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A REPRESENTATION FOR THE LOGARITHMIC DERIVATIVE OF A MEROMORPHIC FUNCTION

MORRIS MARDEN

A new representation is developed for the logarithmic derivative of a meromorphic function f in terms of its zeros and poles, using as parameters some of the critical points of f. Applications are made to locating all but a finite number of critical points of f.

1. The principal result.

THEOREM 1.1. *Let f be a meromorphic function of finite order ρ possessing the finite zeros $a_1, a_2, a_3, \cdots$ and poles $b_1, b_2, b_3, \cdots$. Let $\zeta_1, \zeta_2, \cdots \zeta_n$, be any $n = [\rho]$ distinct zeros of the derivative f' of f which are not also zeros of f. Then for $z \neq a_j, b_j$ $(j = 1, 2, 3, \cdots)$*

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)} \tag{1.1}$$

where $\psi(z) = 1$ for $n = 0$,

$$\psi(z) = (z - \zeta_1)(z - \zeta_2) \cdots (z - \zeta_n) \qquad \text{for } n > 0 . \tag{1.2}$$

In (1.1) the convergence is uniform on every compact set excluding all the a_j and b_j.

In the case that f is a rational function with m zeros and p poles, identity (1.1) reduces to the familiar formula

$$f'(z)/f(z) = \sum_{j=1}^{m} (z - a_j)^{-1} - \sum_{j=1}^{p} (z - b_j)^{-1} .$$

Furthermore, if the second summation is omitted in (1.1), identity (1.1) reduces to one which we had previously obtained [See 1] for entire functions of finite order.

2. *Proof.* Being a meromorphic function, f can be written as a ratio of two entire functions, each of which has an Hadamard representation in terms of its zeros. Thus,

$$f(z) = z^m e^{P(z)} \prod_{j=1}^{\infty} [E(z/a_j, p)/E(z/b_j, q)] \tag{2.1}$$

where m is an integer (positive, negative or zero); $P(z)$ is a polynomial of degree at most $n = [\rho]$; p and q are nonnegative integers not exceeding n and

$$E(u, p) = (1 - u) \exp [u + (1/2)u^2 + \cdots + (1/p)u^p]$$

if $p > 0$ whereas $E(u, 0) = (1 - u)$. Taking the logarithmic derivative of (2.1) and simplifying, one finds that

$$\frac{f'(z)}{f(z)} = \frac{m}{z} + P'(z) + A(z) - B(z) \tag{2.2}$$

where

$$A(z) = \sum_{j=1}^{\infty} \frac{z^p}{a_j^p(z - a_j)}, \quad B(z) = \sum_{j=1}^{\infty} \frac{z^q}{b_j^q(z - b_j)} . \tag{2.3}$$

By hypothesis, $f'(\zeta_k) = 0$, $k = 1, 2, \cdots, n$. Hence, from (2.2), follows that for $k = 1, 2, \cdots, n$

$$P'(\zeta_k) = -(m/\zeta_k) - A(\zeta_k) + B(\zeta_k) . \tag{2.4}$$

Since $P'(z)$ is a polynomial of degree at most $n - 1$, it can be represented by the Lagrange Interpolation Formula as

$$\frac{P'(z)}{\psi(z)} = \sum_{k=1}^{n} \frac{P'(\zeta_k)}{\psi'(\zeta_k)(z - \zeta_k)} .$$

Hence, using (2.3) and (2.4), one finds that

$$\begin{aligned} \frac{P'(z)}{\psi(z)} = &-\sum_{k=1}^{n} \frac{m}{\zeta_k \psi'(\zeta_k)(z - \zeta_k)} - \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{\zeta_k^p}{a_j^p \psi'(\zeta_k)(z - \zeta_k)(\zeta_k - a_j)} \\ &+ \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{\zeta_k^q}{b_j^q \psi'(\zeta_k)(z - \zeta_k)(\zeta_k - b_j)} . \end{aligned} \tag{2.5}$$

