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BANACH ALGEBRAS WHICH ARE IDEALS IN A BANACH ALGEBRA

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In this paper Banach algebras A which are ideals in a Banach algebra B are studied. The main results concern the relationship between the norms of A and B and the relationship between the closed ideals of A and B.

There are many examples of Banach algebras in analysis which are ideals in another Banach algebra. When G is a locally compact group, then the Segal algebras which are studied in H. Reiter's book [7] are ideals in $L^1(G)$. J. Cigler considers more general Banach algebras which are ideals in $L^1(G)$ in [2]. In the theory of operators on a Hilbert space $\mathscr{H}$, the C_p algebras discussed in [4, pp. 1088–1119] are ideals in the algebra of compact operators on $\mathscr{H}$ (C_1 is the ideal of trace class operators and C_2 the ideal of Hilbert-Schmidt operators). Also as we point out in § 4, every full Hilbert algebra is a dense *-ideal in a B^*-algebra.

When A is a Banach algebra which is an ideal in a Banach algebra B, we consider the relationship between the algebras A and B. First we prove that the norms of A and B are related by certain inequalities. As a consequence, if B is semi-simple, then A is a left and right Banach module of B [Theorem 2.3]. Also in this case our results show that A is an abstract Segal algebra with respect to B as defined by J. T. Burnham in [1]. Secondly we relate the closed left and right ideals of A to those of B. Of special interest here is the case where A contains a bounded approximate identity of B [Theorem 3.4]. Finally in § 4 we consider the special case where A is a *-ideal in a B^*-algebra B. The results of this section apply to full Hilbert algebras.

1. **Preliminaries and notation.** When B is any Banach algebra, we denote the Banach algebra norm on B by $\|\cdot\|_B$. If M is a closed left ideal in the Banach algebra B, $B - M = \{b + M \mid b \in B\}$ is the quotient module B modulo M. $B - M$ is normed by the norm

$$\|b + M\|'_B = \inf\{\|b - m\|_B \mid m \in M\} .$$

Throughout this paper A is a given Banach algebra. We always use the term "ideal" to mean two-sided ideal. A is usually an ideal in a Banach algebra B. In this case when E is a subset of A, $\mathrm{cl}(E)$ is the closure of E in B.

At this point we prove a proposition of a purely algebraic nature which is useful in what follows.

PROPOSITION 1.1. *Assume that R is a ring and I is an ideal of R. Assume that M is a modular maximal left* [right] *ideal of R such that $I \not\subset M$. Then*

(1) *I acts strictly irreducibly on $R - M$, and*

(2) *$I \cap M$ is a maximal modular left* [right] *ideal of I.*

Proof. We prove (1) first. Assume $u \in R$ *has the property* $R(1-u) \subset M$. Let $K = \{b \in R \mid Ib \subset M\}$. K is a left ideal of R, $M \subset K$, and $u \notin K$ (if $u \in K$, $I \subset M$, a contradiction). Therefore $K = M$. It follows by the definition of K, that when $b \notin M$, $Ib + M$ properly contains M, and therefore $Ib + M = R$. This suffices to prove (1).

Now consider $I \cap M$. If $a \in I$ and $au \in M$, then $a \in I \cap M$. Therefore $I \cap M = \{a \in I \mid a(u + M) = 0 + M\}$. By (1) we can choose $v \in I$ such that $v(u + M) = u + M$. Then $I(1 - v) \subset I \cap M$ by the characterization of $I \cap M$ given above. Assume that $a \in I$, $a \notin I \cap M$. Given $b \in I$ we can choose $c \in I$ such that $b - ca \in M$ by (1). Then $b = ca + (b - ca) \in Ia + I \cap M$. Therefore $I = Ia + I \cap M$. Which proves (2).

2. **The basic norm inequalities.** In this section we assume that A is a subalgebra of a Banach algebra B. There is a close connection between certain inequalities involving $\|\cdot\|_A$ and $\|\cdot\|_B$ and the algebraic property that A is an ideal in some closed subalgebra of B. The next proposition has been noted by other authors.

