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ON COMPLETELY HAUSDORFF-COMPLETION OF A COMPLETELY HAUSDORFF SPACE

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R. M. Stephenson, Jr. (Trans. Amer. Math. Soc. 133 (1968), 537-546) has established the existence of a completely Hausdorff-closed extension X' of an arbitrary completely Hausdorff space X. Stephenson demonstrates that X' enjoys many interesting properties of the Stone-Čech compactification. This paper shows that, by a modification of the topology, X' is made also to possess a property which is in the line of the celebrated property of the Stone-Čech compactification of a completely regular Hausdorff space that it is the largest amongst all Hausdorff compactifications.

1. **Introduction.** A topological space X is called completely Hausdorff if for every pair x, y of distinct points of X there exists a continuous real valued function f on X such that $f(x) \neq f(y)$. A completely Hausdorff space is called completely Hausdorff-closed if every homeomorphic image of it in any completely Hausdorff space is closed. A space Y is termed a completely Hausdorff-closed extension of a completely Hausdorff space X if X is dense in Y and Y is completely Hausdorff-closed. R. M. Stephenson, Jr. in [4] has established the existence of a completely Hausdorff-closed extension (referred to as the completely Hausdorff-completion) X' of an arbitrary completely Hausdorff space X. If X is completely regular (which, of course, assumes Hausdorff property and is necessarily completely Hausdorff) then X' is the Stone-Čech compactification of X. Stephenson shows {[4], Theorem 4} that, even if X is completely Hausdorff but not necessarily completely regular, X' continues to enjoy many interesting properties of the Stone-Čech compactification. By enlarging the topology of X' we shall, in fact, strengthen Theorem 4 of [4] in the sense that property (vii) therein will be replaced by the following:

X' is a projective maximum in the class of completely Hausdorff-closed extensions Y of X with the property that any element in $F(X)$, the set of all continuous functions on X into $[0, 1]$, admits an extension to $F(Y)$.

The above property is, obviously, akin to the well-known fact that the Stone-Čech compactification is largest among the Hausdorff compactifications of a completely regular Hausdorff space.

2. **Notations and definitions.** We shall try to follow the notations and definitions of [4] as far as possible.

$C(X)$ will stand for the set of all bounded continuous functions on X. If Z is any topological space, we shall denote by $C(X, Z)$ the set of all continuous mappings of X into Z.

A topological space Y is an *extension space* of another space X if X is dense in Y. If T is an extension space of a topological space S, the *tracefilters* of T are the filters $\mathscr{T}(t)$, $t \in T - S$, where $\mathscr{T}(t)$ is the filter on S given by $\{U \cap S: U$ a neighbourhood of t in $T\}$.

Banaschewski [1] introduced the notion of a projective maximum in a set E of extensions of X; an extension Y in E is a projective maximum in E if for each Z in E there is a continuous function from Y onto Z which leaves X pointwise fixed.

A filter $\mathscr{F}$ on a space X is called completely regular provided that it has a base $\mathscr{B}$ of open sets such that for each $B \in \mathscr{B}$, there is a set $B' \subset B$ in $\mathscr{B}$ and a function $f \in F(X)$ such that $f(B') = 0$ and $f = 1$ on $X - B$.

3. **Main result.** Let X be a completely Hausdorff space, and let $\mathscr{M}$ be the set of all maximal completely regular filters on X which have empty adherences. (If $\mathscr{F}$ is a completely regular filter,

$$\cap \{F: F \in \mathscr{F}\} = \cap \{\bar{F}: F \in \mathscr{F}\} = \text{adherence of } \mathscr{F} ,$$

where $\bar{F}$ = closure of F in X. If $\cap \{F: F \in \mathscr{F}\} = \varnothing$, $\mathscr{F}$ is called *free*, otherwise it is called *fixed*.) Put $X' = X \cup \mathscr{M}$. We shall endow X' with a topology as follows:

Any set, open in X, is also open in X'. If $\mathscr{F} \in \mathscr{M}$, basic neighbourhoods of $\mathscr{F}$ are of the form $G \cup \{\mathscr{F}\}$ where $G \in \mathscr{F}$. With this topology (will, henceforth, be called the Katětov topology) X' becomes a completely Hausdorff-closed space and will be called the completely Hausdorff-completion of X. The trace filters of X' are the filters $\{\mathscr{T}(\mathscr{F}): \mathscr{F} \in \mathscr{M}\}$ and for each $\mathscr{F} \in \mathscr{M}$, $\mathscr{T}(\mathscr{F}) = \{U \cap X: U \subset X'$ and U a neighbourhood of $\mathscr{F}\} = \{G: G \in \mathscr{F}\} = \mathscr{F}$. Thus the trace filters of X' are the maximal completely regular filters $\mathscr{F}$ on X such that

$$\cap \{G: G \in \mathscr{F}\} = \varnothing .$$

Now we are in a position to state our main theorem which is identical with Theorem 4 of [4] with the exception of property (vii).

