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THE COHOMOLOGY OF DIVISORIAL VARIETIES

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A well known theorem of Serre states the equivalence between the ampleness of a linear equivalence class of divisors on an algebraic variety and the vanishing of the first cohomology groups related to sufficiently high multiples of such linear equivalence class. In this paper the result of the above theorem is extended in the following direction: given a linear equivalence class on an algebraic variety, does there exist a cohomological characterization of the open subset consisting of points of the variety which belong to affine open complements of effective divisors in the multiples of the given class? The characterization obtained is the main result, and it gives easily Serre's result as a particular case. While in one direction the proof uses the vanishing theorem quoted in the beginning, it is independent of it in the opposite direction. A simple application of the main result gives a first cohomological characterization of divisorial varieties.

We prove first two lemmas, which hold, in our opinion, an intrinsic value, then we prove our main result, namely Theorem 4.

We use throughout this paper the notations and language of [6] and [8]. We deal entirely with schemes of finite type over an algebraically closed groundfield k, and we refer to them, for brevity's sake, simply as schemes. We shall also say "proper schemes" rather than schemes proper over Spec (k).

When we refer to, say, Lemma 3, without any further reference, we mean Lemma 3 of the present work.

We begin with the two lemmas mentioned in the introduction, needed later in the proof of Theorem 4.

LEMMA 1. *Let $f: X \rightarrow Y$ be a projective morphism of schemes. Let $\mathscr{L}$ be an invertible sheaf over X, ample for f, and let*

$$\mathscr{G}' \longrightarrow \mathscr{G} \longrightarrow \mathscr{G}'' \longrightarrow 0$$

be an exact sequence of coherent sheaves over Y. Then, for $n \gg 0$, the sequence of coherent sheaves over Y

$$f_*(\mathscr{L}^{\otimes n} \otimes f^*(\mathscr{G}')) \longrightarrow f_*(\mathscr{L}^{\otimes n} \otimes f^*(\mathscr{G})) \\ \longrightarrow f_*(\mathscr{L}^{\otimes n} \otimes f^*(\mathscr{G}'')) \longrightarrow 0$$

is exact.

Proof. The functor f^* is right exact, and therefore the sequence

$$f^*(\mathscr{G}') \longrightarrow f^*(\mathscr{G}) \longrightarrow f^*(\mathscr{G}'') \longrightarrow 0$$

is exact.

Since the morphism f is projective, and $\mathscr{L}$ is ample for f, we can apply Theorem 2.4.1 of Ch. III of [6] to the sheaves $\mathscr{K}' = \ker [f^*(\mathscr{G}) \longrightarrow f^*(\mathscr{G}'')]$ and $\mathscr{K}'' = \ker[f^*(\mathscr{G}') \longrightarrow \mathscr{K}']$. We obtain that, for $n \gg 0$, the sequence in the statement of the lemma is exact, and the lemma is proved.

In [1] the author defined divisorial schemes as those schemes which have the property that every point is in the affine open complement of an effective Cartier divisor. For divisorial schemes the following lemma holds:

LEMMA 2. *Let $f: X \rightarrow Y$ be a morphism of schemes. Assume that Y is divisorial, and f projective. Let $\mathscr{L}$ be an invertible sheaf over X, ample for f. Then, for every coherent sheaf $\mathscr{F}$ over Y, the natural homomorphism*

$$f_*(\mathscr{L}^{\otimes n}) \otimes \mathscr{F} \longrightarrow f_*(\mathscr{L}^{\otimes n} \otimes f^*(\mathscr{F}))$$

is an isomorphism for $n \gg 0$.

Proof. If $\mathscr{F}$ is locally free over Y, the lemma holds by virtue of 5.4.10 of Ch. 0_I of [6]. Let now $\mathscr{F}$ be arbitrary. Since Y is divisorial, by theorem 3.3 of [2] we have an exact sequence

$$\mathscr{H} \longrightarrow \mathscr{G} \longrightarrow \mathscr{F} \longrightarrow 0\,,$$

where $\mathscr{G}$, $\mathscr{H}$ are locally free sheaves over Y.

