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ON THE LIMITING DISTRIBUTION OF ADDITIVE FUNCTIONS (MOD 1)

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A function $f(n)$, defined on the positive rational integers, is said to be additive if and only if for every pair of coprime integers a and b the relation

$$f(ab) = f(a) + f(b)$$

is satisfied. Thus an additive function is determined by its values on those integers which are prime powers. In an extensive paper Erdos raised the question of characterising those real valued additive functions which have a limiting distribution (mod 1).

It is our present purpose to give such a characterisation.

He proved, in particular, that an additive function $f(n)$ is certainly uniformly distributed in the sense of Weyl if $f(p) \to 0$ as $p \to \infty$, and if the series

$$\sum \frac{f^2(p)}{p}$$

diverges.

For the remainder of this paper we understand a *distribution function* $F(z)$ (mod 1), or more shortly *a distribution function*, to have the properties

(i) *$F(z)$ is increasing in the wide sense*

(ii) *$F(z) = F(z+)$ for all values of z, that is $F(z)$ is right continuous.*

(iii) *$F(z) = 0$ if $z < 0$, and $= 1$ if $z \geqq 1$.*

We say that a sequence of distribution functions $F_n(z)$, $n = 1, 2, \cdots$ has a *limiting distribution* (*mod* 1) if and only if there exists a function $F(z)$, satisfying the above three conditions, so that at every pair of points of continuity (α, β) of $F(z)$, $0 < \alpha < \beta < 1$, we have

$$F_n(\beta) - F_n(\alpha) \to (F(\beta) - F(\alpha)) , \qquad (n \to \infty) .$$

We notice that in the range $0 < z < 1$ any such limiting distribution $F(z)$ is determined only up to an additive constant. When the function $F(z)$ is

$$F(z) = \begin{cases} 1, z \geqq 1 \ , \\ z, 0 < z < 1 \ , \\ 0, z \leqq 0 \ , \end{cases}$$

this definition coincides with Weyl's definition [7] of uniform distribution (mod 1).

We shall say that the sequence of real numbers $x_1, x_2, \cdots$ has a limiting distribution (mod 1) if and only if the sequence of distribution functions defined by

$$F_n(z) = n^{-1} \sum_{\substack{j=1 \\ x_j \leqq z \ (\mathrm{mod}\ 1)}}^{n} 1 \ , \qquad n = 1, 2, \cdots$$

for $0 \leqq z < 1$, and extended in the obvious way outside this interval, have a limiting distribution in the above sense.

In what follows, for each real number α we denote by $\{\alpha\}$ the fractional part of α, that is the least positive representative of the residue class $\alpha(\mathrm{mod}\ 1)$; and by $||\alpha||$ the distance of α from the nearest integer. Thus we have

$$||\alpha|| = \min(\{\alpha\}, 1 - \{\alpha\}) \ .$$

we shall also have occasion to use the function

$$\mathrm{Sign}\, y = \begin{cases} 1 \text{ if } y > 0 \ , \\ 0 \text{ if } y = 0 \ , \\ -1 \text{ if } y < 0 \ . \end{cases}$$

With these definitions, and the above meaning of limiting distribution, we can now state:

THEOREM 1. *A real valued additive number theoretic function $f(n)$ has a limiting distribution (mod 1) if and only if for each integer ν one of the following three conditions is satisfied:*

(i) *For each real value of t the series*

$$\sum_p p^{-1} || f(p) - t \log p ||^2$$

is divergent.

(ii) *For each positive integer r, $\nu f(2^r)$ is half an odd rational integer.*

(iii) *Both of the series*

$$\sum_p p^{-1} || \nu f(p) ||^2, \ \sum_p p^{-1} || \nu f(p) || \ \mathrm{Sign}\, (\tfrac{1}{2} - \{\nu f(p)\})$$

are convergent.

In particular $f(n)$ is uniformly distributed (mod 1) if and only if at least one of the first two conditions is satisfied for each integer.

THEOREM 2. *For each integer* ν *set*

$$\varepsilon_\nu = |(1 + 2^{-1}e^{2\pi i\nu f(2)} + 2^{-2}e^{2\pi i\nu f(2^2)} + \cdots)| .$$

Then a limiting distribution (mod 1) for the function $f(n)$ *is*

(a) *Continuous if and only if*

$$N^{-1}\sum_{\nu\leqq N}\varepsilon_\nu \exp(-2\sum_p \sin^2\pi\nu f(p)) \to 0 , \qquad (N\to\infty) .$$

(b) *Absolutely continuous with a derivative that belongs to the Lebesgue class* $L^2[0, 1]$ *if and only if the series*

$$\sum_{\nu=-\infty}^{\infty}\varepsilon_\nu^2 \exp(-4\sum_p \sin^2\pi\nu f(p))$$

is convergent.

