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MEROMORPHIC ANNULAR FUNCTIONS

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The purpose of this paper is to present a definition of meromorphic annular functions which includes the definition of holomorphic annular functions. Several equivalent conditions for meromorphic annular functions are given.

2. **Preliminary definitions and remarks.** Let D be the disk $|z| < 1$ and C the circle $|z| = 1$. We shall, henceforth, assume that the function $f(z)$ is meromorphic in D.

A boundary path in D is the image of the unit interval $0 \leqq t < 1$ under a continuous function $z = z(t)$ from $0 \leqq t < 1$ into D such that $\lim_{t \to 1} |z(t)| = 1$. A spiral in D is a boundary path with the additional condition that $\lim_{t \to 1} \arg z(t) = +\infty$ or $-\infty$ for any branch of the argument of $z(t)$.

The set $L(\lambda) = \{z \mid |f(z)| = \lambda,\ 0 < \lambda < \infty\}$ is called a level set for the function f and a component of $L(\lambda)$ is called a level curve. It is known [6, Prop. 1] that if $C(\lambda)$ is a level curve which does not contain any zeros of $f'(z)$, then $C(\lambda)$ is either a Jordan curve contained in D or the union of two disjoint boundary paths. If $\lambda = 0$ or $\lambda = \infty$, then $L(\lambda)$ corresponds to the set of zeros or poles, respectively, of $f(z)$.

The function $f(z)$ has the asymptotic value a (allowing $a = \infty$) if there exists a boundary path $z = z(t)$, $0 \leqq t < 1$, such that $\lim_{t \to 1} f(z(t)) = a$.

The following definition will be taken for the definition of meromorphic annular functions.

DEFINITION 1. Let $f(z)$ be a nonconstant meromorphic function in D and let $\{J_n\}$ be a sequence of Jordan curves with J_1 contained in the interior of J_n for $n = 2, 3, 4, \cdots$ such that either

$$\lim_{n \to \infty} \max_{z \in J_n} |f(z) - a| = 0 ,$$

for a finite value a, or, if $a = \infty$,

$$\lim_{n \to \infty} \min_{z \in J_n} |f(z)| = \infty .$$

If, in addition, f has an asymptotic value, then f will be called an annular function with respect to a.

The class of annular functions with respect to a will be denoted by $\mathscr{A}(a)$.

REMARK 1. It is proved in Theorem 1 of [7] that if $f \in \mathscr{A}(a)$, then given any r, $0 < r < 1$, there exists an integer N such that if $n \geqq N$, then the disk $|z| \leqq r$ is contained in the interior of J_n. In such a case the sequence $\{J_n\}$ is said to converge uniformly to the boundary C.

REMARK 2. A subsequence $\{J_{n_k}\}$ can be selected such that if $k \neq j$ then $J_{n_k} \cap J_{n_j} = \phi$.

From these two remarks it may be assumed that the members of the sequence $\{J_n\}$ of Definition 1 are pairwise disjoint and that the sequence tends uniformly to C.

REMARK 3. It is evident that if $f \in \mathscr{A}(a)$ then the asymptotic value assumed to exist in Definition 1 is a.

REMARK 4. If $a \neq b$, then $\mathscr{A}(a) \cap \mathscr{A}(b) = \phi$. The function f is an $\mathscr{A}(a)$ if and only if $1/f \in \mathscr{A}(1/a)$.

REMARK 5. If f is holomorphic and annular in the old sense [1, 340] then there exists a sequence of Jordan curves $\{J_n\}$ which tend to the circle C and on which f tends uniformly to ∞. Since every holomorphic function has an asymptotic value, which in this case must be ∞, it is seen that $f \in \mathscr{A}(\infty)$. Thus there exists a function in $\mathscr{A}(0)$; the reciprocal of any function annular in the old sense.

The following definitions are needed.

DEFINITION 2. If the nonconstant meromorphic function f in D has the asymptotic value a on a spiral asymptotic path, then f is a spiral function with respect to a.

The class of spiral function with respect to a will be denoted by $\mathscr{S}(a)$.