In view of the fact that sums $A(z)$ and $B(z)$ are uniformly and absolutely convergent on every compact set that omits all the a_j and b_j, the order of summation of the double sums in (2.5) can be reversed. Thus the first double sum in (2.5) becomes

$$\sum_{j=1}^{\infty} \frac{1}{a_j^p} \sum_{k=1}^{n} \frac{\zeta_k^p}{\psi'(\zeta_k)(\zeta_k - a_j)(z - \zeta_k)} = \sum_{j=1}^{\infty} \frac{1}{a_j^p(z - a_j)} \left[\frac{S(z)}{\psi(z)} - \frac{S(a_j)}{\psi(a_j)} \right] \tag{2.6}$$

where

$$S(z) = \psi(z) \sum_{k=1}^{n} \frac{\zeta_k^p}{(z - \zeta_k)\psi'(\zeta_k)} . \tag{2.7}$$

Since the polynomial $S(z)$ is of degree at most $n-1$ with $S(\zeta_k) = \zeta_k^p$, the polynomial

$$T(z) = S(z) - z^p \tag{2.8}$$

is of degree at most n such that

$$T(\zeta_k) = 0, \quad \text{for} \quad k = 1, 2, \cdots, n \,.$$

Therefore $T(z) = c\,\psi(z)$, where c is a constant that may be zero. Accordingly,

$$S(z) = z^p + c\,\psi(z)$$

and the sum (2.6) becomes

$$\text{(2.9)} \qquad \sum_{j=1}^{\infty} \frac{z^p}{\psi(z) a_j^p (z - a_j)} - \sum_{j=1}^{\infty} \frac{1}{\psi(a_j)(z - a_j)} \,.$$

Similarly the second double sum in (2.5) reduces to

$$\text{(2.10)} \qquad \sum_{j=1}^{\infty} \frac{z^q}{\psi(z) b_j^q (z - b_j)} - \sum_{j=1}^{\infty} \frac{1}{\psi(b_j)(z - b_j)} \,.$$

Finally, on use of the Lagrange Interpolation Formula for $1/\psi(z)$, the single sum in (2.5) becomes

$$\text{(2.11)} \qquad \frac{m}{z} \sum_{k=1}^{n} \left[\frac{1}{\zeta_k} + \frac{1}{z - \zeta_k} \right] \frac{1}{\psi'(\zeta_k)} = \frac{m}{z} \left[-\frac{1}{\psi(0)} + \frac{1}{\psi(z)} \right] .$$

Substituting from (2.9), (2.10) and (2.11) into (2.5), one reduces (2.2) to

$$\text{(2.12)} \qquad \frac{f'(z)}{f(z)} = \frac{m\psi(z)}{z\psi(0)} + \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(a_j)(z - a_j)} - \sum_{j=1}^{\infty} \frac{\psi(z)}{\psi(b_j)(z - b_j)} \,.$$

However, the first term here may be dropped since it is obtainable from the first or second sum in (2.12) by allowing either m a_j (if $m > 0$) or $-m$ b_j (if $m < 0$) to coalesce at 0. Thus identity (1.1) has been established.

3. Location of critical points. An immediate consequence of Theorem 1.1 is the following:

THEOREM 3.1. *Let f be a meromorphic function of finite order ρ possessing the finite zeros $a_1, a_2, a_3, \cdots$ and poles $b_1, b_2, b_3, \cdots$ and let $\zeta_0, \zeta_1, \cdots, \zeta_n$, be any $n + 1 = [\rho] + 1$ distinct critical points of f which are not also zeros of f. Then*

$$\text{(3.1)} \quad \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - a_j)(\zeta_1 - a_j) \cdots (\zeta_n - a_j)} = \sum_{j=1}^{\infty} \frac{1}{(\zeta_0 - b_j)(\zeta_1 - b_j) \cdots (\zeta_n - b_j)} \,.$$

Equation (3.1) follows from (1.1) on setting $z = \zeta_0$, writing out $\psi(a_j)$ according to (1.2) and cancelling the factor $\psi(\zeta_0) \neq 0$.

As an application of (3.1), the following will now be proved.

THEOREM 3.2. *Let D_a and D_b be two regions with which can be associated a set R of points ζ such that a ray from ζ to some point γ separates $\bar{D}_a$ from $\bar{D}_b$ and such that inequality*

$$0 < \arg [(\gamma - \zeta)/(z - \zeta)] < \pi/(n + 1) \qquad (\text{mod } 2\pi)$$

holds for all z in one of the regions D_a, D_b and inequality

$$-\pi/(n + 1) < \arg [\gamma - \zeta)/(z - \zeta)] < 0 \qquad (\text{mod } 2\pi)$$

holds for all z in the other region. Let f, a meromorphic function of finite order ρ, have all its zeros in D_a and all its poles in D_b. Then at most $n = [\rho]$ critical points of f lie in R.

