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INTEGRATION ON TOPOLOGICAL SEMIFIELDS

HOWARD ANTON AND W. J. PERVIN

T. A. Sarymsakov and H. A. Sarymsakov have considered measures on topological semifields. An integration theory for topological semifields could be based on these measures or, alternatively, on a Daniell approach. In this paper an integration theory for topological semifields will be developed using an analog of the Daniell method.

In the Daniell method of integration for functions on a set Ω, the linear, topological, and lattice structures of R^{Ω} are all used. A topological vector lattice thus has all the essential structure for generalizing the Daniell method. To obtain a measure on subsets of Ω, the ring structure plays a role by relating subsets of Ω to idempotents of R^{Ω}. It will be shown that a topological semifield can be viewed as a topological vector lattice and thus provides a natural setting for generalizing the Daniell method and recovering a measure.

The basic properties of topological semifields were introduced by Antonovskii, Boltyanskii, and Sarymsakov [1, 2, 3]. The reader is referred to these works and to [4] for details; however, for completeness we will summarize some of the essential results needed here.

A topological semifield is an ordered triple $(E, \mathscr{T}, K)$ where E is a commutative ring with at least two elements, $\mathscr{T}$ is Hausdorff topology on E with respect to which E is a topological ring, K is a subring of E such that (i) $K + \bar{K} \subset K$, where $\bar{K}$ is the topological closure of K; (ii) $\bar{K} \cap -\bar{K} = \{0\}$; (iii) $K - K = E$; (iv) $ax = b$ has a solution in K for every $a, b \in K$; and the following axioms are satisfied:

1. If $M \subset E$ is bounded above with respect to the partial order defined by $x \leqq y$ if and only if $y - x \in \bar{K}$, then M has a supremum $(\vee M)$.

2. If $M \subset \nabla$ (the set of idempotents of E) and $\wedge M = 0$, then for each neighborhood U of 0 there exist $e_1, \cdots, e_k$ in M such that $\bigwedge_{i=1}^{k} e_i \in U$.

3. If U is a neighborhood of 0 in E, then there exists a neighborhood Γ of 0 in the relative topology of ∇ such that $E\Gamma \subset U$.

4. If U is a neighborhood of 0 in E, then there exists a neighborhood V of 0 such that $V \subset U$ and V is *saturated* (i.e., $xV \subset V$ when $|x| \leqq 1$).

It follows from the axioms that there is a unique multiplicative identity, 1, satisfying $1 > 0$. Further, the minimal subsemifield containing 1 is semifield isomorphic to R and is called the *axis* of E. If R is identified with the axis, E becomes a vector lattice and an ordered vector space over R with positive cone $\bar{K}$. A nonzero idempotent e is called *minimal* (or *irreducible*) if given any idempotent e' satisfying $0 \leqq e' \leqq e$, it follows that $e' = 0$ or $e' = e$.

It should be noted that the above definition of a topological semifield is the one given in [3]. The original notion of a semifield given in [1] is a special case and is now called a semifield of the first kind or a "Tychonoff" semifield. Semifields of the first kind are isomorphic to subsemifields of the topological semifield $R^{\varDelta}$, where $\varDelta$ is the set of minimal idempotents, and are characterized by the property that $\vee \varDelta = 1$.

It is possible that there are no minimal idempotents; e.g., if E is the set of equivalence classes of measurable functions on $[0, 1]$ with the usual coset operations, $\mathscr{T}$ is the topology of convergence in measure, and K consists of those classes whose representatives are almost everywhere positive. The idempotents are the cosets represented by characteristic functions of measurable sets, and clearly no minimal idempotents exist. If there are no minimal idempotents, the semifield is said to be of the second kind. Every semifield which is not of the first or second kind is representable as a direct sum $E_I \oplus E_{II}$ where E_I is a topological semifield of the first kind and E_{II} is a topological semifield of the second kind.

2. Elementary integrals on integration lattices. Much of the following integration theory will be developed for ordered topological vector spaces and topological vector lattices (see [6] for definitions). As previously noted, every topological semifield is an ordered topological vector space over its axis. We will now show that it is also a topological vector lattice.

Recall that if A is a subset of an ordered topological vector space E, the *full hull*, $[A]$, of A is defined by

$$[A] = \{z \in E: x \leqq z \leqq y \text{ for some } x, y \in A\} .$$

If $A = [A]$, then A is said to be *full*.