THEOREM 1. *Let X be a completely Hausdorff space. The completely Hausdorff-completion X' of X has the following properties*:

(i) *If Z is a compact Hausdorff space, then each function in $C(X, Z)$ has a unique extension in $C(X', Z)$.*

(ii) *The Stone-Weierstrass theorem holds for X'.*

(iii) *X' is locally connected if and only if X is locally connected and each trace filter of X' has a base consisting of connected open sets.*

(iv) *X' is locally connected only if X is locally connected and pseudocompact.*

(v) *X' is connected if and only if X is connected.*

(vi) *$C(X')$ and $C(X)$ are isomorphic, and if R is the real line, $C(X')$ and $C(X, R)$ are isomorphic only if X is pseudocompact.*

(vii) *Suppose Y is a completely Hausdorff-closed space containing X as a dense subset and each element of $F(X)$ has an extension to $F(Y)$. Then there exists a one-to-one function $g \in C(X', Y)$ such that $g(X') = Y$ and g is identity on X. In short, X' is a projective maximum in the class of completely Hausdorff-closed extensions Y of X with the property that any element in $F(X)$ admits an extension to $F(Y)$.*

Proof. Proofs of (i) – (vi) are omitted as they are same as those given in Theorem 4 of [4] (page 540). We shall only give a proof for (vii). Let Y be a completely Hausdorff-closed topological space containing X as a dense subset and such that every function in $F(X)$ admits an (unique) extension to $F(Y)$. If $\mathscr{F}$ is a nonconvergent maximal completely regular filter on X(i.e., $\mathscr{F} \in \mathscr{M}$) define $Z = \{f \in F(X)$: for some $G', G \in \mathscr{F}$ with $G' \subset G$, one has $f(G') = 0$ and $f(X - G) = 1\}$. Z is nonvoid as $\mathscr{F}$ is completely regular. For $f \in F(X)$ let f' denote its extension in $F(Y)$. Put $Z' = \{f' : f \in Z\}$. Take $\mathscr{S} = \{V(f', t) = f'^{-1}[0, t) : f' \in Z', 0 < t \leqq 1\}$. The empty set does not belong to $\mathscr{S}$. Consider, $V(f_i', t_i) \in \mathscr{S}$, $i = 1, 2, \cdots, n$ and choose, for each i, $0 < s_i < t_i$. By using the normality of $[0, 1]$ we can get $g_i \in F(Y)$ such that $g_i(V(f_i', s_i)) = 0$ and $g_i [Y - V(f_i', t_i)] = 1$ for $i = 1, 2, \cdots, n$. Put $g = \max_{1 \leqq i \leqq n} g_i$. Then $g \in F(Y)$ and $g[\bigcap_{i=1}^{n} V(f_i', s_i)] = 0$ and

$$g[Y - \bigcap_{j=1}^{n} V(f_j', t_j)] = 1 .$$

Note also that $\bigcap_{j=1}^{n} V(f_j', s_j) \subset \bigcap_{j=1}^{n} V(f_j', t_j)$. Thus, we have shown that finite intersections of sets of $\mathscr{S}$ form a completely regular filter base on Y. Let $\mathscr{G}$ be the completely regular filter on Y generated by $\mathscr{S}$ and let $\mathscr{U}$ denote a maximal completely regular filter on Y such that $\mathscr{G} \subset \mathscr{U}$. Since Y is completely Hausdorff-closed every completely regular filter on Y has nonempty adherence (See [4] Theorem 1, and [2]). Consequently adherence of $\mathscr{U}$ $(= ad\ (\mathscr{U}))$ is nonempty and maximality of $\mathscr{U}$ will make $\mathscr{U}$ converge to each point in $ad(\mathscr{U})$. But Y is Hausdorff, so $ad(\mathscr{U})$ must contain exactly one point, i.e., $\cap U = \cap \{U : U \in \mathscr{U}\}$ is a singleton. We now claim that $\mathscr{F} = \{U \cap X : U \in \mathscr{U}\}$.