In the statement of this theorem is to be understood that if a series

$$\sum_p p^{-1}\sin^2\pi\nu f(p)$$

diverges, then the corresponding number

$$\exp(-2\sum_p\cdots)$$

si defined to be zero.

We note that in either of the circumstances (a) or (b) of Theorem 2 we can assert that there exists a distribution $F(z)$ so that

$$n^{-1}\sum_{f(n)\leqq z} 1 \to F(z) , \qquad (h\to\infty) ,$$

holds for every real value of z.

For the proofs of these theorems we need essentially two lemmas. Before stating the first of these we discuss some results of Halász [4].

A number theoretic function $g(n)$ is said to be multiplicative if for every pair of coprime integers a, b, the relation

$$g(ab) = g(a)g(b)$$

is satisfied. In his paper of 1968 Halász gives necessary and sufficient criteria that multiplicative functions of wide classes have mean-value theorems of the type

$$\lim_{n\to\infty} n^{-1} \sum_{m\leqq n} g(m) \quad \textit{exists} .$$

It is convenient to restate some of his results. We shall adopt for the moment the notation of his paper [4], save that in place of $f(n)$ we set $g(n)$, $n = 1, \cdots$. For a fixed value of $P(\geqq 3)$ we define a multiplicative function $g^*(n)$ by

$$g^*(p^k) = \begin{cases} 0 & \text{for } p \leqq P, k = 1, 2, \cdots \\ (g(p))^k & \text{for } p > P, k = 1, 2, \cdots . \end{cases}$$

We note here that no essential use is made of the size of P during any of the proofs of the theorems in Halász' paper, it being a parameter introduced as a technical convenience to ensure the non-vanishing of certain products (see pp. 369–370 of [4]). We shall also need the function

$$\lambda(n) = \frac{1}{k}\begin{cases} \text{if } n = p^k, p \textit{ prime}, k = 1, 2, \cdots \\ 0 \textit{ otherwise.} \end{cases}$$

If now $g(n)$ is assumed to satisfy the inequality $|g(n)| \leqq 1$ for every integer n, then as Theorem 2 of his paper Halász proves that

$$x^{-1} \sum_{m\leqq x} g(m) = C_0 \frac{H(1 + ia_0)}{1 + ia_0} L_0(\log x) x^{1+ia_0} + o(x) , \qquad (x \to \infty) ,$$

with the following understanding:

If for every value of t the series

$$\sum_{n=1}^{\infty} n^{-1} \lambda(n)(1 - Re\ g^*(n)n^{-it})$$

diverges ([4] p. 380), or if

$$(1 + g(2)2^{-1} + g(2^2)2^{-2} + \cdots) = 0$$

([4] p. 369) then $C_0 L_0(\log x)$ is to be replaced by zero.

On the other hand, if for some values of t (which is in fact unique) the above series converges, then we set $a_0 = t$, and have ([4] p. 382),

$$C_0 = \exp\Big(- \sum_{n=1}^{\infty} n^{-1} \lambda(x)(1 - Re\ g^*(n)n^{-it})\Big) .$$

The function $L_0(\log x)$ is defined by

$$L_0\Big(\frac{1}{\sigma - 1}\Big) = \exp\Big(i \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}} \operatorname{Im}(g^*(n)n^{-it})\Big) , \qquad \sigma > 1 ,$$

so that as $\sigma \to 1+$;

$$C_0 L_0\left(\frac{1}{\sigma - 1}\right) \sim \exp\left(- \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^{\sigma}} (1 - g^*(n) n^{-it})\right) .$$

Here $H(s)$ is the function defined for complex numbers s by

$$H(s) = \prod_{p} \left(1 + \frac{g(p)}{p^s} + \frac{g(p^2)}{p^{2s}} + \cdots\right) \prod_{p>P} \left(1 - \frac{g(p)}{p^s}\right) ,$$

which is absolutely convergent for $\sigma = \text{Re } s \geqq 1$.