Now, by tensoring the above exact sequence with $f_*(\mathscr{L}^{\otimes n})$, and then applying to the same exact sequence above lemma 1, we obtain two exact sequences which, by comparison and our first remark, give us the proof of the lemma.

As is well known (See [8], or Ch. III of [6]), the vanishing of the first cohomology group $H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$, $\mathscr{F}$ an arbitrary coherent sheaf over X, and $n \gg 0$, is equivalent, when X is a proper scheme, to saying that the invertible sheaf $\mathscr{L}$ is ample on X. Theorem 4 below generalizes the above result.

We begin by recalling, for completeness sake, the definition of $\mathscr{L}$-projective open subsets, given by the author in [3], where certain properties of such subsets are studied.

DEFINITION 3. Let Y be a sheme, $\mathscr{L}$ an invertible sheaf over Y, U an open subscheme of Y. The subscheme U is called $\mathscr{L}$-projective if every point $y \in U$ is contained in the affine complement of the zeros of a section $s \in H^{\circ}(Y, \mathscr{L}^{\otimes n})$, for some integer $n > 0$.

Our main result is the following characterization of $\mathscr{L}$-projective open subschemes.

THEOREM 4. *Let Y be a proper, normal, divisorial, integral scheme. Let $\mathscr{L}$ be an invertible sheaf over Y, and U an open subscheme of Y. Then U is $\mathscr{L}$-projective if, and only if, there exists a sequence of coherent sheaves of ideals $\{\mathscr{I}_n\}_{n>0}$ of $\mathscr{O}_Y$ with supports off U and such that*

(i) $\mathscr{I}_n \supset \mathscr{I}_{n+1}$ *for all* $n > 0$

(ii) $\mathscr{I}_n \mathscr{I}_m \subset \mathscr{I}_{n+m}$ *for all* $n, m > 0$

(iii) *For every coherent sheaf $\mathscr{F}$ over Y there exists an integer N such that, for all $n \geqq N$, and all $i > 0$*

$$H^i(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \otimes \mathscr{F}) = 0 .$$

Proof. Let U be $\mathscr{L}$-projective. By Proposition 2.7 of [3] there exists a proper scheme X, and a morphism $f: X \longrightarrow Y$ such that:

(a) f is projective, birational, and surjective.

(b) $f \mid f^{-1}(U): f^{-1}(U) \longrightarrow U$ is an isomorphism.

(c) There exist two invertible sheaves of ideals $\mathscr{I}$, $\mathscr{J}$ of $\mathscr{O}_X$, with supports off $f^{-1}(U)$, and an integer $t > 0$, such that, for $n \gg 0$, the invertible sheaf $f^*(\mathscr{L}^{\otimes nt}) \otimes \mathscr{I}^{\otimes n} \otimes \mathscr{J}$ is ample for X.

Fix one such n, say $n_{\circ}$. By property (c) above we have that, for every coherent sheaf $\mathscr{F}$ over Y there exists an integer $\bar{n}$ such that, for all $n \geqq \bar{n}$, and all j, $0 \leqq j < n_{\circ}t$, and all $i > 0$,

$$H^i(X, f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F})) = 0$$

and also

$$R^q f_*[f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F})] = 0 \qquad q > 0 . \tag{4.1}$$

By Lemma 2 we may furthermore assume that $\bar{n}$ is chosen so that, for all $n \geqq \bar{n}$, and all $j, 0 \leqq j < n \circ t$

$$\begin{aligned} &f_*(f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F})) \\ &\cong f_*(f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n}) \otimes \mathscr{F} \\ &\cong \mathscr{L}^{\otimes nn_{\circ}t+j} \otimes f_*(\mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n}) \otimes \mathscr{F}. \end{aligned} \tag{4.2}$$

Consider now the Leray spectral sequence of the morphism f (See [7] or [4]),

$$\begin{aligned} E_2^{p,q} &= H^p[Y, R^q f_*(f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F}))] \\ &\Longrightarrow H^{p+q}(X, f^*(\mathscr{L}^{\otimes nn_{\circ}t+j}) \otimes \mathscr{J}^{\otimes nn_{\circ}} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F})) . \end{aligned}$$