THEOREM 2.1. *Every topological semifield is a topological vector lattice over its axis.*

Proof. By Proposition 4.7, page 104 of [6], a topological vector space which is a vector lattice is a topological vector lattice if and only if the positive cone is normal (i.e., there is a neighborhood basis at 0 consisting of full sets) and the lattice operations are continuous. It follows from Theorem 19.4 of [3] that the lattice operations of a topological semifield are continuous. Let W be any neighborhood of 0. It follows from Axiom 4 that there exists a saturated neighborhood V of 0 such that $V + V \subset W$. Choose a saturated neighborhood

U of 0 such that $U + U \subset V$. We will show that $[U]$ is the desired full neighborhood of 0 in W. Let $z \in [U]$. There exist $x, y \in U$ such that $x \leqq z \leqq y$. Thus $0 \leqq z - x \leqq y - x \in U - U = U + U \subset V$. Since V is saturated, $z - x \in V$ (see Theorem 19.2 of [3]). Thus $z \in x + V \subset V + V \subset W$ so that there is a neighborhood basis at 0 consisting of full sets.

A nonempty subset L of an ordered topological vector space E is called an *integration lattice* if (i) L is a vector lattice and (ii) for every $x \in E$ there exists an element $\hat{x} \in L$ such that $x \leqq \hat{x}$. For example, in the semifield of equivalence classes of essentially bounded measurable functions on [0, 1], the simple functions form an integration lattice.

In the classical Daniell approach to integration, extended real valued functions are considered in order to assure the existence of limits of increasing sequences. In the present abstract setting, it is desirable to avoid adjoining ideal elements. Condition (ii) insures that the original lattice is sufficiently large so that no such adjunction will be necessary.

If L is an integration lattice, a strictly positive linear functional $I: L \to R$ is called an *elementary integral* if $I(x_n) \to 0$ whenever $\{x_n\}$ is a sequence in E such that $x_n \downarrow 0$; i.e., $x_{n+1} \leqq x_n$ and $x_n \to 0$ in the vector space topology.

We will assume, now, that L is a fixed integration lattice in a topological vector lattice E, and that I is an elementary integral on L. A point $x \in E$ will be called an *upper element* if there exists a sequence $\{x_n\}$ in L such that $x_n \uparrow x$. The class of all upper elements will be denoted by U.

LEMMA 2.2. *U contains L and is closed under addition, multiplication by nonnegative reals, and the lattice operations.*

Proof. This result follows from the continuity of the linear and lattice operations.

LEMMA 2.3. *If $x_n \uparrow x$ and $y_n \uparrow x$ where $\{x_n\}$ and $\{y_n\}$ are sequences in L, then* $\lim_{n\to\infty} I(x_n) = \lim_{n\to\infty} I(y_n)$.

Proof. It suffices to show that $\lim_{k\to\infty} I(y_k) \leqq \lim_{n\to\infty} I(x_n)$. Since $y_k \leqq x$ for each fixed k, $x \wedge y_k = y_k$. Now $x_n \wedge y_k \leqq x_n$ so that

$$I(x_n \wedge y_k) \leqq I(x_n) \ .$$

Further, by the continuity of $\wedge$, $x_n \wedge y_k \uparrow x \wedge y_k = y_k$ as $n \to \infty$. Therefore, $y_k - (x_n \wedge y_k) \downarrow 0$ as $n \to \infty$. From the definition of an elementary integral we have $I(y_k) - I(x_n \wedge y_k) \to 0$ as $n \to \infty$. Thus $I(y_k) = \lim_{n\to\infty} I(x_n \wedge y_k) \leqq \lim_{n\to\infty} I(x_n)$. Since this holds for each fixed k, we have $\lim_{k\to\infty} I(y_k) \leqq \lim_{n\to\infty} I(x_n)$.

Following the usual Daniell approach, we extend I from L to U by defining $I(x) = \lim_{n\to\infty} I(x_n)$ where $\{x_n\}$ is any sequence in L such that $x_n \uparrow x$. From the previous Lemma, I is a well-defined function which extends the original integral. It follows from property (ii) of integration lattices that I is finite valued. It is also easy to see that $I(x + y) = I(x) + I(y)$ and $I(cx)$ for $c \geqq 0$ and x, y in U.

