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COUNTEREXAMPLES TO A CONJECTURE OF G. N. DE OLIVEIRA

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G. N. de Oliveira gives the following conjecture.
CONJECTURE. Let A be an n $\times$ n doubly stochastic irreducible matrix. If n is even, then $f(z)=$perm $(Iz-A)$ has no real roots; if n is odd, then $f(z)=$perm $(Iz-A)$ has one and only one real root.

In this paper we give counter examples to this conjecture.

Results:

EXAMPLE 1. Let

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

$f(z) =$ perm $(Iz - A)$ is such that $f(0) < 0$ and $f(1) > 0$. Consider $f(z)\cdot(z-1) = g(z)$. Note that $g(0) > 0$ and since there is a $\xi(0 < \xi < 1)$ for which $f(\xi) > 0$ we see that $g(\xi) < 0$. Now consider

$$B(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} - \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 1-\varepsilon \end{bmatrix}.$$

If $0 \leqq \varepsilon \leqq \frac{3}{4}$, $B(\varepsilon)$ is doubly stochastic. Further if $g_\varepsilon(z) =$ perm $[Iz - B(\varepsilon)]$ then for each z, $g(z) = \lim_{\varepsilon\to 0} g_\varepsilon(z)$. Since $g_\varepsilon(0) > 0$ for each ε and $g(\xi) = \lim_{\varepsilon\to 0} g_\varepsilon(\xi) < 0$ we see that for sufficiently small ε, say ε_0, $g_{\varepsilon_0}(z)$ has a real root and $B(\varepsilon_0)$ is irreducible. This yields the counter-example. Note also that $g_{\varepsilon_0}(z) > 0$ for $z > 1$ [see 1], hence $g_{\varepsilon_0}(z)$ has at least two real roots.

EXAMPLE 2. For simplification let $B(\varepsilon_0) = B$ and $g_{\varepsilon_0}(z) = g(z)$. Recall

(a) $g(0) > 0$ and
(b) $g(\xi) < 0$. By direct calculation we see that
(c) $g(1) > 0$ and hence for some η, $\xi < \eta < 1$
(d) $g(\eta) > 0$.

Now consider $f(z) = g(z)\cdot(z-1)$. Note that

(a) $f(0) < 0$
(b) $f(\xi) > 0$

(c) $f(1) = 0$
(d) $f(\eta) < 0$.

Consider

$$A(\varepsilon) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} - \varepsilon_0 & \varepsilon_0 & 0 \\ 0 & 0 & \varepsilon_0 & 1 - \varepsilon_0 - \varepsilon & \varepsilon \\ 0 & 0 & 0 & \varepsilon & 1 - \varepsilon \end{bmatrix}$$

where $0 < \varepsilon < 1 - \varepsilon_0$.
Let $f_\varepsilon(z) = \text{perm}\,[Iz - A(\varepsilon)]$. Note that for each z, $\lim_{\varepsilon \to 0} f_\varepsilon(z) = f(z)$. Therefore for ε sufficiently small, say ε_1

(a) $f_{\varepsilon_1}(0) < 0$
(b) $f_{\varepsilon_1}(\xi) > 0$
(c) $f_{\varepsilon_1}(\eta) < 0$
(d) $f_{\varepsilon_1}(z) > 0$ for $z > 1$. Further $A(\varepsilon_1)$ is doubly stochastic and irreducible. Hence $f_{\varepsilon_1}(z)$ has at least three real roots. This yields a counter-example to the conjecture in the case n is odd.

References

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