By (4.1) above we obtain, for $n \geqq \bar{n}$, and $i > 0$

$$\begin{aligned} &H^i(Y, f_*(f^*(\mathscr{L}^{\otimes nn_\circ t+j}) \otimes \mathscr{J}^{\otimes nn_\circ} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F}))) \\ &= H^j(X, f^*(\mathscr{L}^{\otimes nn_\circ t+j}) \otimes \mathscr{J}^{\otimes nn_\circ} \otimes \mathscr{I}^{\otimes n} \otimes f^*(\mathscr{F})) = 0 \ , \end{aligned}$$

and therefore, by (4.2),

$$H^i(Y, \mathscr{L}^{\otimes nn_\circ t+j} \otimes f_*(\mathscr{J}^{\otimes nn_\circ} \otimes \mathscr{I}^{\otimes n}) \otimes \mathscr{F}) = 0 \tag{4.3}$$

for all $n \geqq \bar{n}$, all j, $0 \leqq j < n_\circ t$, and all $i > 0$.

For any integer $m > 0$ let $i(m)$ denote the greatest integer $\leqq m/n_0 t$. Define

$$\mathscr{I}_m = f_*(\mathscr{J}^{\otimes n_\circ i(m)} \otimes \mathscr{I}^{\otimes i(m)}) \ .$$

Using that Y is normal, f_* left exact, and (4.3), one easily verifies that the sequence of coherent sheaves $\{\mathscr{I}_m\}_{m>0}$ is indeed a sequence of sheaves of ideals of $\mathscr{O}_Y$ having the desired properties (i), (ii), (iii) of the statement of the theorem. Furthermore, since $\mathscr{I}$, $\mathscr{J}$ have supports off $f^{-1}(U)$, the sheaves of ideals $\mathscr{I}_m$, $m = 1, 2, \cdots$ have supports off U. The necessity of our condition is proved.

To prove the sufficiency we have to show that every point $y \in U$ belongs to the open affine complement Y_s of some section $s \in H^\circ(Y, \mathscr{L}^{\otimes n})$, for some integer $n > 0$. It suffices to do this for closed points.

Let $\mathscr{I}_y$ be the sheaf of ideals of $\mathscr{O}_Y$ defining the reduced scheme structure on the closed subscheme $\{y\}$. Choose the integer $n > 0$, using condition (iii), so that

$$H^1(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \otimes \mathscr{I}_y) = 0 \ .$$

Since y does not belong to the support of $\mathscr{I}_n$ we have an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \otimes \mathscr{I}_y &\longrightarrow \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \\ &\longrightarrow \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \otimes (\mathscr{O}_Y/\mathscr{I}_y) \longrightarrow 0 \ . \end{aligned}$$

By our choice of n we obtain that the homomorphism

$$H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n) \longrightarrow H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n \otimes (\mathscr{O}_Y/\mathscr{I}_y)) = k$$

is surjective. Therefore there exists a section $s' \in H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n)$ which maps onto the element $1 \in k$ under the above homomorphism. The inclusion $\mathscr{I}_n \rightarrow \mathscr{O}_Y$ gives rise to a monomorphism

$$H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{I}_n) \longrightarrow H^\circ(Y, \mathscr{L}^{\otimes n})$$

and the section $s \in H^\circ(Y, \mathscr{L}^{\otimes n})$ which corresponds to s' under the above monomorphism is such that $y \in Y_s$.