LEMMA 2.4. *I is monotone and strictly positive on U.*

Proof. Suppose $x_n \uparrow x$, $y_n \uparrow y$, and $x \leqq y$ with x, y in U and $\{x_n\}$, $\{y_n\}$ in L. Then $x_k \leqq y$ for all k, and so $y \wedge x_k = x_k$. Now $y_n \wedge x_k \leqq y_n$ so that $I(y_n \wedge x_k) \leqq I(y_n)$ since I was given to be a positive linear functional on L. By the continuity of $\wedge$, $y_n \wedge x_k \uparrow y \wedge x_k = x_k$ as $n \to \infty$. Thus $I(y_n \wedge x_k) \uparrow I(x_k)$ as $n \to \infty$. Hence

$$I(x_k) = \lim_{n\to\infty} I(y_n \wedge x_k) \leqq \lim_{n\to\infty} I(y_n) = I(y)$$

and, taking limits as $k \to \infty$, we have $I(x) \leqq I(y)$. Further, if $0 \leqq x \in U$ with $I(x) = 0$, there exists a sequence $\{x_n\}$ in L such that $x_n \uparrow x$. Then $0 \leqq x_n \vee 0 \uparrow x$ so that $0 \leqq I(x_n \vee 0) \uparrow I(x) = 0$. Thus, since I is strictly positive on L, $x_n \vee 0 = 0$ for all n, so that $x = 0$.

LEMMA 2.5. *If $\{x_n\}$ is a sequence in U such that $x_n \uparrow x$, then $x \in U$ and $I(x_n) \to I(x)$.*

Proof. For each index m there is a sequence $\{y_{n,m}\}$ in L such that $y_{n,m} \uparrow x_m$ as $n \to \infty$. Let $y_n = y_{n,1} \vee y_{n,2} \vee \cdots \vee y_{n,n} \in L$. Since $y_{n,m} \leqq x_m \leqq x_n$ for all $m \leqq n$, we have $y_n = \bigvee_{m=1}^{n} y_{n,m} \leqq x_n$ for all n. Also $y_n \leqq y_{n+1}$ since

$$y_n = \bigvee_{m=1}^{n} y_{n,m} \leqq \bigvee_{m=1}^{n} y_{n+1,m} \leqq \bigvee_{m=1}^{n+1} y_{n+1,m} = y_{n+1} \,.$$

We will show that $y_n \uparrow x$. Let W be a full neighborhood of x. There exists an index $N = N(W)$ such that $x_n \in W$ for all $n \geqq N$. Since $y_{n,N} \uparrow x_N$, there exists an integer $i = i(N(W))$ such that $y_{i,N} \in W$. Further, we may choose $i \geqq N$ so that $x_i \in W$. Thus, since $y_{i,N} \leqq y_i \leqq x_i \leqq x$ and W is full, we have $y_k \in W$ for each $k \geqq i$. Thus $x \in U$ and $I(y_n) \to I(x)$.

Since $I(y_{n,m}) \leqq I(y_n) \leqq I(x_n)$, by taking limits as $n \to \infty$ we have $I(x_m) \leqq I(x) \leqq \lim_{n\to\infty} I(x_n)$. Finally, letting $m \to \infty$, the lemma follows.

Define $-U$ by $-U = \{x \in E: -x \in U\}$. It is easy to show that

Lemma 2.2 holds for $-U$ and that $x \in -U$ if and only if there exists a sequence $\{x_n\}$ in L such that $x_n \downarrow x$. If $x \in -U$, we define $I(x)$ by $I(x) = -I(-x)$. This definition extends I as a monotone and strictly positive function from U to $U \cup -U$ and for $x, y \in -U$ and $c \geqq 0$ we have $I(x + y) = I(x) + I(y)$ and $I(cx) = cI(x)$. Further, if $x_n \in -U$ and $x_n \downarrow x$, then $x \in -U$ and $I(x_n) \to I(x)$ which is analogous to Lemma 2.5.

3. Summable elements and the monotone convergence theorem. An element x in the topological vector lattice E is said to be *I-summable* if given any $\varepsilon > 0$ there exists a pair of elements $y \in -U$ and $z \in U$ such that $y \leqq x \leqq z$ with $I(z) - I(y) < \varepsilon$. The class of I-summable elements will be denoted by $L^1(I)$.

It is clear that if $x \in L^1(I)$ then $\underline{I}(x) = \sup\{I(y) \colon y \in -U, y \leqq x\}$ and $\bar{I}(x) = \inf\{I(z) \colon z \in U, x \leqq z\}$ are equal. We define $I(x)$ to be this common (finite) value. We note that if $x \in U \cup -U$, then $x \in L^1(I)$ and the new definition of I agrees with the old. In particular, $L \subset L^1(I)$.

