also less inclusive than the term in (3) and hence

$$(3)' \qquad \sum_J s(J, M)[1 - \sum_I s(J, I)s(I, M)]\,|z(J)| < \varepsilon/2 \,.$$

Relations (2)′ and (3)′ show that z is quasi additive on M.

PROPOSITION 4.3. *Assume that each interval $I \in \{I\}$ is connected. If z is t-quasi additive on M and if $\int[|z|, M]$ is finite, then $F(M^t) = F(M)$.*

Proof. Let $\varepsilon > 0$ be given and let $\mu = \mu(\varepsilon/2, M)$, $D_0 = [I]$ with $\delta(D_0) < \mu$, $\lambda = \lambda(\varepsilon/2, M, D_0)$, and $D = [J]$ with $\delta(D) < \lambda$ be as in the t-quasi additivity version of Proposition (1.4). Then

$$\begin{aligned} &|F(M) - S[|z|, M, D_0]| < \varepsilon/2 \,, \\ &|F(M) - S[|z|, M, D]| < \varepsilon/2 \,, \\ &\sum_I s(I, M)\,|\sum_J s(J, I^t)\,|z(J)| - |z(I)|| < \varepsilon/2 \,. \end{aligned}$$

Let M' denote the union of the sets I^t such that $I \in D_0$ and $I \subset M$. Then $M' \subset M^t \subset M$. Since each I^t is contained in the interior of I and since the intervals J are connected, each interval $J \in D$ with $J \subset M'$ is contained in I^t for one and only one interval $I \in D_0$ with $I \subset M$. Hence

$$\sum_J s(J, M') \, |z(J)| = \sum_I s(I, M) \sum_J s(J, I^t) \, |z(J)|$$

and therefore

$$\begin{aligned}
&|F(M) - \sum_J s(J, M') \, |z(J)|| \\
&\leqq |F(M) - \sum_I s(I, M) \, |z(I)|| \\
&\quad + |\sum_J s(J, M') \, |z(J)| - \sum_I s(I, M) \, |z(I)|| \\
&\quad < \varepsilon/2 + \sum_I s(I, M) \, |\sum_J s(J, I^t) \, |z(J)| - |z(I)|| \\
&\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon \, .
\end{aligned}$$

Thus, given $\varepsilon > 0$, there is a set $M' = M'(\varepsilon)$, $M' \subset M^t \subset M$, and a number $\lambda = \lambda(\varepsilon) > 0$ such that

$$F(M) - \varepsilon < S[|z|, M', D] \leqq S[|z|, M^t, D] \leqq S[|z|, M, D] < F(M) + \varepsilon$$

for every system D with $\delta(D) < \lambda$. Thus, $F(M^t) = F(M)$.

PROPOSITION 4.4. *If z is quasi additive on M, $\int [|z|, M]$ is finite, and $F(M^t) = F(M)$, then $\mathscr{F}(M^t) = \mathscr{F}(M)$ and similarly for $\mathscr{F}_r$, F_r, F_r^+, and F_r^-, $r = 1, \cdots, m$.*

Proof. We show that $\mathscr{F}(M^t) = \mathscr{F}(M)$; the other parts are proved in an analogous manner. All limits below are taken as $\delta(D) \to 0$, $D = [I]$. Since

$$\begin{aligned}
F(M) &= \lim \sum_I s(I, M) \, |z(I)| \\
&= \lim \{\sum_I s(I, M^t) \, |z(I)| + \sum_I s(I, M)[1 - s(I, M^t)] \, |z(I)|\}
\end{aligned}$$

and

$$F(M) = F(M^t) = \lim \sum_I s(I, M^t) \, |z(I)| < \infty \, ,$$

we conclude that

$$\lim \sum_I (I, M)[1 - s(I, M^t)] \, |z(I)| = 0 \, ,$$

and therefore

$$\lim \sum_I s(I, M)[1 - s(I, M^t)] z(I) = 0 \, .$$

Hence,

$$\begin{aligned}
\mathscr{F}(M) &= \lim \{\sum_I s(I, M^t) z(I) + \sum_I s(I, M)[1 - s(I, M^t)] z(I)\} \\
&= \mathscr{F}(M^t) \, .
\end{aligned}$$

THEOREM 4.5. *Assume that each interval $I \in \{I\}$ is connected, z is quasi additive on M, and $\int [|z|, M]$ is finite. Then z is t-quasi additive on M if and only if $F(I^t) = F(I)$ for each interval $I \subset M$.*