Finally, we need the fact, also proved in [4], that $L_0(u)$ is a slowly oscillating function. In other words, $|L_0(u)| = 1$ for all values of $u > 0$, and

$$\frac{L_0(y)}{L_0(u)} \longrightarrow 1$$

holds uniformly for $u < y \leqq 2u$, as $u \longrightarrow \infty$.

We can now state our first lemma.

LEMMA 1. *Let $g(n)$ be a complex valued multiplicative number theoretical function which satisfies*

$$|g(n)| \leqq 1 , \qquad (n = 1, 2, \cdots) .$$

Then

$$\lim_{n\to\infty} n^{-1} \sum_{m \leqq n} g(m) = C$$

exists under the following circumstances:

(i) *with* $C = 0$:
Either

$$-1 = g(2) = g(2^2) = \cdots$$

Or, the series

$$\sum_{p} p^{-1} (1 - \text{Re } g(p) p^{-it})$$

diverges for each real value of t.

(ii) *with* $C \neq 0$:
The series

$$\sum_{p} p^{-1} (g(p) - 1)$$

converges.

The second of these two assertions was first proved by Delange [1]. The first assertion was proved for real valued functions, in particular, by Wirsing [8], and in its full generality by Halász [4].

Proof. If for any positive integer r, Re $g(2^r) > -1$, then

$$\mathrm{Re}\Big(1 + \sum_{m=1}^{\infty} 2^{-m} g(2^m)\Big) > 1 - \sum_{m=1}^{\infty} 2^{-m} = 0 ,$$

so that in our present circumstances the series

$$1 + \sum_{m=1}^{\infty} 2^{-m} g(2^m)$$

can vanish only if Im $g(2^r) = 0$, $(r = 1, 2, \cdots)$. The first assertion now follows from the remarks concerning Halász' paper [4] which were made preeceding the statement of Lemma 1, provided we note that uniformly for all integers $N > P$,

$$\Big| \sum_{p<n\leqq N} n^{-1} \lambda(n)(1 - \mathrm{Re}\ g^*(n)n^{-it}) - \sum_{p<p\leqq N} p^{-1}(1 - \mathrm{Re}\ g(p)p^{-it}) \Big|$$
$$\leqq \sum_{p} \sum_{k=2}^{\infty} \frac{2}{kp^k} \leqq \sum_{p} \frac{1}{p(p-1)} < \infty .$$

In order to prove the second assertion we note that if the nonzero mean-value exists then, (in the notation of the earlier remarks), $C_0 \neq 0$, so that for some value of t the series

$$\sum_{n=1}^{\infty} n^{-1} \lambda(n)(1 - \mathrm{Re}\ g^*(n)n^{-it})$$

converges. Moreover, as $x \to \infty$,

$$L_0(\log x)x^{it} \to \mathrm{A} \neq 0 , \tag{1}$$

say.

We next note that we can find an unbounded sequence of positive real numbers $z_1, z_2, \cdots$ so that $z_n^{it} \to 1$ as $n \to \infty$. For, given any positive real number ε we can apply Dirichlet's theorem on Diophantine approximation (see for example Hardy and Wright [5] pp. 156–157) to deduce that there exists a nonzero integer m so that

$$\left\| \frac{mt}{2\pi} \right\| < \varepsilon .$$

Setting $z = e^n$ we see that

$$|z^{it} - 1| = |\exp(imt) - 1| \leqq 2\pi\varepsilon \exp(\varepsilon) .$$

If $t/2\pi$ is irrational our assertion is justified by choosing a sequence of ε converging to zero. It $t/2\pi$ is rational it is clear that we can even choose a sequence $z_1, z_2, \cdots$ so that $z_n^{it} = 1$ holds for all members of the sequence.

It follows that

$$L_0(\log z_n) \to A , \qquad n \to \infty .$$

Suppose now that $t \neq 0$. Then because of the slowly-oscillating nature of the function $L_0(n)$,

$$L_0(\log(z_n \exp(\pi t^{-1}))) \to A , \qquad (n \to \infty) ,$$

and therefore from (1)

$$(z_n \exp(\pi t^{-1}))^{it} \to 1 , \qquad (n \to \infty) .$$

Since by the construction of the z_n the left-hand side converges to -1, we obtain a contradiction. It follows that $t = 0$, and that

$$\lim_{\sigma \to 1+} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^\sigma} (1 - g^*(n))$$

exists, and is finite. By a standard Tauberian theorem of Hardy and Littlewood we deduce that the series

$$\sum_{n=1}^{\infty} n^{-1} \lambda(n)(1 - g^*(n)) \text{ and } \sum_p p^{-1}(1 - g(p))$$

converge.