PROPOSITION 2.1. *Assume that*

(1) *there exists $D > 0$ such that $D\|a\|_A \geqq \|a\|_B$ for all $a \in A$, and*

(2) *there exists $C > 0$ such that $\|ab\|_A \leqq C \max\{\|a\|_A \|b\|_B, \|a\|_B \|b\|_A\}$ for all a, $b \in A$.*

Then A is an ideal in $\mathrm{cl}(A)$.

Proof. Assume that $a \in A$ and $b \in \mathrm{cl}(A)$ are given. Choose $\{b_n\} \subset A$ such that $\|b_n - b\|_B \to 0$. Then $\|ab_n - ab_m\|_A \leqq C\|a\|_A \|b_n - b_m\|_B$, so that $\{ab_n\}$ is Cauchy in A. Then there exists $c \in A$ such that $\|ab_n - c\|_A \to 0$. By (1) $\|ab_n - c\|_B \to 0$, and since $\|ab_n - ab\|_B \to 0$, we have $ab = c$. This proves that A is a right ideal of B. The proof that A is a left ideal of B is similar.

Together the next two results establish a converse to Proposition 2.1.

PROPOSITION 2.2. *Assume that A is a dense ideal in a semi-simple Banach algebra B. Then there exists $D > 0$ such that $D\|a\|_A \geqq \|a\|_B$ for all $a \in A$.*

Proof. We prove that the embedding $(A, \|\cdot\|_A) \to (B, \|\cdot\|_B)$ is a closed, and hence continuous, map. Assume that $\{a_n\} \subset A$, $b \in B$, $\|a_n\|_A \to 0$ and $\|a_n - b\|_B \to 0$. Let M be a modular maximal left ideal of B with $A \not\subset M$, and let $u \in B$ have the property that $B(1-u) \subset M$. Given $a \in A$, let T_a act on $B - M$ by $T_a(b + M) = ab + M$. By Proposition 1.1 (1), $a \to T_a$ is a stricly irreducible representation of A on $B - M$. Let P be the kernel of this representation. P is a primitive ideal of A, and therefore P is closed in A. A/P is a Banach algebra with norm $\|a + P\|'_A$, $a \in A$. Given $a \in A$, define $S_{a+P}(b+M) = ab + M$, $b \in M$. Then $a + P \to S_{a+P}$ is a faithful strictly irreducible representation of A/P into the bounded operators on $B - M$. Then a theorem of B. E. Johnson [6, Theorem 1, p. 537] implies that $a + p \to S_{a+P}$ is a continuous map. Since $\|a_n + P\|'_A \to 0$, then $\|a_n u + M\|'_B = \|S_{a_n+P}(u + M)\|'_B \to 0$. Also $\|(a_n - b)(u + M)\|'_B \to 0$. It follows that $bu + M = 0$, and thus $b = bu + (b - bu) \in M$. Then b must be in every modular maximal left ideal of B, so that by the semi-simplicity of B, $b = 0$.

THEOREM 2.3. *Assume that A is an ideal in a Banach algebra B. Assume that there exists $D > 0$ such that $D\|a\|_A \geqq \|a\|_B$ for all $a \in A$. Then there exists $C > 0$ such that*

(1) $\|ab\|_A \leqq C\|a\|_A \|b\|_B$ *for all* $a \in A$, $b \in B$, *and*

(2) $\|ab\|_A \leqq C\|a\|_B \|b\|_A$ *for all* $a \in B$, $b \in A$.

Proof. We prove only (1). Let L_a, $a \in A$ be the operator mapping B into A given by $L_a(b) = ab$, $b \in B$. We prove that L_a is continuous by showing that L_a is a closed map from B into A. Assume that $\{b_n\} \subset B$, $c \in A$, and $\|b_n\|_B \to 0$, $\|L_a(b_n) - c\|_A \to 0$. Then $\|ab_n - c\|_A \to 0$, and since the A-norm dominates the B-norm, $\|ab_n - c\|_B \to 0$. Also $\|ab_n\|_B \to 0$, and therefore $c = 0$.