THEOREM 3.1. *$L^1(I)$ is an integration lattice and I is a strictly positive linear functional on $L^1(I)$.*

Proof. Arguments analogous to those used in the classical Daniell development will show that $L^1(I)$ is a linear subspace of E and I is a strictly positive linear functional. To show that $L^1(I)$ is closed under the lattice operations, let $x, y \in L^1(I)$ and suppose $\varepsilon > 0$ is given. Choose x_1, y_1 in $-U$ and x_2, y_2 in U such that $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$ and so that $I(x_2) - I(x_1) < \varepsilon/2$, $I(y_2) - I(y_1) < \varepsilon/2$. By Lemma 2.2 it follows that $x_1 \vee y_1$ and $x_1 \wedge y_1$ are in $-U$ and that $x_2 \vee y_2$ and $x_2 \wedge y_2$ are in U. Since $(x_i \vee y_i) + (x_i \wedge y_i) = x_i + y_i$ for $i = 1, 2$, we have

$$I(x_1 \vee y_1) + I(x_1 \wedge y_1) = I(x_1) + I(y_1) > I(x) + I(y) - \varepsilon$$

and

$$I(x_2 \vee y_2) + I(x_2 \wedge y_2) = I(x_2) + I(y_2) < I(x) + I(y) + \varepsilon\ .$$

Now

$$x_1 \vee y_1 \leqq x \vee y \leqq x_2 \vee y_2$$

and

$$x_1 \wedge y_1 \leqq x \wedge y \leqq x_2 \wedge y_2$$

so that, by the above inequalities,

$$\bar{I}(x \vee y) + \bar{I}(x \wedge y) \leqq I(x) + I(y) + \varepsilon$$

and

$$\underline{I}(x \vee y) + \underline{I}(x \wedge y) \geqq I(x) + I(y) - \varepsilon\ .$$

Since ε is arbitrary, we have

$$\bar{I}(x \vee y) + \bar{I}(x \wedge y) \leqq I(x) + I(y) \leqq \underline{I}(x \vee y) + \underline{I}(x \wedge y) .$$

This implies that $\bar{I}(x \vee y) = \underline{I}(x \vee y)$ and $\bar{I}(x \wedge y) = \underline{I}(x \wedge y)$. Thus $x \vee y$ and $x \wedge y$ are in $L^1(I)$. Finally, since $L \subset L^1(I)$, property (ii) of integration lattices holds.

A topological vector lattice is said to have the *monotone convergence property* if every monotone increasing sequence which is bounded above converges.

THEOREM 3.2. (Monotone Convergence) *If E is a topological vector lattice with the monotone convergence property, and if $\{x_n\}$ is a sequence in $L^1(I)$ such that $x_n \uparrow x$, then $x \in L^1(I)$ and $I(x_n) \to I(x)$.*

Proof. Let $\varepsilon > 0$ be given. We may suppose, by subtracting x_0 if necessary, that $x_0 = 0$. We have $x_n - x_{n-1} \in L^1(I)$ so there exists an element y_n in U such that $x_n - x_{n-1} \leqq y_n$ and

$$(*) \qquad I(y_n) \leqq I(x_n - x_{n-1}) + \varepsilon/2^n .$$

By property (ii) of integration lattices, there is an element $\hat{x}$ in L such that $x \leqq \hat{x}$. Let

$$z_n = \hat{x} \wedge \sum_{i=1}^{n} y_i .$$

Then $\{z_n\}$ is an increasing sequence in U bounded above by $\hat{x}$. From the monotone convergence property, there is an element z such that $z_n \uparrow z$. By Lemma 2.5, $z \in U$. Now

$$x_n = \hat{x} \wedge x_n = \hat{x} \wedge \sum_{i=1}^{n} (x_i - x_{i-1}) \leqq \hat{x} \wedge \sum_{i=1}^{n} y_i = z_n$$

so that $x \leqq z$. From inequality (*),

$$\sum_{i=1}^{n} I(y_i) \leqq I(x_n) + \varepsilon .$$

Since $z_n \uparrow z$, we have

$$I(z) = \lim_{n\to\infty} I(z_n) \leqq \lim_{n\to\infty} I\Big(\sum_{i=1}^{n} y_i\Big) = \lim_{n\to\infty} \sum_{i=1}^{n} I(y_i) \leqq \lim_{n\to\infty} I(x_n) + \varepsilon .$$

By a similar argument, there is an element y in $-U$ such that $y \leqq x$ and $I(y) \geqq \lim_{n\to\infty} I(x_n) - \varepsilon$, so that x is I-summable.