Proof. The condition is necessary by (4.3). Assume now that $F(I^t) = F(I)$ for each interval $I \subset M$. We must show that z is t-quasi additive on M. Let $D_0 = [I]$ and $D = [J]$ be any two systems. Then

$$\begin{aligned}
&\sum_I s(I, M)\, |\sum_J s(J, I^t)z(J) - z(I)| \\
&\quad \leqq \{\sum_I s(I, M)\, |\sum_J s(J, I^t)z(J) - \mathscr{F}(I)|\} \\
&\qquad + \{\sum_I s(I, M)\, |\mathscr{F}(I) - \sum_J s(J, I)z(J)|\} \\
&\qquad + \{\sum_I s(I, M)\, |\sum_J s(J, I)z(J) - z(I)|\} \\
&\quad = s_1 + s_2 + s_3 ,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_J s(J, M)[1 - \sum_I s(J, I^t)s(I, M)]\, |z(J)| \\
&\quad = \sum_J s(J, M)\, |z(J)| - \sum_J \sum_I s(J, I^t)s(I, M)\, |z(J)| \\
&\quad \leqq \{|\sum_J s(J, M)\, |z(J)| - \sum_J \sum_I s(J, I)s(I, M)\, |z(J)||\} \\
&\qquad + \{|\sum_J \sum_I s(J, I)s(I, M)\, |z(J)| - \sum_J s(I, M)F(I)|\} \\
&\qquad + \{|\sum_I s(I, M)F(I) - \sum_J \sum_I s(J, I^t)s(I, M)\, |z(J)||\} \\
&\quad = s_4 + s_5 + s_6 .
\end{aligned}$$

Let $\varepsilon > 0$ be given and let $\eta_1 = \eta(z, \varepsilon/3, M)$ and $\eta_2 = \eta(|z|, \varepsilon/3, M)$ be as in Definition (1.1). Let $\eta = \min\,[\eta_1, \eta_2]$ and let $D_0 = [I]$ satisfy $\delta(D_0) < \eta$. Let $\lambda_1 = \lambda(z, \varepsilon/3, M, D_0)$ and $\lambda_2 = \lambda(|z|, \varepsilon/3, M, D_0)$ be as in Definition (1.1) and let $\lambda = \min\,[\lambda_1, \lambda_2]$. Let N be the number of intervals $I \in D_0$ and let $\varepsilon' = \varepsilon/N$. For each $I \in D_0$ with $I \subset M$ let $\mu_I = \mu(\varepsilon'/3, I)$ and $\mu_I^t = \mu(\varepsilon'/3, I^t)$ be as in (1.4). Let $\mu = \min\,\{\lambda, \mu_I, \mu_I^t : I \in D_0, I \subset M\}$ and let $D = [J]$ satisfy $\delta(D) < \lambda$. Then

(4) $|\mathscr{F}(I^t) - S[z, I^t, D]| < \varepsilon'/3$ for each $I \in D_0$ with $I \subset M$,

(5) $|\mathscr{F}(I) - S[z, I, D]| < \varepsilon'/3$ for each $I \in D_0$ with $I \subset M$,

(6) $\sum_I s(I, M)\, |\sum_J s(J, I)z(J) - z(I)| < \varepsilon/3$,

(7) $\sum_J s(J, M)[1 - \sum_I s(J, I)s(I, M)]\, |z(J)| < \varepsilon/3$,

(8) $|F(I) - S[|z|, I, D]| < \varepsilon'/3$ for each $I \in D_0$ with $I \subset M$,

(9) $|F(I^t) - S[|z|, I^t, D]| < \varepsilon'/3$ for each $I \in D_0$ with $I \subset M$.

Hence,

$$
\begin{aligned}
0 \leqq s_1 &= \sum_I s(I, M) \,|\sum_J s(J, I^t)z(J) - \mathscr{F}(I)| \\
&= \sum_I s(I, M) \,|\sum_J s(J, I^t)z(J) - \mathscr{F}(I^t)| \\
&\quad < N(\varepsilon'/3) = \varepsilon/3 \text{ by (4.4) and (4),} \\
0 \leqq s_2 &= \sum_I s(I, M) \,|\mathscr{F}(I) - \sum_J s(J, I)z(J)| \\
&\quad < N(\varepsilon'/3) = \varepsilon/3 \text{ by (5) ,} \\
0 \leqq s_3 &= \sum_I s(I, M) \,|\sum_J s(J, I)z(J) - z(I)| < \varepsilon/3 \text{ by (6) ,} \\
0 \leqq s_4 &= |\sum_J s(J, M) |z(J)| - \sum_J \sum_I s(J, I)s(I, M) |z(J)|| \\
&\leqq \sum_J s(J, M)[1 - \sum_I s(J, I)s(I, M) \,|z(J)| < \varepsilon/3 \text{ by (7) ,} \\
0 \leqq s_5 &= |\sum_J \sum_I s(J, I)s(I, M) |z(J)| - \sum_I s(I, M)F(I)| \\
&\leqq \sum_I s(I, M) \,|\sum_J s(J, I) |z(J)| - F(I)| \\
&\quad < N(\varepsilon'/3) = \varepsilon/3 \text{ by (8) ,} \\
0 \leqq s_6 &= |\sum_I s(I, M)F(I) - \sum_J \sum_I s(J, I^t)s(I, M) |z(J)|| \\
&\leqq \sum_I s(I, M) \,|F(I) - \sum_J s(J, I^t) |z(J)|| \\
&= \sum_I s(I, M) \,|F(I^t) - \sum_J s(J, I^t) |z(J)|| \\
&< N(\varepsilon'/3) = \varepsilon/3 \text{ by (9) .}
\end{aligned}
$$