Now since L_a is continuous, for each $a \in A$ there exists $M_a > 0$ such that $\|ab\|_A \leqq M_a\|b\|_B$, $b \in B$. Given $b \in B$, let R_b be the operator mapping A into A defined by $R_b(a) = ab$, $a \in A$. We prove that R_b is a closed, and hence continuous, map from A to A. Assume that $\{a_n\} \subset A$, $c \in A$, $\|a_n\|_A \to 0$, and $\|R_b(a_n) - c\|_A \to 0$. Then $\|a_n\|_B \to 0$ and $\|a_n b - c\|_B \to 0$. Thus $c = 0$. Therefore for each $b \in B$, R_b is a continuous operator. Set $|R_b| = \sup\{\|R_b(a)\|_A \mid a \in A, \|a\|_A \leqq 1\}$. Let $\mathscr{S} = \{R_b \mid b \in B, \|b\|_B \leqq 1\}$. We have that $\|R_b(a)\|_B \leqq M_a$ for each $a \in A$ and $R_b \in \mathscr{S}$. Then by the Uniform Boundedness Theorem there exists $C > 0$ such that $|R_b| \leqq C$ for all $R_b \in \mathscr{S}$. Thus

$$\frac{\|R_b(a)\|_A}{\|a\|_A} \leqq C$$

for all $a \in A$, $a \neq 0$, and all $b \in B$, $\|b\|_B \leqq 1$. Finally it follows that

$$\|ab\|_A \leqq C\|a\|_A \|b\|_B$$

for all $a \in A$ and $b \in B$.

We remark that if A satisfies the hypotheses of Theorem 2.3, then by (1) and (2) A is a left and right Banach module over B; see [5, Definition (32.14), p. 263].

3. Closed left and right ideals. Now assuming that A is an ideal of B, we relate the closed left and right ideals of A to those of B. The most comprehensive results in this direction are obtained when A has an approximate identity. However in the general case we do have the following theorem concerning modular closed left and right ideals of A.

THEOREM 3.1. *Assume that A is a dense ideal of a Banach algebra B and that there exists $D > 0$ such that $D\|a\|_A \geqq \|a\|_B$ for all $a \in A$. Let M be a closed modular left* [right] *ideal of A. Then $M = A \cap \mathrm{cl}(M)$.*

Proof. By Theorem 2.3 there exists $C > 0$ such that $\|ab\|_A \leqq C\|a\|_B\|b\|_A$ for all $a, b \in A$. Assume that M is a closed modular left ideal of A. Then there exists $u \in A$ such that $A(1-u) \subset M$. Given $a \in A$, $a = au + (a - au)$ and $a - au \in M$. Therefore $\|a+M\|'_A = \|au+M\|'_A$. Also $\|au + M\|'_A \leqq \|au - bu\|_A$ for any $b \in M$ (note that when $b \in M$, then $bu \in M$). Therefore for all $b \in M$,

$$\|a + M\|'_A \leqq \|au - bu\|_A \leqq C\|a - b\|_B \|u\|_A .$$

Then $\|a + M\|'_A \leqq (C\|u\|_A) \|a + M\|'_B$.

Assume that $a \in A \cap \mathrm{cl}(M)$. Choose $\{a_n\} \subset M$ such that $\|a_n - a\|_B \to 0$. Then $\|(a_n - a) + M\|'_B \to 0$, and therefore $\|(a_n - a) + M\|'_A \to 0$. Thus there exists $\{b_n\} \subset M$ such that $\|(a_n - a) - b_n\|_A \to 0$. Since $\{a_n - b_n\} \subset M$, we have $a \in M$. Thus $A \cap \mathrm{cl}(M) \subset M$. The opposite inclusion is immediate, so that $M = A \cap \mathrm{cl}(M)$.