Proof of the claim. Since $\mathscr{U}$ is a maximal completely regular

open filter it has a completely regular filter base $\mathscr{V}$ consisting of open sets. As X is dense in Y, it is easy to see that $\mathscr{U} \cap X = \{U \cap X: U \in \mathscr{U}\}$ is an open filter on X with an open base given by $\mathscr{V} \cap X = \{V \cap X: V \in \mathscr{V}\}$. Let $V \cap X \in \mathscr{V} \cap X$. Since $V \in \mathscr{V}$ there exist $V' \in \mathscr{V}$ with $V' \subset V$ and $h \in F(Y)$ such that $h(V') = 0$ and $h(Y - V) = 1$. Obviously, $h(V' \cap X) = 0$ and $h(X - V \cap X) = 1$. Let f denote the restriction of h to X. Then $f \in F(X)$ and $f(V' \cap X) = 0$ and $f(X - V \cap X) = 1$ i.e., $\mathscr{V} \cap X$ is a completely regular filter base on X for $\mathscr{U} \cap X$. Therefore $\mathscr{U} \cap X$ is a completely regular filter on X. Again $\mathscr{F}$ is a completely regular filter on X, so $F \in \mathscr{F}$ implies that there exist $F' \in \mathscr{F}$ with $F' \subset F$ and $f \in F(X)$ such that $f(F') = 0$ and $f(X - F) = 1$. This gives $F' \subset f^{-1}[0, 1) \subset F$. Hence $f \in Z$ and $F' \subset f'^{-1}[0, 1) \cap X \subset F$ where $f' \in Z'$. Now, $f'^{-1}[0, 1) \in \mathscr{S} \subset \mathscr{U}$. Thus $X \cap f'^{-1}[0, 1) \in \mathscr{U} \cap X$ and $F \supset X \cap f'^{-1}[0, 1)$ implies $F \in \mathscr{U} \cap X$ (since it is a filter). We get $\mathscr{F} \subset \mathscr{U} \cap X$ and maximality of $\mathscr{F}$ forces $\mathscr{F} = \mathscr{U} \cap X$. Immediately we have from the above fact, $(\cap U) \cap X = \cap (U \cap X) = \cap \{F: F \in \mathscr{F}\} = \varnothing$ as $\mathscr{F}$ is a free maximal completely regular filter. So the single point contained in $\cap U$ is actually in $Y - X$. Let the point be denoted by $y(\mathscr{F})$. Next we show that if $\mathscr{F}_1$ and $\mathscr{F}_2$ are two distinct points in $\mathscr{M}$, the points $y(\mathscr{F}_1)$ and $y(\mathscr{F}_2)$ are distinct points of $Y - X$. Since $\mathscr{F}_1$ and $\mathscr{F}_2$ are two distinct free maximal completely regular filters there must exist $G_1 \in \mathscr{F}_1$ and $G_2 \in \mathscr{F}_2$ such that G_1 and G_2 are open in X and $G_1 \cap G_2 = \varnothing$. As shown earlier, we can associate two maximal completely regular filters $\mathscr{U}_1$ and $\mathscr{U}_2$ on Y with $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively. By definition $\{y(F_i)\} = \cap \{U: U \in \mathscr{U}_i\}$, $i = 1, 2$ and we also know that $\mathscr{F}_i = \mathscr{U}_i \cap X$. Consequently there exists $U_i \in \mathscr{U}_i$ such that $U_i \cap X = G_i$ and U_i is open $(i = 1, 2)$. Since $G_1 \cap G_2 = \varnothing$ and X is dense in Y we have $U_1 \cap U_2 = \varnothing$. Since $y(\mathscr{F}_i) \in \mathscr{U}_i$ for $i = 1, 2$ we get $y(\mathscr{F}_1) \neq y(\mathscr{F}_2)$. So far we have shown that $\mathscr{F} \mapsto y(\mathscr{F})$ is a one-to-one map of $\mathscr{M}$ into $Y - X$. Let i denote the identity map on X into Y.
Define $\bar{i}: X' \to Y$ as follows:

$$\bar{i}(x) = i(x) = x \qquad \text{if } x \in X, \text{ and}$$
$$\bar{i}(\mathscr{F}) = y(\mathscr{F}) \qquad \text{if } \mathscr{F} \in \mathscr{M} = X' - X.$$

Claim: $\bar{i}$ is continuous.

We shall establish the continuity by showing the continuity at each point.

(i) Suppose $x \in X$. Then $\bar{i}(x) = x$. Let W be an open neighbourhood of x in Y, then $\bar{i}^{-1}(W) \cap X = i^{-1}(W) = G$, an open neighbourhood of x in X and hence open in X' and also $\bar{i}\,(G) \subset W$.