That these conditions are indeed sufficient follows from Theorem 1 of Halász [4].

This completes the proof of Lemma 1.

LEMMA 2. *A sequence of distribution functions $F_n(z)$ (mod 1) $n = 1, 2, \cdots$ has a limiting distribution (mod 1) if and only if for each integer ν*

$$\alpha_\nu = \lim_{n \to \infty} \int_0^1 e^{2\pi i \nu z} \, dF_n(z)$$

exists.

Moreover, the limiting distribution, if it exists, is continuous if and only if

$$N^{-1} \sum_{|\nu| \leqq N} |\alpha_\nu| \to 0 , \qquad (N \to \infty) ,$$

and absolutely continuous with a derivative which belongs to the class $L^2[0, 1]$ if and only if the series

$$\sum_{\nu=-\infty}^{\infty} |\alpha_\nu|^2$$

converges.

Proof. The results of this lemma are well known to workers in the field. A proof of the main assertion can be sketched briefly as follows:

The necessity of the condition is clear from integration by parts and an application of Lebesgue's theorem of dominated convergence.

For sufficiency, we note that the sequence $\alpha_1, \alpha_2, \cdots$ satisfies

$$\sum_{u=1}^{m}\sum_{v=1}^{m} \alpha_{u-v} z_u \bar{z}_v = \lim_{n\to\infty}\int_0^1 \left|\sum_{u=1}^{m} z_n e^{2\pi i u x}\right|^2 dF_n(x) \geqq 0$$

for all integers m and complex numbers $z_1, \cdots, z_m$. In the classical terminology it is positive definite. Then by a theorem of Herglotz [6] there is a Borel measure μ on $[0, 1]$, and so a corresponding distribution function $F(x) = \mu[0, x]$, so that

$$\alpha_\nu = \int_0^1 e^{2\pi i \nu x}\, dF(x) , \qquad (\nu = 0, \pm 1, \pm 2, \cdots) .$$

If now α and β satisfy $0 < \alpha < \beta < 1$, then by the stone-Weierstrass theorem the characteristic function of the interval $(\alpha, \beta]$ can be uniformly approximated on the unit interval $0 \leqq x < 1$ by polynomials in $\exp(2\pi i x)$. If α and β are points of continuity of $F(x)$ it follows easily from the monotonicity of distribution functions that

$$F_n(\alpha) - F_n(\alpha) \to (F(\beta) - F(\alpha)) , \qquad (n \to \infty) .$$

The second and third results of the lemma are special cases of results from the theory of Fourier series. Both can be found for example, in Edwards [2]. In its present form the assertion concerning the possible continuity of a limiting distribution is due to Wiener.

Proof of Theorem 1. It is clear from Lemma 2 that the distributions

$$F_n(x) = n^{-1} \sum_{\substack{m \leqq n \\ f(m) \leqq x \ (\mathrm{mod}\ 1)}} 1$$

have a limiting distribution (mod 1) if and only if the limits

$$\lim_{n\to\infty} n^{-1} \sum_{m=1}^{n} e^{\pi i f(m)\nu} = \lim_{n\to\infty}\int_0^1 e^{2\pi i \nu x}\, dF_n(x) , \quad (\nu = 0, \pm 1, \pm 2, \cdots)$$

exists. We can then apply Lemma 1 to deduce that for each integer ν one of the following three conditions is to be satisfied:

(i) For each value of t the series

$$\sum_p p^{-1}(1 - \mathrm{Re}(e^{2\pi i \nu f(p)} p^{-it}))$$

diverges

(ii) $-1 = e^{2\pi i f(2)\nu} = e^{2\pi i f(2^2)\nu} = \cdots$

(iii) The series

$$\sum_p p^{-1}(1 - e^{2\pi i\nu f(p)})$$

converges.

Of these conditions only the first and third call for comment.

Since for each real number y

$$\mathrm{Re}(1 - e^{2\pi iy}) = -2\sin^2\pi y = -2(\sin\pi\,\|y\|)^2$$

and

$$2y/\pi \leqq \sin y \leqq y \text{ if } 0 \leqq y \leqq \pi/2 ,$$

the first condition is equivalent to the series

$$\sum p^{-1}\,\|{}^{\nu}f(p) - t\log p\|^2$$

being divergent for each value of t.