Proof. If on the contrary $n+1$ distinct critical points $\zeta_0, \zeta_1, \cdots, \zeta_n$ were in R, identity (3.1) holds for them in relation to the zeros and poles of f. By hypothesis, one can associate with each ζ_k, a point γ_k such that for $j = 1, 2, 3, \cdots$ the inequalities

$$(3.2) \qquad 0 < \arg \frac{\gamma_k - \zeta_k}{a_j - \zeta_k} < \frac{\pi}{n + 1} \qquad (\text{mod } 2\pi)$$

$$(3.3) \qquad -\frac{\pi}{n + 1} < \arg \frac{\gamma_k - \zeta_k}{b_j - \zeta_k} < 0 \qquad (\text{mod } 2\pi)$$

hold (or those with a_j and b_j interchanged).
Setting

$$T(z) = \prod_{k=0}^{n} [(\gamma_k - \zeta_k)/(z - \zeta_k)] ,$$

one infers that

$$0 < \arg T(a_j) < \pi, \qquad -\pi < \arg T(b_j) < 0 ,$$

for all j. This means that

$$0 < \arg \sum_{j=1}^{\infty} T(a_j) < \pi ,$$

$$-\pi < \arg \sum_{j=1}^{\infty} T(b_j) < 0 .$$

Consequently,

$$\sum_{j=1}^{\infty} T(a_j) \neq \sum_{j=1}^{\infty} T(b_j) ,$$

in contradiction to (3.1). Consequently, at most n distinct critical points ζ_k can lie on R, as was to be proved.

As an illustration, let f be a meromorphic function of order ρ,

$1 \leqq \rho < \infty$, and let

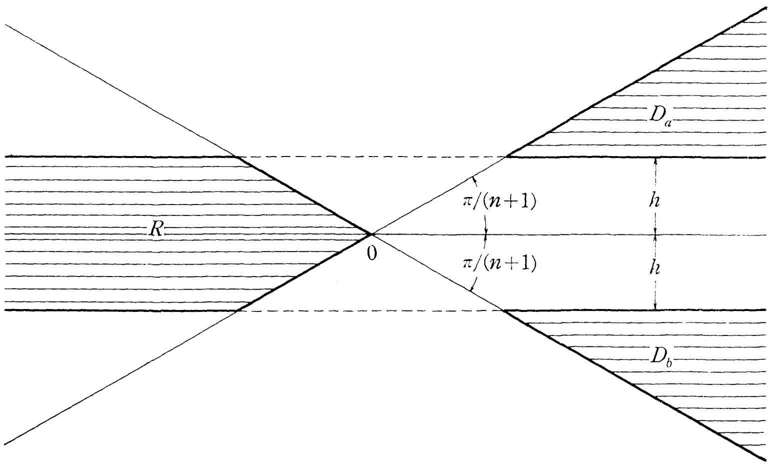

FIGURE 1

$$D_a = \{z = x + iy: x > 0, \qquad 0 \leqq h < y < x \tan [\pi/(n+1)]\}$$
$$D_b = \{z = x + iy: x > 0, \qquad -h > y > -x \tan [\pi/(n+1)]\} .$$

Then, according to Theorem 3.2, at most n critical points of f lie in the region

$$R = \{z = x + iy: x < 0, \qquad |y| < \min [h, |x| \tan [\pi/(n+1)]\}$$

REMARK. In identity (1.1) and the subsequent theorems, $f'(z)$ may be replaced by the linear combination $f'(z) + \lambda f(z)$ or more generally by

$$F_1(z) = f'(z) + f(z)g'(z)$$

where $g(z)$ is an arbitrary polynomial of degree at most n, provided $\zeta_0, \zeta_1, \zeta_2, \cdots \zeta_n$ are taken as the zeros of $F_1(z)$. This follows from the fact that the meromorphic function $F(z) = e^{g(z)}f(z)$ is also of order ρ, has the same zeros and poles as f and $F'(z) = e^{g(z)}F_1(z)$.

REFERENCE

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UNIVERSITY OF WISCONSIN-MILWAUKEE