DEFINITION 3. If the nonconstant and meromorphic function f in D is bounded away from a on a spiral boundary path, then f is said to be in the Valiron class with respect to a, provided f has the asymptotic value a.

The class of such functions will be denoted by $\mathscr{V}(a)$.

REMARK 6. $\mathscr{V}(a) \subset \mathscr{S}(a)$.

DEFINITION 4. The function $f(z)$ is in the class $\mathscr{L}'(a)$ if f is nonconstant and meromorphic in D and has the asymptotic value a as well as the following property: In the case of a finite value a, every level curve of $f(z) - a$ which is disjoint from the zeros of $f'(z)$ is a compact set in D, or, in the case of $a = \infty$, every level curve of $f(z)$ which is disjoint from the zeros of $f'(z)$ is a compact set in D.

DEFINITION 5. Let $\mathscr{L}(a)$ be the class of functions f such that f is in $\mathscr{L}'(a)$ and such that every level curve of $f(z) - a$ (or $f(z)$ if $a = \infty$) is a compact set in D.

It will be shown that $\mathscr{L}(a) = \mathscr{L}'(a)$.

DEFINITION 6. A tract $\{D(\in), a\}$ for the meromorphic function f in D associated with the value a is a set of non-void domains $D(\varepsilon)$ each of which is a component of $\{z \mid |f(z) - a| < \varepsilon\}$, or $\{z \mid |f(z)| > 1/\varepsilon\}$ if $a = \infty$, such that $D(\varepsilon) \subset D(\varepsilon')$ if $\varepsilon < \varepsilon'$ and $\bigcap_{\varepsilon<0} D(\varepsilon) = \phi$.

3. Equivalences for $\mathscr{A}(a)$. The following theorem gives the main equivalences for the class $\mathscr{A}(a)$ and corresponds to Theorems 1 and 3 of [6].

THEOREM 1. $\mathscr{A}(a) = \mathscr{S}(a) - \mathscr{T}(a) = \mathscr{L}'(a) = \mathscr{L}(a)$.

Proof. To prove the theorem in the most economical way we prove the chain of containments $\mathscr{A}(a) \subset \mathscr{S}(a) - \mathscr{T}(a) \subset \mathscr{L}'(a) \subset \mathscr{A}(a) \subset \mathscr{L}(a) \subset \mathscr{L}'(a)$.

First, let $f \in \mathscr{A}(a)$, let T be the asymptotic path on which f tends to a (see Remark 3), and let $\{J_n\}$ be the sequence of Jordan curves of Definition 1 on which f tends uniformly to a. Using the same construction as in Theorem 2 of [6] a spiral may be constructed on which f has the asymptotic value a. Thus $f \in \mathscr{S}(a)$. Evidently every boundary path intersects members of $\{J_n\}$ for all sufficiently large n so that f cannot be bounded away from a on any spiral. Since f has the asymptotic value a, f is not in $\mathscr{T}(a)$ and is in $\mathscr{S}(a) - \mathscr{T}(a)$.

Now let $f \in \mathscr{S}(a)$ and let $C(\lambda)$ be a level curve of $f(z) - a$ which contains no zeros of $f'(z)$. If $C(\lambda)$ is not a Jordan curve in D, then it consists of two boundary paths (spirals) on which f is bounded away from a, and we may conclude that $f \in \mathscr{T}(a)$. Therefore, if $f \in \mathscr{S}(a) - \mathscr{T}(a)$, then each level curve $C(\lambda)$ of $f(z) - a$ containing no zeros of $f'(z)$ must be a Jordan curve in D, and hence $f \in \mathscr{L}'(a)$ and we obtain $\mathscr{S}(a) - \mathscr{T}(a) \subset \mathscr{L}'(a)$.

Let $f \in \mathscr{L}'(a)$. Because f has the asymptotic value a, there is a tract $\{D(\varepsilon), a\}$ associated with a. Choose a sequence $\{\varepsilon_n\}$ of positive numbers such that $\varepsilon_n \downarrow 0$ as $n \to \infty$ and the level set $\{z \mid |f(z) - a| = \varepsilon_n\}$, or $\{z \mid |f(z)| = 1/\varepsilon_n\}$ if $a = \infty$, does not contain any zeros of $f'(z)$

for $n = 1, 2, \cdots$.