Likewise, in (iii) the series

$$\sum_p p^{-1}(1 - \mathrm{Re}(e^{2\pi i\nu f(p)})) \text{ and } \sum_p p^{-1}\,\|\nu f(p)\|^2$$

converge and diverge together. Moreover, for each real number y

$$|\mathrm{Sin}\, y - y| \leqq \frac{|y|^3}{6}$$

so that

$$\begin{aligned}\mathrm{Sin}\, 2\pi\nu f(p) &= \mathrm{Sin}\, 2\pi\{\nu f(p)\} = \mathrm{Sin}\, 2\pi\,\|\nu f(p)\|.\ \ \mathrm{Sign}(\tfrac{1}{2} - \{\nu f(p)\}) \\ &= \mathrm{Sign}(\tfrac{1}{2} - \{\nu f(p)\})\,(2\pi\,\|\nu f(p)\| + 0(\|\nu f(p)\|^3))\end{aligned}$$

and uniformly for all $P > 0$

$$\begin{aligned}&\Big|\sum_{p\leqq P} p^{-1}\,\mathrm{Sin}\, 2\pi\nu f(p) - 2\pi\sum_{p\leqq P}\mathrm{Sign}(\tfrac{1}{2} - \{\nu f(p)\})p^{-1}\,\|\nu f(p)\|\Big| \\ &\quad\leqq \text{constant}\sum_p p^{-1}\,\|\nu f(p)\|^2 .\end{aligned}$$

It is now clear from the previous remark that the series

$$\sum p^{-1}(1 - e^{2\pi i\nu f(p)})$$

and the pair of series

$$\sum_p p^{-1}\,\|\nu f(p)\|\,\mathrm{Sign}(\tfrac{1}{2} - \{\nu f(p)\}),\ \sum p^{-1}\,\|\nu f(p)\|^2$$

converge and diverge together.

This completes the proof of Theorem 1.

Proof of Theorem 2. To prove Theorem 2 we prove that uniformly for all integers ν, $|\alpha_\nu|$ lies between two positive constant multiples of

$$\varepsilon_\nu \exp\Bigl(-2\sum_p p^{-1}\operatorname{Sin}^2\pi\nu f(p)\Bigr)\,.$$

We note from the remarks preceeding Lemma 1 that if $\alpha_\nu \neq 0$ then it has the form

$$\alpha_\nu = \frac{H(1+ia_0)}{1+ia_0}\exp\Bigl(-\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}(1-g^*(n)n^{-ia_0})\Bigr)$$

where

$$\begin{aligned} &H(i+ia_0)\\ &\quad= \prod_{p>P}\Bigl(1+\frac{g(p)}{p}+\frac{g(p^2)}{p^2}+\cdots\Bigr)\Bigl(1-\frac{g(p)}{p}\Bigr)\prod_{p\leqq P}\Bigl(1+\frac{g(p)}{p}+\cdots\Bigr)\\ &\quad= \prod_{p\leqq P}\Bigl(1+\frac{g(p)}{p}+\cdots\Bigr)\prod_{p>P}\Bigl(1+\frac{f(p^2)-(f(p))}{p^2}\\ &\qquad+\frac{f(p^3)-f(p)f(p^2)}{p^3}+\cdots\Bigr)\,. \end{aligned}$$

It is clear that since $g(2^r)\neq -1$, for every integer r (since α_ν is nonzero),

$$c_1 \leqq |H(1+ia_0)\varepsilon_\nu^{-1}(1+ia_0)^{-1}| \leqq c_2$$

for suitable positive constants c_1, c_2 depending at most upon $f(n)$. Moreover,

$$\Bigl|\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}(1-g^*(n)n^{-it}) - \sum_p \frac{1}{p}(1-g^*(p)p^{-it})\Bigr| \leqq 1\,,$$

and from these two facts the desired inequalities (2) follow.

If $\alpha_\nu = 0$ then either $\varepsilon_\nu = 0$, or $\varepsilon_\nu \neq 0$ but

$$\sum_p p^{-1}\operatorname{Sin}^2\pi\nu f(p)$$

diverges, so that with our earlier convention

$$\exp(-\sum_p p^{-1}\operatorname{Sin}^2\pi\nu f(p)) = 0\,,$$

and the inequalities (2) are still valid.

Theorem 2 now follows immediately from Lemma 2.

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