Thus $s_1 + s_2 + s_3 < \varepsilon$, $s_4 + s_5 + s_6 < \varepsilon$, and we conclude that z is t-quasi additive on M.

We remark that the connectedness of the intervals $I \in \{I\}$ is not used in the sufficiency part of the above proof.

5. **Remarks.** Definition (1.1) was used under axioms (a)–(d) by Cesari [2] for the case $M = A$. In [3] Cesari extended the notions of B–C integral and quasi additivity to a class $\{G\}$ of nonempty subsets of A as follows. For each G in $\{G\}$ let $\{D\}_G$ be the family of all systems $D_G = \{I \in D \colon I \subset G\}$ obtained as D ranges over the family $\{D\}$, and let δ_G be a mesh satisfying axiom (d) relative to $\{D\}_G$. In addition, assume the axiom

(e): given $\tau > 0$, there exists $\nu = \nu(\tau, G) > 0$ such that if D is any system in $\{D\}$ with $\delta(D) < \nu$, then the associated system $D_G = \{I \in D \colon I \subset G\}$ is nonempty and $\delta_G(D_G) < \tau$.

Cesari then defined B–C integrals over G and quasi additivity on G relative to the directed system $(\{D\}_G, \delta_G)$; axiom (e) was used to obtain properties of the B–C integrals as set functions.

To see that Cesari's formulation is contained in that of the present paper we observe the following two statements.

(i) $\lim_{\delta(D)\to 0} S[z, G, D] = \lim_{\delta_G(D_G)\to 0} S[z, G, D_G]$ whenever axiom (e) holds and the limit on the right exists.

(ii) If z is quasi additive [subadditive] on G relative to $(\{D\}_G, \delta_G)$ and if axiom (e) holds, then z is quasi additive [subadditive] on G relative to $(\{D\}, \delta)$.

Statement (i) was proved by Cesari [3, p. 117], and (ii) may be proved in an analogous manner. Since simple examples show that if $\int[|z|, G] = 0$ there may no mesh δ_G satisfying axiom (e), some improvement is gained by formulating all quasi additivity relations relative to the single directed system $(\{D\}, \delta)$ as in the present paper. The theorems proved by Cesari [3] carry over to the present setting, moreover, with only the obvious changes in the mesh conditions required. For the sake of completeness, we shall next restate the most important of these theorems.

Let $\mathscr{T}$ denote the topology on A, $\mathscr{G}$ be a topology on A coarser than $\mathscr{T}$, and $\mathscr{B}$ be the σ-algebra on A generated by $\mathscr{G}$. In addition to the axioms (a)-(d) of § 1, assume the following four additional hypotheses.

(H_1): z is quasi additive on A and $\int[|z|, A]$ is finite.

(H_2): Each interval $I \in \{I\}$ is $\mathscr{T}$-connected.

(H_3): If $G = \cup_n G_n$ is a countable union of sets $G_n \in \mathscr{G}$, then $F(G) \leqq \sum_n F(G_n)$, and analogously for F_r, F_r^+, and F_r^-, $r = 1, \cdots, m$.

(H_4): If $G \in \mathscr{G}$, then $F(G) = \sup F(G')$ where the supremum is taken over all sets $G' \in \mathscr{G}$ whose $\mathscr{G}$-closure is contained in G, and analogously for F_r, F_r^+, and F_r^-, $r = 1, \cdots, m$.

Neither (H_3) nor (H_4) is a consequence of the axioms or preceding hypotheses. Finally, for each subset M of A, define

$$\begin{aligned} \mu(M) &= \inf F(G) , & \mu_r(M) &= \inf F_r(G) , \\ \mu_r^+(M) &= \inf F_r^+(G) , & \mu_r^-(M) &= \inf F_r^-(G) , \\ \nu_r(M) &= \mu_r^+(M) - \mu_r^-(M) , & \nu(M) &= (\nu_1(M), \cdots, \nu_m(M)) , \end{aligned}$$

where the infima are taken over all sets $G \in \mathscr{G}$ with $M \subset G$.