For each n, $D(\varepsilon_n)$ contains an asymptotic path with asymptotic value a so that it cannot be contained within a Jordan curve in D. Thus the set $D(\varepsilon_n)$ is $|z| < 1$ with a countable or finite number of Jordan domains removed. Otherwise the boundary of $D(\varepsilon_n)$ would contain a level curve which is not a compact set in D, contrary to the hypothesis that $f \in \mathscr{L}'(a)$.

If $D(\varepsilon_n) = D$ for every n then $|f(z) - a| < \varepsilon_n$ (or $|f(z)| > 1/\varepsilon_n$) in D for every n and $f(z)$ is identically constant contrary to the definition of $\mathscr{L}'(a)$. Thus there exists an integer n_1 and a point $z_1 \in D$ which is not in $D(\varepsilon_{n_1})$. Let J_1 be the Jordan curve in D which contains z_1 in its interior and which is a component of the boundary of $D(\varepsilon_{n_1})$. Let J_2 be the Jordan curve which contains z_1 in its interior and is a boundary component of $D(\varepsilon_{n_1+1})$. Because of the definition of tract, $D(\varepsilon_{n_1+1}) \subset D(\varepsilon_{n_1})$ which implies that J_2 contains J_1 in its interior. Continuing in the same manner we obtain a sequence of Jordan curves $\{J_n\}$ such that J_n contains J_1 in its interior for $n = 2, 3, 4, \cdots$ and such that $|f(z) - a| = \varepsilon_n$ (or $|f(z)| = 1/\varepsilon_n$) for all $z \in J_n$, $n = 1, 2, \cdots$. Since f is not a constant, J_n tends uniformly to C, or $\min_{z \in J_n} |z| \to 1$ as $n \to \infty$, because of [7, Theorem 1]. Thus f has an asymptotic value and has the sequence $\{J_n\}$ with all the properties of Definition 1 so that f is in $\mathscr{A}(a)$.

If $f \in \mathscr{A}(a)$, then it is easy to see that any level curve of $f(z) - a$ is contained inside one of the Jordan curves J_n of Definition 1 and is thus compact in D. Thus $f \in \mathscr{L}(a)$.

Finally, $\mathscr{L}(a) \subset \mathscr{L}'(a)$ by definition, and the proof of Theorem 1 is complete. There is one other characterization of the set $\mathscr{A}(a)$ which was suggested to me by J. Choike.

COROLLARY. *The function f is in $\mathscr{A}(a)$ if and only if f has an asymptotic value and every boundary path contains a sequence of points a_n such that $\lim_{n\to\infty} f(z_n) = a$ and $\lim_{n\to\infty}|z_n| = 1$*

Proof. If $f \in \mathscr{A}(a)$ the conclusion follows immediately.

Let $f \notin \mathscr{A}(a)$. Then by Theorem 1 there exists a level curve $C(\lambda)$ of the function $f(z) - a$, or $f(z)$ if $a = \infty$, which is disjoint from the zeros of $f'(z)$ and is not a Jordan curve in D. Hence, $C(\lambda)$ contains a boundary path T on which $|f(z) - a| = \lambda$, or $|f(z)| = \lambda$ if $a = \infty$, and so there does not exist a sequence $z_n \in T$ such that $\lim_{n\to\infty} f(z_n) = a$ and f fails to satisfy the conditions of the corollary. This completes the proof.

4. A short proof of a corollary of McMillan. The proof given in this section is elementary in the sense that it uses only the classical results of Fatou and F. and M. Riesz. The theorem of McMillan [4, p. 151] to which this corollary refers is very complicated and uses many measure theoretic concepts.

The method of proof uses a result of MacLane [3, p. 13] which was used to prove several results in [7].