The next theorem provides a sufficient condition on A that every closed left [right] ideal of A is the intersection of A with a closed left [right] ideal of B. This theorem is proved by J. T. Burnham in [1, Theorem 1.1] (Theorem 2.3 removes one of Burnham's hypotheses).

THEOREM 3.2. *Assume that A is a dense ideal of B with the* *property that there exists* $D > 0$ *such that* $D\|a\|_A \geqq \|a\|_B$ *for all* $a \in A$. *Furthermore assume that for all* $a \in A$, $a \in \overline{Aa}$ $[a \in \overline{aA}]$ *where* "—" *denotes closure in* A. *Then*

(1) *if* N *is a closed left* [right] *ideal of* B, *then* $N \cap A$ *is a closed left* [right] *ideal of* A, *and*

(2) *if* M *is a closed left* [right] *ideal of* A, *then* $M = A \cap \mathrm{cl}(M)$.

In many of the examples in harmonic analysis A is an ideal in $L^1(G)$ which contains a bounded approximate identity of $L^1(G)$. We prove that under these circumstances A has an approximate identity.

PROPOSITION 3.3. *Assume that* A *is a dense ideal in a Banach algebra* B *and that there exists* $D > 0$ *such that* $D\|a\|_A \geqq \|a\|_B$ *for all* $a \in A$. *Then if* $\{e_\alpha\}$ *is a left* [right] *bounded approximate identity for* B *and* $\{e_\alpha\} \subset A$, $\{e_\alpha\}$ *is a left* [right] *approximate identity for* A.

Proof. By Theorem 2.3 A is a left Banach module of B. Therefore by Cohen's Theorem [5, Theorem (32.22), pp. [268] given $a \in A$ there exists $b \in B$ and $c \in A$ such that $a = bc$. Then

$$\|bc - e_\alpha bc\|_A \leqq C\|b - e_\alpha b\|_B \|c\|_A \to 0 .$$

Therefore $\{e_\alpha\}$ is a left approximate identity for A.

Combining several previous results, we have the following theorem which applies to many interesting examples in harmonic analysis.

THEOREM 3.4. *Assume that* A *is a dense ideal in a semi-simple Banach algebra* B. *Assume that* A *contains a bounded approximate identity of* B. *Then*

(1) *for every closed left* [right] *ideal* M *of* A, $M = A \cap \mathrm{cl}(M)$, *and*

(2) *if* B *has the property that every proper closed left* [right] *ideal of* B *is contained in a modular maximal left* [right] *ideal of* B, *then* A *has the property that every proper closed left* [right] *ideal of* A *is contained in a modular maximal left* [right] *ideal of* A.

Proof. (1) follows from Proposition 2.2, Proposition 3.3, and Theorem 3.2. Then (1) and Proposition 1.1 imply (2).

4. *-ideals in a B-*algebra. Assume that A is a full Hilbert algebra; see [9]. Then A is a pre-Hilbert space with the corresponding (linear) norm $\|\cdot\|_2$ on A. Also given $a \in A$, the operator U_a defined by $U_a(b) = ab$ for $b \in A$ is a bounded operator on $(A, \|\cdot\|_2)$. For $a \in A$ left $|a|$ denote the operator bound of U_a. Then $|\cdot|$ is an algebra norm on A with the B^*-property. Let $\|a\|_A = \|a\|_2 + |a|$. M. Rieffel proves that $\|\cdot\|_A$ is a complete algebra norm on A [9, Proposition 1.15, p. 270]. Certainly $\|a\|_A \geqq |a|$ for all $a \in A$. Also for all $a, b \in A$,

$$\begin{aligned} \|ab\|_A &= \|ab\|_2 + |ab| \\ &\leqq |a|\,\|b\|_2 + |a|\,|b| \\ &\leqq |a|\,\|b\|_A . \end{aligned}$$

Similarly $\|ab\|_A \leqq \|a\|_A |b|$ for all $a, b \in A$. Let B be the completion of A in the norm $|\cdot|$. B is a B^*-algebra and A is a *-subalgebra of B. Then by Proposition 2.1 A is a *-ideal in B. Therefore every full Hilbert algebra is a *-ideal in a B^*-algebra. In this section we consider briefly algebras A which are *-ideals in B^*-algebras.