(ii) For $\mathscr{F} \in \mathscr{M}$, we have $\bar{i}(\mathscr{F}) = y(\mathscr{F})$. By construction of

$y(\mathscr{F})$ we know that it is the point of convergence of a maximal completely regular filter $\mathscr{U}$ on Y such that $\mathscr{F} = \mathscr{U} \cap X$.

If W is an open neighbourhood of $y(\mathscr{F})$ in Y then $W \in \mathscr{U}$ i.e., $W \cap X \in \mathscr{F}$. But $W \cap X$ is open in X and hence $(W \cap X) \cup \{\mathscr{F}\}$ is an open neighbourhood of $\mathscr{F}$ in X' such that

$$\begin{aligned}\bar{i}[(W \cap X) \cup \{\mathscr{F}\}] &= \bar{i}(W \cap X) \cup \bar{i}(\mathscr{F}) = i(W \cap X) \cup \{y(\mathscr{F})\} \\ &= (W \cap X) \cup \{y(\mathscr{F})\} \subset W .\end{aligned}$$

Thus the continuity of $\bar{i}$ has been proved. But X' is, in particular, completely Hausdorff-closed and $\bar{i}$ is a continuous function on X' into a completely Hausdorff space Y in which X is dense. Consequently, from the following fact it will follow that $\bar{i}$ is *onto* Y.

Fact. Let X be a completely Hausdorff-closed space and let Y be a completely Hausdorff space such that there is a continuous function $f: X \to Y$. Then $f(X)$ is a completely Hausdorff closed subspace of Y.

Let us put $g = \bar{i}$. Then $g \in C(X', Y)$ with $g(X') = Y$ and g restricted to X equals i, the identity map on X.

COROLLARY 1. *Suppose Y is completely Hausdorff-closed space satisfying the conditions stated in Theorem* 1(vii) *and f is a homeomorphism of X onto X, then there exists a one-to-one function $g \in C(X', Y)$ such that $g(X') = Y$ and g restricted to X equals f.*

Proof. We first note that if $\mathscr{F}$ is a nonconvergent maximal completely regular filter on X, $f(\mathscr{F})$ is a nonconvergent maximal completely regular filter on X. Then the proof follows by a reasoning similar to one presented in the proof of Theorem 1(vii) where i is replaced by f.

4. REMARKS. The completely Hausdorff-completion X' of X in Theorem 1 is essentially unique, i.e., if T is any completely Hausdorff closed extension of X and T satisfies the properties of Theorem 1 then X' and T are homeomorphic. For there exists $g \in C(X', T)$ such that $g(X') = T$ and g is identity on X. Also, there exists $h \in C(T, X')$ such that $h(T) = X'$ and h is identity on X. Therefore by the following result {[3], page 5} we can assert that X' and T are homeomorphic.

Result. Let X be dense in each of the Hausdorff spaces S and T. If the identity mapping on X has continuous extensions s from

S into T, and t from T into S, then s is a homeomorphism onto, and $s^{-1} = t$.

One can raise the following two questions regarding Theorem 1: (a) Is a Y satisfying the condition (vii) of Theorem 1 homeomorphic to X'? (b) Is X' a one-to-one continuous image of such Y? We shall answer both the questions in the negative. Let N denote the set of natural numbers with discrete topology. On N any free maximal completely regular filter is nothing but a free ultrafilter. Thus $\beta N = NU\mathscr{M}$ where $\mathscr{M}$ is the set of all free ultrafilters on N. The topology by which βN is the Stone-Čech compactification of N will be called Stone-Čech topology ($S - \check{C}$ topology) for βN. Its open sets are generated by $\{V' \colon V \text{ open in } N\}$ where $V' = V \cup \{\mathscr{F} \in \mathscr{M} \colon V \in \mathscr{F}\}$. But, according to our definition, βN endowed with the Katětov topology is the completely Hausdorff-completion of N and in this topology $\mathscr{M} = \beta N - N$ is a closed, discrete infinite subspace of βN and, thus, cannot be compact. While in the $S - \check{C}$ topology of βN, $\mathscr{M}$ is closed, no doubt, and hence compact. Clearly, the $S - \check{C}$ topology is strictly weaker than the Katětov topology. As $S - \check{C}$ topology of βN is compact, *no continuous map from* βN with $S - \check{C}$ *topology onto* βN *with Katětov topology can exist*. So homeomorphism is ruled out. But the Stone-Čech compactification βN *satisfies all the conditions enjoyed by a* Y *in Theorem* 1(vii).

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