DEFINITION 7. The end of the tract $\{D(\varepsilon), a\}$ is $\bigcap_{\varepsilon>0} \bar{D}(\varepsilon)$ where $\bar{D}(\varepsilon)$ represents the closure of $D(\varepsilon)$.

THEOREM 2 (McMillan). *If $f(z)$ is a holomorphic function in D which has a finite number of tracts, the union K of the ends of the tracts associated with ∞ is the circle C: $|z| = 1$.*

Proof. Let $T_1(\infty) = \{D_1(\varepsilon), \infty\}$, $T_2(\infty) = \{D_2(\varepsilon), \infty\}, \cdots, T_m(\infty) = \{D_m(\varepsilon), \infty\}$, $T_1(a_1)$, $T_2(a_2), \cdots, T_p(a_p)$, be the tracts for f where $a_i \neq \infty$, $i = 1, 2, \cdots, p$.

Assume the contrary of the conclusion: that is $K \neq C$. Because K is closed there exists a disk N about a point of C such that $K \cap \bar{N} = \phi$. If for some i and every $\varepsilon > 0$, $\bar{D}_i(\varepsilon) \cap \bar{N} \neq \phi$, where $D_i(\varepsilon)$ is the set of domains for $T_i(\infty)$, select a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ and a sequence $\{z_n\}$ such that $z_n \in \bar{D}_i(\varepsilon_n) \cap \bar{N}$. By Definition 6 $\{z_n\}$ has a limit point $\zeta \in C$ which is also in $\bar{N}$. Then $\zeta \in \bigcap_{j=1}^{\infty} \bar{D}_i(\varepsilon_j) \subset K$, in violation of $K \cap \bar{N} = \phi$. Thus for each $n = 1, 2, \cdots, m$ there exists an $\varepsilon_n > 0$ such that $\bar{D}_n(\varepsilon_n) \cap \bar{N} = \phi$. For $\varepsilon = \mathrm{Min}\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_m\}$, we have $\bar{D}_n(\varepsilon) \cap \bar{N} = \phi$, for each n, $n = 1, 2, \cdots, m$.

If $f(z)$ were bounded in $N \cap D$, then $f(z)$ has radial limit almost everywhere on $N \cap C$ by the theorem of Fatou [2]. These limits must be selected from $\{a_1, a_2, \cdots, a_p\}$. Let A_i be the set of $\zeta \in N \cap C$ for which f has radial limit a_i. By the F. and M. Riesz theorem [5] the measure of A_i is 0. Hence the measure of $\bigcup_{i=1}^{p} A_i$ is also 0 and $f(z)$ has radial limit on at most a set of measure 0 on $N \cap C$. Thus $f(z)$ is unbounded in $N \cap D$ and it is possible to choose $z_0 \in N \cap D$ such that $|f(z_0)| > 1/\varepsilon$, $f'(z_0) \neq 0$, and $f(z_0) \neq a_i$ for $i = 1, 2, \cdots, p$.

By methods of MacLane [3, p. 13] there exists an arc T from $f(z_0)$ on the Riemann surface of f^{-1} which ends at ∞. The arc T can be chosen so that its projection in the w-plane is a ray on which $|w| \geqq |f(z_0)|$. The inverse image of T has a component γ which contains z_0. Because $|f(z)| \geqq |f(z_0)| \geqq 1/\varepsilon$ for all $z \in \gamma$ and because γ is an asymptotic path with ∞ as asymptotic value, $\gamma \subset D_n(\varepsilon)$ for some n between 1 and m. This implies $z_0 \in D_n(\varepsilon) \cap N$. But it has been established that $D_n(\varepsilon) \cap N = \phi$. This contradiction implies that the assumption $K \neq C$ is false and the theorem is proved.

REMARK. The proof just given goes through for holomorphic functions with finite tracts associated with ∞ and a countable number of tracts associated with finite values. By another corollary of McMillan [4, p. 151] no such function exists. If $f(z)$ has finite tracts associated with ∞ and infinite tracts, then f has point asymptotic values (values which are approached along a path that ends at a point of C) on a set of positive measure.

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