The next proposition is true in much more generality than we present here. When C is a Banach algebra, we denote the spectrum in C of an element $a \in C$ by $Sp_C(a)$. Also for $a \in C$ we let

$$\nu_C(a) = \inf(\|a^n\|_C^{1/n}) .$$

PROPOSITION 4.1 *Assume that A is a dense *-ideal in a semi-simple Banach *-algebra B. Then every *-representation of A on a Hilbert space $\mathscr{H}$ extends uniquely to a *-representation of B on $\mathscr{H}$.*

Proof. First note that by Johnson's Uniqueness of Norm Theorem [6, Theorem 2, p. 539] there exists $K > 0$ such that

$$\|b^*\|_B \leqq K^2 \|b\|_B \text{ for all } b \in B .$$

Assume that $a \to \pi(a)$ is a *-representation of A into the bounded operators on a Hilbert space $\mathscr{H}$. If T is a bounded operator on $\mathscr{H}$, we denote the operator norm of T by $|T|$. By [8, Lemma (4.4.6), p. 208] $|\pi(a)|^2 \leqq \nu_A(a^*a)$ for all $a \in A$. Since A is an ideal of B, then $Sp_A(a) \cup \{0\} = Sp_B(a) \cup \{0\}$ for all $a \in A$. Then $|\pi(a)|^2 \leqq \nu_A(a^*a) = \nu_B(a^*a) \leqq \|a^*a\|_B \leqq K^2 \|a\|_B^2$ for all $a \in A$. Thus $|\pi(a)| \leqq K\|a\|_B$ for all $a \in A$. Therefore π extends uniquely to a *-representation of B on $\mathscr{H}$.

Now we prove the main result of this section.

THEOREM 4.2. *Assume that A is a dense *-ideal in a B^*-algebra B. Then*

(1) *A has an approximate identity consisting of self-adjoint elements.*

(2) *For every closed left* [right] *ideal M of A, $M = A \cap \mathrm{cl}(M)$.*

(3) *Every proper closed left* [right] *ideal M of A in the inter-*

section of modular maximal left [right] *ideals of* A.

(4) *Every* **-representation of* A *on a Hilbert space* $\mathscr{H}$ *extends uniquely to a* **-representation of* B *on* $\mathscr{H}$.

Proof. Construct the net $\{d_\lambda\}$, $\lambda \in \Lambda$, in A as in the proof of [8, Theorem (4.8.14), p. 245]. Then by this theorem and the fact that A is dense in B, $\{d_\lambda\}$, $\lambda \in \Lambda$, is a self-adjoint bounded approximate identity for B. Then by Proposition 3.3, $\{d_\lambda\}$, $\lambda \in \Lambda$, is an approximate identity for A. This proves (1). (2) follows from (1), Proposition 2.2, and Theorem 3.2.

Assume that M is a closed left ideal of A. Then by (2) $M = A \cap \mathrm{cl}(M)$. By [3, Theorem 2.9.5, p. 48] $\mathrm{cl}(M) = \bigcap_{\gamma \in \Gamma} N_\gamma$ where Γ is an index set and each N_γ is a modular maximal left ideal of B. By Proposition 1.1 $A \cap N_\gamma$ is a modular maximal left ideal of A for each $\gamma \in \Gamma$. Then $M = A \cap (\mathrm{cl}(M)) = \bigcap_{\gamma \in \Gamma} (A \cap N_\gamma)$. This proves (3). Finally (4) follows from Proposition 4.1.

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