We now proceed to show that Y_s is affine. Let $V = \mathrm{Spec}(A)$ be a sufficiently small open subscheme of Y, and let, for every positive integer m,

$$I_m = H^\circ(V, \mathscr{J}_{nm}) .$$

Since the section $s \in H^\circ(Y, \mathscr{L}^{\otimes n})$ is the image of the section $s' \in H^\circ(Y, \mathscr{L}^{\otimes n} \otimes \mathscr{J}_n)$ we see that the local equation of s over U is an element $f \in A$ which belongs to I_1. By condition (ii) of the theorem we have $f^m \in I_m$ for every $m > 0$. Consequently we have inclusions $A \subseteqq (1/f^m)A \otimes I_m \subseteqq A_f$, which show that, for every $m > 0$, we have inclusions

$$\mathscr{O}_Y \subseteqq \mathscr{L}^{\otimes nm} \otimes \mathscr{J}_{nm} \subseteqq j_*\mathscr{O}_{Y_s} , \tag{4.4}$$

where $j\colon Y_s \longrightarrow Y$ is the canonical inclusion.

Let now $\mathscr{F}$ be an arbitrary coherent sheaf of $\mathscr{O}_{Y_s}$-modules. Since the complement of Y_s in Y is locally given by a single equation, the inclusion j is an affine morphism, and theorefore $H^1(Y_s, \mathscr{F}) = H^1(Y, j_*(\mathscr{F}))$. To prove that Y_s is affine it therefore suffices, by a well known theorem of Serre (See [9]), to show that $H^1(Y, j_*(\mathscr{F})) = 0$.

Now $j_*(\mathscr{F})$ is a quasi-coherent sheaf of $\mathscr{O}_Y$-modules, and therefore it is the union of an increasing sequence $\{\mathscr{F}_\nu\}_{\nu>0}$ of coherent subsheaves. The inclusions (4.4) give natural homomorphisms

$$\mathscr{F}_\nu \longrightarrow \mathscr{L}^{\otimes nm} \otimes \mathscr{J}_{nm} \otimes \mathscr{F}_\nu \longrightarrow j_*(\mathscr{F})$$

for every $m > 0$, whose composition is the inclusion $\mathscr{F}_\nu \rightarrow j_*(\mathscr{F})$. For $m \gg 0$ (depending possibly on ν) we have, by hypothesis (iii) of the theorem,

$$H^1(Y, \mathscr{L}^{\otimes nm} \otimes \mathscr{J}_{nm} \otimes \mathscr{F}_\nu) = 0 .$$

Therefore the natural homomorphism $H^1(Y, \mathscr{F}_\nu) \longrightarrow H^1(Y, j_*(\mathscr{F}))$ is the zero homomorphism. Taking a direct limit over ν we see that the identity homomorphism of $H^1(Y, j_*(\mathscr{F}))$ into itself is the zero homomorphism. Therefore $H^1(Y, j_*(\mathscr{F}))$ must vanish, and the theorem is proved.

The author is grateful to the referee for suggesting the correct proof of the vanishing of $H^1(Y, j_*(\mathscr{F}))$.

REMARKS 1. As usual, only the case $i = 1$ of hypothesis (iii) was used in the proof of the sufficiency of the condition. Note also that the condition is sufficient for any integral scheme Y, normal or not, proper or not.

2. The hypothesis of normality of Y can be relaxed in Theorem 4. However, one does no longer obtain that the sheaves $\mathscr{J}_n$, $n = 1, 2, \cdots$ are sheaves of ideals of $\mathscr{O}_Y$. More precisely:

COROLLARY 5. *Let Y be a proper, divisorial, integral scheme. Let U be an open subscheme of Y, and $\mathscr{L}$ an invertible sheaf over Y. Then U is $\mathscr{L}$-projective if, and only if, there exists a descending sequence $\{\mathscr{I}_n\}_{n \geqq 0}$ of coherent $\mathscr{O}_Y$-submodules of $\mathscr{R}(Y)$ such that:*

(i) $\mathscr{O}_Y \subseteqq \mathscr{I}_0$, $\mathscr{I}_n \cap \mathscr{O}_Y \mid U = \mathscr{O}_Y \mid U$ $n \geqq 0$

(ii) $\mathscr{I}_n \mathscr{I}_m \subset \mathscr{I}_{n+m}$ $n, m \geqq 0$

(iii) *For every coherent sheaf of $\mathscr{I}_0$-modules $\mathscr{F}$*

$$H^i(Y, \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_Y} \mathscr{I}_n \otimes_{\mathscr{I}_0} \mathscr{F}) = 0$$

for $n \gg 0$, and all $i > 0$.