With the help of (H_1) and Theorem (1.4), we see that

$$\begin{aligned} &\mu_r(M) = \mu_r^+(M) + \mu_r^-(M) , \\ &|\nu_r(M)| \leqq \mu_r(M) \leqq \mu(M) , \\ &|\nu(M)| \leqq [\sum_r \mu_r^2(M)]^{1/2} \leqq \mu(M) \leqq \sum_r \mu_r(M) , \end{aligned}$$

for each r and M. Moreover, the set functions μ, μ_r, μ_r^+, μ_r^-, ν_r, and ν agree on $\mathscr{G}$ with the set functions F, F_r, F_r^+, F_r^-, $\mathscr{F}_r$, and $\mathscr{F}$, respectively.

PROPOSITION 5.1. *Under hypotheses* (H_1)-(H_4), *the set functions* μ, μ_r, μ_r^+, *and* μ_r^-, $r = 1, \cdots, m$, *are outer measures on* A *and are finite measures on* $\mathscr{B}$.

It follows that the set functions ν_r are signed measures on $\mathscr{B}$ which are absolutely continuous with respect to the measure μ. Thus we may define the Radon-Nikodym derivatives

$$\theta_r = d\nu_r/d\mu \, , \qquad \theta = (\theta_1, \cdots, \theta_m) \, .$$

As Cesari observed, the relations $\nu_r = \mu_r^+ - \mu_r^-$ need not represent Jordan decompositions of the signed measures ν_r. This situation is rectified by replacing (H_1) by the slightly stronger hypothesis

$(H_1)'$: z is t-quasi additive on A and $\int[|z|, A]$ is finite, where t denotes the interior operator for the topology $\mathscr{G}$.

By (4.5), $(H_1)'$ and (H_2) are equivalent to (H_1), (H_2), and the statement that $F(I) = F(I^t)$ for every interval I.

PROPOSITION 5.2. *Assume* $(H_1)'$ *and* (H_2)-(H_4). *Then*

(i) $\nu_r = \mu_r^+ - \mu_r^-$ *represent Jordan decompositions,*

(ii) $\mu(B) = \sup \sum_j [\sum_r \mu_r^2(B_j)]^{1/2} = \sup \sum_j |\nu(B_j)|$, *where* B *is an arbitrary set in* $\mathscr{B}$ *and the suprema are taken over all finite decompositions* $B = \cup_j B_j$ *of* B *into sets* B_j *in* $\mathscr{B}$,

(iii) $|\theta| = 1$ $\mu - a.e.$ *in* A.

We turn now to Cesari's theorem on the existence and representation of Cesari-Weierstrass integrals. Let $T: A \rightarrow K$, $x = T(w)$, be a mapping from A into a metric space K, and let $f: K \times E_m \rightarrow E_1$, $a = f(x, q)$, be a real-valued function defined on the product space $K \times E_m$. Let $S^{m-1} = \{q \in E_m: |q| = 1\}$ be the unit sphere in E_m. Finally, let w_I denote an arbitrary point of I for each interval $I \in \{I\}$.

THEOREM 5.3. *Suppose*

(i) $\max$ $\{$*diameter* $T(I): I \in D\} \leqq \delta(D)$ *for each* $D \in \{D\}$,

(ii) f *is bounded and uniformly continuous on* $K \times S^{m-1}$,

(iii) $f(x, tq) = tf(x, q)$ *for all* $t \geqq 0$, $x \in K$, *and* $q \in E_m$.

Then, under hypothesis (H_1), *the real-valued interval function*

$$Z(I) = f[T(w_I)z(I)]$$

is quasi additive on A, *the parameters of Definition* (1.1) *can be determined independently of the choice of* $w_I \in I$, *and the value of the* $B - C$ *integral*

$$\int[f(T, z), A] = \int[Z, A]$$

is independent of the choice of $w_I \in I$. *Further, under the hypotheses* $(H_1)'$ *and* (H_2)-(H_4), *the function* $f[T(w), \theta(w)]$, $w \in A$, *is* μ-*integrable on* A *and*

$$\int [f(T, z), A] = \int_A f[T(w), \theta(w)] d\mu .$$

The proof, given in [2, 3] for K a subset of some Euclidean space, is valid if K is simply a metric space.

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Received September 1, 1970. A portion of this paper was taken from the author's doctoral dissertation written under the direction of Professor L. Cesari at the University of Michigan, 1969. (AFOSR Project 69-1662.)

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