COROLLARY 3.3. *If E is a topological vector lattice with the monotone convergence property, and if $\{x_n\}$ is a sequence in $L^1(I)$ such that $x_n \to x$, then $x \in L^1(I)$ and $I(x_n) \to I(x)$.*

This corollary is the abstract analog of the Dominated Convergence

Theorem of Lebesgue. Although the usual domination property does not explicitly appear, it is satisfied as a consequence of condition (ii) of integration lattices.

From Theorem 2.1, the Monotone Convergence Theorem holds for topological semifields which have the monotone convergence property. We note that every topological semifield of the first kind has the monotone convergence property. Since almost everywhere convergence implies convergence in measure for finite measure spaces, the topological semifield S of measurable functions on $[0, 1]$ also has the monotone convergence property. It follows that the direct sum of a semifield of the first kind with S also has the monotone convergence property. It is a consequence of an unpublished result of D. A. Vladimirov [4, p. 187] that there exist semifields which are not of this form. An open problem is to characterize those topological semifields which have the monotone convergence property.

4. The confinal of measurable idempotents. It is shown in [3] that the set ∇ of all idempotents in a topological semifield forms a (topological) Boolean algebra. We define a *Boolean ring* of idempotents to be a set Ξ of idempotents which is closed under finite suprema and proper differences (i.e., if $x, y \in \Xi$, then $x - (x \wedge y) \in \Xi$). It should be noted that a Boolean ring is closed under finite infima and is itself a Boolean algebra if and only if $1 \in \Xi$.

As an immediate consequence of Theorem 3.1 we have the following.

THEOREM 4.1. *The set $J = \nabla \cap L^1(I)$ of all I-summable idempotents in a topological semifield forms a Boolean ring. Further, if $1 \in L$, then J is a Boolean algebra.*

In [5], T. A. Sarymsakov and H. A. Sarymsakov define a subset K of ∇ to be a *confinal* if it is closed under finite infima, suprema, and complementation. It is evident that a confinal is a Boolean algebra. They give the following definition. A nonnegative, continuous function m defined on a confinal K, admitting finite or infinite values, finitely additive for disjoint elements of the confinal K, and equal to zero only on the idempotent 0 is called a *measure*.

Let $e \in \Delta$. If there exist $\sup\{m(a): a \leqq e, a \in K\} = m_*(e)$ and $\inf\{m(a): a \geqq e, a \in K\} = m^*(e)$, and they are equal, then the element e is called *measurable* and $m(e) = m_*(e) = m^*(e)$ is its measure. The set of all measurable elements in ∇, denoted K_∇, is called a *maximal confinal*.

THEOREM 4.2. *In a topological semifield with the monotone convergence property, $m(e) = I(e)$ defines a finite measure on J. Further,*

$J = J_r$ is a maximal confinal with respect to this measure.

Proof. The continuity of m follows immediately from Theorem 3.2. Since $J \subset J_r$, to show that J is maximal, it suffices to show that $J_r \subset L^1(I)$. If $e \in J_r$, then from the definition of an integration lattice, there is an element $\hat{e}$ in $L^1(I)$ such that $e \leqq \hat{e}$. Thus $m^*(e) = m_*(e) = \sup\{m(a): a \leqq e, a \in J\} \leqq I(\hat{e}) < +\infty$. It follows that there exist e_1, $e_2 \in J$ and $z \in U$, $y \in -U$ such that $y \leqq e_1 \leqq e \leqq e_2 \leqq z$ and $I(e_2) - I(e_1) < \varepsilon/2$, $I(z) < I(e_2) + \varepsilon/4$, $I(y) > I(e_1) - \varepsilon/4$. So that $e \in L^1(I)$.

In [5] an element $x \in E$ is called measurable if its carrier e_x (see [3], p. 68) is in K_r and $\bigvee\{e: xe < \lambda e\} \in K_r$ for every real number λ. The relationship between these measurable elements and the I-summable elements obtained in the above Daniell approach will not be considered here.

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DREXEL UNIVERSITY