Proof. Note that condition (ii) makes $\mathscr{I}_0$ into a coherent sheaf of $\mathscr{O}_Y$-algebras, and $\mathscr{I}_n$ into a coherent sheaf of $\mathscr{I}_0$-ideals. Therefore condition (iii) makes sense.

Let U be $\mathscr{L}$-projective. With the same notations as in the proof of Theorem 4, let $X \xrightarrow{g} Z \xrightarrow{h} Y$ be the Stein factorization of the morphism $f: X \to Y$. Apply the same argument given in the proof of Theorem 4 to the morphism g. Using 1.4.8.1 of Ch. II of [6] we easily verify that the sequence $\{\mathscr{I}_m\}_{m>0}$ defined as in the proof on the theorem, together with $\mathscr{I}_0 = h_*(\mathscr{O}_Z)$, obeys conditions (i), (ii), and (iii) of the statement of the corollary. The necessity is proved.

To prove the sufficiency, let $\mathscr{B} = \mathscr{I}_0$, $Z = \mathrm{Spec}(\mathscr{B})$, and let $g: Z \to Y$ be the canonical morphism. By hypothesis (i) we have that $g \mid g^{-1}(U): g^{-1}(U) \to U$ is an isomorphism. Also, from 1.4.8.1 of Ch. II of [6] and the remark after the proof of Theorem 4, we obtain that $g^{-1}(U)$ is $g^*(\mathscr{L})$-projective. Let $\mathscr{K}$ be the conductor of $\mathscr{B}$ over $\mathscr{O}_Y$, and let, with the notations of § 1.4 of Ch. II of [6], $\mathscr{K}' = \widetilde{\mathscr{K}}$.

Since $\mathscr{K}' \mid g^{-1}(U) = \mathscr{O}_Z \mid g^{-1}(U)$, an easy argument shows that U is $\mathscr{L}$-projective. (See, for example, the proof of 2.6.2.5 of Ch. III of [6]). The corollary is proved.

REMARKS 2. Both in Theorem 4 and Corollary 5 the proof of the necessity of the condition uses the vanishing of $H^1(X, \mathscr{L}^{\otimes n} \otimes \mathscr{F})$ when $\mathscr{L}$ is an ample invertible sheaf over X. However, the proof of the sufficiency of the condition (s) is independent of the corresponding sufficiency statement for the ampleness of $\mathscr{L}$. In fact, the latter result can be easily obtained from the former by setting $\mathscr{I}_n = \mathscr{O}_Y$.

2. Note that in the proof of the sufficiency in Theorem 4 we have actually shown that Y-Supp$(\mathscr{I}_n)$ is $\mathscr{L}$-projective. In fact one easily checks that the properties stated for the sequence $\{\mathscr{I}_n\}_{n>0}$ imply that the sheaves of ideals $\mathscr{I}_n$, $n = 1, 2, \cdots$ have all the same radical, hence the same support.

COROLLARY 6. *Let X be a normal, proper, integral scheme. Then X is divisorial if and only if, there exists a finite number of decreasing sequences of sheaves of ideals $\{\mathscr{I}_n^{(i)}\}_{n>0}$, $i = 1, \cdots, t$, of $\mathscr{O}_X$, and invertible sheaves $\mathscr{L}_1, \cdots, \mathscr{L}_t$ over X such that:*

(i) $\mathscr{I}_n^{(i)} \mathscr{I}_m^{(i)} \subset \mathscr{I}_{n+m}^{(i)}\ n, m > 0, 1 \leqq i \leqq t$

(ii) $\bigcap_{i=1}^{t} \operatorname{Supp}(\mathscr{I}_n^{(i)}) = \varnothing$

(iii) *For every coherent sheaf $\mathscr{F}$ over X and every i, $1 \leqq i \leqq t$, $H^p(X, \mathscr{L}_i^{\otimes n} \otimes \mathscr{I}_n^{(i)} \otimes \mathscr{F}) = 0$ for all $p > 0$, and $n \gg 0$.*

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